

A DARBOUX THEOREM FOR DERIVED SCHEMES WITH SHIFTED SYMPLECTIC STRUCTURE

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1. INTRODUCTION

In the context of Toën and Vezzosi’s theory of derived algebraic geometry [27–29], Pantev, Toën, Vaquié, and Vezzosi [24, 32] defined a notion of *k-shifted symplectic structure* ω on a derived scheme or stack \mathbf{X} , for $k \in \mathbb{Z}$. If \mathbf{X} is a derived scheme and ω a 0-shifted symplectic structure, then $\mathbf{X} = X$ is a smooth classical scheme and $\omega \in H^0(\Lambda^2 T^*X)$ a classical symplectic structure on X .

Pantev et al. [24] introduced a notion of Lagrangian $\mathbf{i} : \mathbf{L} \rightarrow \mathbf{X}$ in a *k-shifted symplectic derived stack* (\mathbf{X}, ω) and showed that the fibre product $\mathbf{L} \times_{\mathbf{X}} \mathbf{M}$ of Lagrangians $\mathbf{i} : \mathbf{L} \rightarrow \mathbf{X}$, $\mathbf{j} : \mathbf{M} \rightarrow \mathbf{X}$ is $(k - 1)$ -shifted symplectic. Thus, (derived) intersections $L \cap M$ of Lagrangians L, M in a classical algebraic symplectic manifold (S, ω) are -1 -shifted symplectic. They also proved that if Y is a Calabi–Yau m -fold, then the derived moduli stacks \mathcal{M} of (complexes of) coherent sheaves on Y carry a natural $(2 - m)$ -shifted symplectic structure.

The main aim of this paper is to prove a *Darboux theorem*, Theorem 5.18 below, which says that if \mathbf{X} is a derived scheme and $\tilde{\omega}$ a *k-shifted symplectic structure* on \mathbf{X} for $k < 0$ with $k \not\equiv 2 \pmod{4}$, then $(\mathbf{X}, \tilde{\omega})$ is Zariski locally equivalent to $(\mathbf{Spec} A, \omega)$, for $\mathbf{Spec} A$ an affine derived scheme in which the commutative differential graded algebra (cdga) A is smooth in degree 0 and quasi-free in negative degrees, and that has Darboux-like coordinates x_j^i, y_j^{k-i} with respect to which the symplectic form $\omega = \sum_{i,j} d_{dR} x_j^i d_{dR} y_j^{k-i}$ is standard, and in which the differential in A is given by a Poisson bracket with a Hamiltonian function Φ of degree $k + 1$.

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When $k < 0$ with $k \equiv 2 \pmod{4}$ we give two statements, one Zariski local in \mathbf{X} in which the symplectic form ω on $\mathbf{Spec} A$ is standard except for the part in the degree $k/2$ variables, which depends on some functions q_i , and one étale local in \mathbf{X} in which ω is entirely standard. In the case $k = -1$, Theorem 5.18 implies that a -1 -shifted symplectic derived scheme $(\mathbf{X}, \tilde{\omega})$ is Zariski locally equivalent to the derived critical locus $\mathbf{Crit}(\Phi)$ of a regular function $\Phi : U \rightarrow \mathbb{A}^1$ on a smooth scheme U .

1.1. This series of papers. This is the second in a series of eight papers [16], [7], [9], [3], [8], [5], [17], with more to come. The previous paper [16] defined *algebraic d-critical loci* (X, s) , which are classical schemes X with an extra (classical, not derived) geometric structure s that records information on how X may locally be written as a classical critical locus $\mathbf{Crit}(\Phi)$ of a regular function $\Phi : U \rightarrow \mathbb{A}^1$ on a smooth scheme U .

Our second main result, Theorem 6.6 below, says that if (\mathbf{X}, ω) is a -1 -shifted symplectic derived scheme, then the underlying classical scheme $X = t_0(\mathbf{X})$ extends naturally to an algebraic d-critical locus (X, s) . That is, we define a truncation functor from -1 -shifted symplectic derived schemes to algebraic d-critical loci.

The third and fourth papers [7, 9] will show that if (X, s) is an algebraic d-critical locus with an *orientation*, then we can define a natural perverse sheaf $P_{X,s}^\bullet$, a \mathcal{D} -module $D_{X,s}^\bullet$, a mixed Hodge module $M_{X,s}^\bullet$ (when X is over \mathbb{C}), and a motive $MF_{X,s}$ on X , such that if (X, s) is locally modelled on $\mathbf{Crit}(\Phi : U \rightarrow \mathbb{A}^1)$, then $P_{X,s}^\bullet, D_{X,s}^\bullet, M_{X,s}^\bullet$ are locally modelled on the perverse sheaf, \mathcal{D} -module, and mixed Hodge module of vanishing cycles of Φ , and $MF_{X,s}$ is locally modelled on the motivic vanishing cycle of Φ .

Combining these with Theorem 6.6 and results of Pantev et al. [24] gives natural perverse sheaves, \mathcal{D} -modules, mixed Hodge modules, and motives on classical moduli schemes \mathcal{M} of simple (complexes of) coherent sheaves on a Calabi–Yau 3-fold with orientations, and on intersections $L \cap M$ of spin Lagrangians L, M in an algebraic symplectic manifold (S, ω) . These will have applications to categorified and motivic Donaldson–Thomas theory of Calabi–Yau 3-folds, and to defining *Fukaya categories* of Lagrangians in algebraic symplectic manifolds using perverse sheaves.

The fifth paper [3] will extend the results of this paper and [7, 9] from (derived) schemes to (derived) Artin stacks. The sixth [8] will prove a complex analytic analogue of Corollary 6.8 below, saying that the intersection $X = L \cap M$ of complex Lagrangians L, M in a complex symplectic manifold (S, ω) extends naturally to a complex analytic d-critical locus (X, s) .

The seventh paper [5] uses Theorem 5.18 to show that any -2 -shifted symplectic derived \mathbb{C} -scheme (\mathbf{X}, ω) can be given the structure of a derived smooth manifold \mathbf{X}_{dm} . If \mathbf{X}_{dm} is compact and oriented, it has a virtual cycle $[\mathbf{X}_{\text{dm}}]$ in bordism or homology. Using this, we propose to define Donaldson–Thomas style invariants *counting* (semi)stable coherent sheaves on Calabi–Yau 4-folds.

The eighth paper [17] proves a *Lagrangian neighbourhood theorem* which gives local models for Lagrangians \mathbf{L} in k -shifted symplectic derived \mathbb{K} -schemes (\mathbf{X}, ω) , relative to the Darboux form local models for (\mathbf{X}, ω) in Theorem 5.18.

1.2. Organisation of this paper. We begin with background material from algebra in §2, and from derived algebraic geometry in §3. Section 4, which is not particularly original, proves that any derived \mathbb{K} -scheme \mathbf{X} is near any point $x \in \mathbf{X}$ Zariski

locally equivalent to $\mathbf{Spec} A$ for A a *standard form* cdga which is minimal at x , and explains how to compare two such local presentations $\mathbf{Spec} A \simeq \mathbf{X} \simeq \mathbf{Spec} B$.

The heart of the paper is §5, which defines our Darboux form local models A, ω , and proves our main result, Theorem 5.18, that any k -shifted symplectic derived \mathbb{K} -scheme $(\mathbf{X}, \tilde{\omega})$ for $k < 0$ is locally of the form $(\mathbf{Spec} A, \omega)$. Finally, §6 discusses algebraic d-critical loci from [16], and proves our second main result Theorem 6.6, defining a truncation functor from -1 -shifted symplectic derived \mathbb{K} -schemes to algebraic d-critical loci.

Bouaziz and Grojnowski [6] have independently proved their own Darboux theorem for k -shifted symplectic derived schemes for $k < 0$, similar to Theorem 5.18, by a different method. See Joyce and Safronov [17, Rem. 2.15] for an explanation of how this paper and [6] are related.

1.3. Conventions. Throughout, \mathbb{K} will be an algebraically closed field with characteristic 0. All cdgas will be graded in nonpositive degrees (i.e., they are *connective* cdgas). All classical \mathbb{K} -schemes are assumed locally of finite type, and all derived \mathbb{K} -schemes \mathbf{X} are assumed to be locally finitely presented.

2. BACKGROUND FROM ALGEBRA

We begin by reviewing some fairly standard facts about cdgas, and their cotangent complexes. Some references on cdgas in derived algebraic geometry are Toën and Vezzosi [28, §2.3], [29, §2], and Lurie [23, §7.1]; on cdgas from other points of view are Gelfand and Manin [11, §5.3] and Hess [14]; and on cotangent complexes are Toën and Vezzosi [28, §1.4], [27, §4.2.4–§4.2.5, §3.1.7], and Lurie [23, §7.3].

2.1. Commutative graded algebras. In this section we introduce general definitions and conventions about commutative graded algebras.

Definition 2.1. Fix a ground field \mathbb{K} of characteristic 0. Eventually we shall wish to standardise a quadratic form, at which point we need to assume in addition that \mathbb{K} is algebraically closed. We shall work with *connective commutative graded algebras* A over \mathbb{K} so A has a decomposition $A = \bigoplus_{i \leq 0} A^i$ (“connective” means concentrated in nonpositive degrees) and an associative product $m : A^i \otimes A^j \rightarrow A^{i+j}$ satisfying $fg = (-1)^{|f||g|}gf$ for homogeneous elements $f, g \in A$. Given a graded left module M , we consider it as a *symmetric bimodule* by setting

$$m \cdot f = (-1)^{|f||m|} f \cdot m$$

for homogeneous elements $f \in A, m \in M$.

Define a *derivation of degree k* from A to a graded module M to be a \mathbb{K} -linear map $\delta : A \rightarrow M$ that is homogeneous of degree k and satisfies

$$\delta(fg) = \delta(f)g + (-1)^{k|f|} f\delta(g).$$

Just as for (ungraded) commutative algebras, there is a universal derivation into a (graded) module of *Kähler differentials* Ω_A^1 , which can be constructed as I/I^2 for $I = \text{Ker}(m : A \otimes A \rightarrow A)$. The universal derivation $\delta : A \rightarrow \Omega_A^1$ is then computed as $\delta(a) = a \otimes 1 - 1 \otimes a \in I/I^2$. One checks that indeed δ is a universal degree 0 derivation, so that $\circ\delta : \underline{\text{Hom}}_A(\Omega_A^1, M) \rightarrow \underline{\text{Der}}(A, M)$ is an isomorphism of graded modules. Here the underline denotes that we take a sum over homogeneous morphisms of each degree.

In the particular case when $M = A$, one sometimes refers to a derivation $X : A \rightarrow A$ of degree k as a *vector field of degree k* . Define the *graded Lie bracket* of two homogeneous vector fields X, Y by

$$[X, Y] := XY - (-1)^{|X||Y|}YX.$$

One checks that $[X, Y]$ is a homogeneous vector field of degree $|X| + |Y|$. On any commutative graded algebra, there is a canonical degree 0 *Euler vector field* E which acts on a homogeneous element $f \in A$ via $E(f) = |f|f$. In particular, it annihilates functions $f \in A^0$ of degree 0.

Define the *de Rham algebra* of A to be the free commutative graded algebra over A on the graded module $\Omega_A^1[1]$:

$$(2.1) \quad \text{DR}(A) := \text{Sym}_A(\Omega_A^1[1]) = \bigoplus_p \Lambda^p \Omega_A^1[p].$$

We endow $\text{DR}(A)$ with the de Rham operator d_{dR} , which is the unique square-zero derivation of degree -1 on the commutative graded algebra $\text{DR}(A)$ such that for $f \in A$, $d_{dR}(f) = \delta(f)[1] \in \Omega_A^1[1]$. Thus $d_{dR}(fg) = d_{dR}(f)g + (-1)^{|f|}f d_{dR}(g)$ for $f, g \in A$ and $d_{dR}(\alpha \cdot \beta) = d_{dR}(\alpha)\beta + (-1)^{|\alpha|}\alpha d_{dR}(\beta)$ for any two $\alpha, \beta \in \text{DR}(A)$.

Remark 2.2. The de Rham algebra $\text{DR}(A)$ has two gradings, one induced by the grading on A and on the module $\Omega^1[1]$ and the other given by p in the decomposition $\text{DR}(A) := \text{Sym}_A(\Omega_A^1[1]) \cong \bigoplus_p \Lambda^p \Omega_A^1[p]$. We shall refer to the first grading as *degree* and the second grading as *weight*. Thus the de Rham operator d_{dR} has degree -1 and weight $+1$.

Note that the condition that d_{dR} be a derivation of degree -1 *does not* take into account the additional grading by weight, but nevertheless this convention does recover the usual de Rham complex in the classical case when the algebra A is concentrated in degree 0.

Example 2.3. For us, a typical commutative graded algebra $A = \bigoplus_{i \leq 0} A^i$ will have A^0 smooth over \mathbb{K} and will be free over A^0 on graded variables $x_1^{-1}, \dots, x_{m_1}^{-1}, x_1^{-2}, \dots, x_{m_2}^{-2}, \dots, x_1^{-n}, \dots, x_{m_n}^{-n}$ where x_j^{-i} has degree $-i$. Localising A^0 if necessary, we may assume that there exist $x_1^0, \dots, x_{m_0}^0 \in A^0$ so that

$$d_{dR}x_1^0, \dots, d_{dR}x_{m_0}^0$$

form a basis of $\Omega_{A^0}^1$.

The graded A -module $\Omega_A^1[1]$ then has a basis

$$d_{dR}x_1^0, \dots, d_{dR}x_{m_0}^0, d_{dR}x_1^{-1}, \dots, d_{dR}x_{m_1}^{-1}, \dots, d_{dR}x_1^{-n}, \dots, d_{dR}x_{m_n}^{-n}.$$

Since $\text{DR}(A) = \text{Sym}_A(\Omega_A^1[1])$ is free commutative on $\Omega_A^1[1]$ over A and since $d_{dR}x_s^{-i}$ and $d_{dR}x_t^{-j}$ have degrees $-i - 1$ and $-j - 1$, respectively, we have

$$d_{dR}x_s^{-i}d_{dR}x_t^{-j} = (-1)^{(-i-1)(-j-1)}d_{dR}x_t^{-j}d_{dR}x_s^{-i}.$$

Thus, for example, $d_{dR}x_s^{-i}$ and $d_{dR}x_t^{-i}$ anticommute when $-i$ is even and commute when $-i$ is odd. In particular, $d_{dR}x_s^{-i}d_{dR}x_s^{-i} = 0$ when $-i$ is even and $d_{dR}x_s^{-i}d_{dR}x_s^{-i} \neq 0$ when $-i$ is odd.

Definition 2.4. Given a homogeneous vector field X on A of degree $|X|$, the contraction operator ι_X on $\mathrm{DR}(A)$ is defined to be the unique derivation of degree $|X| + 1$ such that $\iota_X f = 0$ and $\iota_X d_{\mathrm{dR}}(f) = X(f)$ for all $f \in A$. We define the Lie derivative L_X along a vector field X by

$$L_X = [\iota_X, d_{\mathrm{dR}}] = \iota_X d_{\mathrm{dR}} - (-1)^{|X|+1} d_{\mathrm{dR}} \iota_X = \iota_X d_{\mathrm{dR}} + (-1)^{|X|} d_{\mathrm{dR}} \iota_X.$$

It is a derivation of $\mathrm{DR}(A)$ of degree $|X|$. In particular, the Lie derivative along E is of degree 0. Given $f \in A$, we have

$$L_E f = \iota_E d_{\mathrm{dR}} f = E(f) = |f|f, \quad L_E d_{\mathrm{dR}} f = |f|d_{\mathrm{dR}} f.$$

In particular, the de Rham differential $d_{\mathrm{dR}} : A \rightarrow \Omega_A^1[1]$ is injective except in degree 0. Furthermore, we see that for a homogeneous form $\alpha \in \Lambda^p \Omega_A^1[p]$,

$$L_E \alpha = \iota_E d_{\mathrm{dR}} \alpha + d_{\mathrm{dR}} \iota_E \alpha = (|\alpha| + p)\alpha.$$

If α is in addition de Rham closed, then $d_{\mathrm{dR}} \iota_E \alpha = (|\alpha| + p)\alpha$, so if $|\alpha| + p \neq 0$, then α is in fact de Rham exact:

$$\alpha = d_{\mathrm{dR}} \left(\frac{\iota_E \alpha}{|\alpha| + p} \right).$$

The top degree of $\Lambda^p \Omega_A^1[p]$ is $-p$ and the elements of top degree are of the form $\alpha \in \Lambda^p \Omega_{A^0}^1[p]$, so a de Rham closed form α can fail to be exact only if it lives on $A^0 \subset A$.

The following relations between derivations of $\mathrm{DR}(A)$ can be checked by noting that both sides of an equation have the same degree and act in the same way on elements of weight 0 and weight 1 in $\mathrm{DR}(A)$.

Lemma 2.5. *Let X, Y be homogeneous vector fields on A . Then we have the following equalities of derivations on $\mathrm{DR}(A)$:*

$$[d_{\mathrm{dR}}, L_X] = 0, \quad [\iota_X, \iota_Y] = 0, \quad [L_X, \iota_Y] = \iota_{[X, Y]}, \quad [L_X, L_Y] = L_{[X, Y]}.$$

Remark 2.6. The last statement ensures that the map $X \mapsto L_X$ determines a dg-Lie algebra homomorphism $\underline{\mathrm{Der}}(A) \rightarrow \underline{\mathrm{Der}}(\mathrm{DR}(A))$.

2.2. Commutative differential graded algebras.

Definition 2.7. A commutative differential graded algebra or *cdga* (A, d) is a commutative graded algebra A over \mathbb{K} , as in §2, endowed with a square-zero derivation d of degree 1. Usually we write A rather than (A, d) , leaving d implicit. Note that the cohomology $H^*(A)$ of A with respect to the differential d is a commutative graded algebra. Note too that by default all of our cdgas are *connective* (that is, concentrated in nonpositive cohomological degrees).

A *morphism of cdgas* is a map of complexes $f : A \rightarrow B$ respecting units and multiplication. It is a *quasi-isomorphism* or *weak equivalence* if the underlying map of complexes is so, it is a *fibration* if it is a degree-wise surjection, and it is a *cofibration* if it has the left lifting property with respect to trivial cofibrations. A standard argument (see for instance Goerss and Schemmerhorn [12, Th. 3.6]) shows that in characteristic 0 this choice of weak equivalences and fibrations define a model structure on cdgas.

We shall assume that all cdgas in this paper are (homotopically) of *finite presentation* over the ground field \mathbb{K} . As for finitely presented classical \mathbb{K} -algebras, this can be formulated in terms of the preservation of filtered (homotopy) colimits.

For the precise notion, see Toën and Vezzosi [28, §1.2.3]. A classical \mathbb{K} -algebra that is of finite presentation in the classical sense *need not be* of finite presentation when considered as a cdga. Indeed, Theorem 4.1 will show that a cdga A of finite presentation admits a very strong form of finite resolution.

In much of the paper, we will work with cdgas of the following form:

Example 2.8. We will explain how to inductively construct a sequence of cdgas $A(0), A(1), \dots, A(n)$, where $A(0)$ is a smooth \mathbb{K} -algebra, and $A(k)$ has underlying commutative graded algebra free over $A(0)$ on generators of degrees $-1, \dots, -k$.

Begin with a commutative algebra $A(0)$ smooth over \mathbb{K} . To make Proposition 2.12 below hold (which says that the cotangent complex $\mathbb{L}_{A(k)}$ has a simple description) we assume the cotangent module $\Omega_{A(0)}^1$ is a free $A(0)$ -module, which can always be achieved by Zariski localising $A(0)$. Choose a free $A(0)$ -module M^{-1} of finite rank together with a map $\pi^{-1} : M^{-1} \rightarrow A(0)$. Define a cdga $A(1)$ whose underlying commutative graded algebra is free over $A(0)$ with generators given by M^{-1} in degree -1 and with differential d determined by the map $\pi^{-1} : M^{-1} \rightarrow A(0)$. By construction, we have $H^0(A(1)) = A(0)/I$, where the ideal $I \subseteq A(0)$ is the image of the map $\pi^{-1} : M^{-1} \rightarrow A(0)$.

Note that $A(1)$ fits in a homotopy pushout diagram of cdgas

$$\begin{array}{ccc} \mathrm{Sym}_{A(0)}(M^{-1}) & \xrightarrow{\quad 0_* \quad} & A(0) \\ \downarrow \pi_*^{-1} & & \downarrow \\ A(0) & \xrightarrow{\quad f^{-1} \quad} & A(1), \end{array}$$

with morphisms $\pi_*^{-1}, 0_*$ induced by $\pi^{-1}, 0 : M^{-1} \rightarrow A(0)$. Write $f^{-1} : A(0) \rightarrow A(1)$ for the resulting map of algebras.

Next, choose a free $A(1)$ -module M^{-2} of finite rank together with a map $\pi^{-2} : M^{-2}[1] \rightarrow A(1)$. Define a cdga $A(2)$ whose underlying commutative graded algebra is free over $A(1)$ with generators given by M^{-2} in degree -2 and with differential d determined by the map $\pi^{-2} : M^{-2}[1] \rightarrow A(1)$. Write f^{-2} for the resulting map of algebras $A(1) \rightarrow A(2)$.

As the underlying commutative graded algebra of $A(1)$ was free over $A(0)$ on generators of degree -1 , the underlying commutative graded algebra of $A(2)$ is free over $A(0)$ on generators of degrees $-1, -2$. Since $A(2)$ is obtained from $A(1)$ by adding generators in degree -2 , we have $H^0(A(1)) \cong H^0(A(2)) \cong A(0)/I$.

Note that $A(2)$ fits in a homotopy pushout diagram of cdgas

$$\begin{array}{ccc} \mathrm{Sym}_{A(1)}(M^{-2}[1]) & \xrightarrow{\quad 0_* \quad} & A(1) \\ \downarrow \pi_*^{-2} & & \downarrow \\ A(1) & \xrightarrow{\quad f^{-2} \quad} & A(2), \end{array}$$

with morphisms $\pi_*^{-2}, 0_*$ induced by $\pi^{-2}, 0 : M^{-2}[1] \rightarrow A(1)$.

Continuing in this manner inductively, we define a cdga $A(n) = A$ with $A^0 = A(0)$ and $H^0(A) = A(0)/I$, whose underlying commutative graded algebra is free over $A(0)$ on generators of degrees $-1, \dots, -n$.

Definition 2.9. A cdga $A = A(n)$ over \mathbb{K} constructed inductively as in Example 2.8 from a smooth \mathbb{K} -algebra $A(0)$ with $\Omega_{A(0)}^1$ free of finite rank over $A(0)$, and free finite rank modules $M^{-1}, M^{-2}, \dots, M^{-n}$ over $A(0), \dots, A(n-1)$, will be called a *standard form cdga*.

Equivalently, A is of standard form if it is finitely presented, and $A(0)$ is smooth with $\Omega_{A(0)}$ a free $A(0)$ -module, and the underlying graded algebra of A is freely generated over $A(0)$ by finitely many generators in negative degrees.

If A is of standard form, we will call a cdga A' a *localisation* of A if $A' = A \otimes_{A^0} A^0[f^{-1}]$ for $f \in A^0$; that is, A' is obtained by inverting f in A . Then A' is also of standard form, with $A'^0 \cong A^0[f^{-1}]$. If $p \in \text{Spec } H^0(A)$ with $f(p) \neq 0$, we call A' a *localisation of A at p* .

Standard form cdgas are a mild generalisation of *connective semifree cdgas*, which are standard form cdgas in which $A(0)$ is a free commutative \mathbb{K} -algebra. Semifree cdgas are among the cofibrant objects in the model structure on $\mathbf{cdga}_{\mathbb{K}}^{\leq 0}$, and Proposition 2.12 below is standard for semifree cdgas.

2.3. Cotangent complexes of cdgas.

Definition 2.10. Let (A, d) be a cdga. Then as in Definition 2.1, to the underlying commutative graded algebra A we associate the module of Kähler differentials Ω_A^1 with universal degree 0 derivation $\delta : A \rightarrow \Omega_A^1$, and the de Rham algebra $\text{DR}(A) = \text{Sym}_A(\Omega_A^1[1])$ in (2.1), with degree -1 de Rham differential $d_{dR} : \text{DR}(A) \rightarrow \text{DR}(A)$.

The differential d on A induces a unique differential on Ω_A^1 , also denoted d , satisfying $d \circ \delta = \delta \circ d : A \rightarrow \Omega_A^1$, and making (Ω_A^1, d) into a dg-module. According to our sign conventions, the differential d on $\Omega_A^1[1]$ is that on Ω_A^1 multiplied by -1 , so d on $\Omega_A^1[1]$ anticommutes with the de Rham operator $d_{dR} : A \rightarrow \Omega^1[1]$. We extend the differential d uniquely to all of $\text{DR}(A)$ by requiring it to be a derivation of degree 1 with respect to the multiplication on $\text{DR}(A)$.

When (A, d) is sufficiently nice (as in Example 2.8), then the Kähler differentials (Ω_A^1, d) give a model for the *cotangent complex* $\mathbb{L}_{(A,d)}$ of (A, d) . In practice, we shall always work with such cdgas, so we shall freely identify $\mathbb{L}_{(A,d)}$ and (Ω_A^1, d) . Usually, when dealing with cdgas and their cotangent complexes, we leave d implicit and write \mathbb{L}_A, Ω_A^1 rather than $\mathbb{L}_{(A,d)}$ and (Ω_A^1, d) .

Similarly, given a map $A \rightarrow B$ of cdgas, we can define the *relative Kähler differentials* $\Omega_{B/A}^1$, and when the map $A \rightarrow B$ is nice enough (for example, B is obtained from A adding free generators of some degree and imposing a differential, as in Example 2.8), then the relative Kähler differentials give a model for the *relative cotangent complex* $\mathbb{L}_{B/A}$.

We recall some basic facts about cotangent complexes:

- (i) Given a map of cdgas $\alpha : A \rightarrow B$, there is an induced map $\mathbb{L}_\alpha : \mathbb{L}_A \rightarrow \mathbb{L}_B$ of A -modules, and we have a (homotopy) fibre sequence

$$\mathbb{L}_A \otimes_A B \xrightarrow{\mathbb{L}_\alpha \otimes \text{id}_B} \mathbb{L}_B \longrightarrow \mathbb{L}_{B/A}.$$

(ii) Given a (homotopy) pushout square of cdgas

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A' & \longrightarrow & B', \end{array}$$

we have a *base change* equivalence

$$(2.2) \quad \mathbb{L}_{B'/A'} \simeq \mathbb{L}_{B/A} \otimes_B B'.$$

(iii) If $B = \text{Sym}_A(M)$ for a module M over a cdba A , with inclusion morphism $\iota : A \hookrightarrow B = A \oplus M \oplus S^2M \oplus \dots$, we have a canonical equivalence

$$\mathbb{L}_{B/A} \simeq B \otimes_A M.$$

Also, any A -module morphism $\rho : M \rightarrow A$ (such as $\rho = 0$) induces a unique cdba morphism $\rho_* : B \rightarrow A$ with $\rho_*|_{\text{Sym}^0(M)} = \text{id}_A$ and $\rho_*|_{\text{Sym}^1(M)} = \rho$. Then we have a canonical equivalence

$$(2.3) \quad \mathbb{L}_{A/B} \simeq M[1].$$

Example 2.11. In Example 2.8 we constructed inductively a sequence of cdgas $A(0) \rightarrow A(1) \rightarrow \dots \rightarrow A(n) = A$, where $A(k)$ was obtained from $A(k-1)$ by adjoining a module of generators M^{-k} in degree $-k$ and imposing a differential. The cdba $A(k)$ then fits into a (homotopy) pushout diagram

$$(2.4) \quad \begin{array}{ccc} B(k-1) := \text{Sym}_{A(k-1)}(M^{-k}[k-1]) & \xrightarrow{0_*} & A(k-1) \\ \downarrow \pi_*^{-k} & & \downarrow \\ A(k-1) & \xrightarrow{f^{-k}} & A(k), \end{array}$$

in which $\pi_*^{-k}, 0_*$ are induced by maps $\pi_*^{-k}, 0 : M^{-k}[k-1] \rightarrow A(k-1)$.

We will describe the relative cotangent complexes $\mathbb{L}_{A(k)/A(k-1)}$ and hence, inductively, the cotangent complex \mathbb{L}_A . By equation (2.3), the relative cotangent complex $\mathbb{L}_{A(k-1)/B(k-1)}$ for $0_* : B(k-1) \rightarrow A(k-1)$ in (2.4) is given by

$$\mathbb{L}_{A(k-1)/B(k-1)} \simeq M^{-k}[k].$$

Thus by base change (2.2), the relative cotangent complex $\mathbb{L}_{A(k)/A(k-1)}$ of $f^{-k} : A(k-1) \rightarrow A(k)$ in (2.4) satisfies

$$\mathbb{L}_{A(k)/A(k-1)} \simeq A(k) \otimes_{A(k-1)} M^{-k}[k].$$

Note in particular that when $i > -k$, we have

$$(2.5) \quad H^i(\mathbb{L}_{A(k)/A(k-1)}) = 0.$$

Commutative diagrams of the following form, in which the rows and columns are fibre sequences, are very useful for computing the cohomology of \mathbb{L}_A :

$$(2.6) \quad \begin{array}{ccccc} & & & & A \otimes_{A(k)} \mathbb{L}_{A(k)/A(k-1)} \\ & & & & \downarrow \\ A \otimes_{A(k-1)} \mathbb{L}_{A(k-1)} & \longrightarrow & \mathbb{L}_A & \longrightarrow & \mathbb{L}_{A/A(k-1)} \\ \downarrow & & \parallel & & \downarrow \\ A \otimes_{A(k)} \mathbb{L}_{A(k)} & \longrightarrow & \mathbb{L}_A & \longrightarrow & \mathbb{L}_{A/A(k)} \\ \downarrow & & & & \\ A \otimes_{A(k)} \mathbb{L}_{A(k)/A(k-1)} & & & & \end{array}$$

Proposition 2.12. *Let $A = A(n)$ be a standard form cdga constructed inductively as in Example 2.8. Then the restriction of the cotangent complex \mathbb{L}_A to $\text{Spec } H^0(A)$ is naturally represented as a complex of free $H^0(A)$ -modules*

$$(2.7) \quad 0 \longrightarrow V^{-n} \xrightarrow{d^{-n}} V^{1-n} \xrightarrow{d^{1-n}} \dots \xrightarrow{d^{-2}} V^{-1} \xrightarrow{d^{-1}} V^0 \longrightarrow 0,$$

where $V^{-k} = H^{-k}(\mathbb{L}_{A(k)/A(k-1)})$ is in degree $-k$ and the differential $d^{-k} : V^{-k} \rightarrow V^{1-k}$ is identified with the composition

$$H^{-k}(\mathbb{L}_{A(k)/A(k-1)}) \longrightarrow H^{1-k}(\mathbb{L}_{A(k-1)}) \longrightarrow H^{1-k}(\mathbb{L}_{A(k-1)/A(k-2)}),$$

in which $H^{-k}(\mathbb{L}_{A(k)/A(k-1)}) \rightarrow H^{1-k}(\mathbb{L}_{A(k-1)})$ is induced by the fibre sequence

$$A(k) \otimes_{A(k-1)} \mathbb{L}_{A(k-1)} \longrightarrow \mathbb{L}_{A(k)} \longrightarrow \mathbb{L}_{A(k)/A(k-1)},$$

and $H^{1-k}(\mathbb{L}_{A(k-1)}) \rightarrow H^{1-k}(\mathbb{L}_{A(k-1)/A(k-2)})$ is induced by the fibre sequence

$$A(k-1) \otimes_{A(k-2)} \mathbb{L}_{A(k-2)} \longrightarrow \mathbb{L}_{A(k-1)} \longrightarrow \mathbb{L}_{A(k-1)/A(k-2)}.$$

Proof. The proof is almost immediate by induction on n . When $n = 0$, $\mathbb{L}_{A(0)} = \Omega_{A(0)}^1$ is free of finite rank over $A(0)$ by definition of standard form cdgas, so $V^0 = \Omega_{A(0)}^1 \otimes_{A(0)} H^0(A)$ is free of finite rank over $H^0(A)$. For $n = 1$, tensor the fibre sequence $A(1) \otimes_{A(0)} \mathbb{L}_{A(0)} \rightarrow \mathbb{L}_{A(1)} \rightarrow \mathbb{L}_{A(1)/A(0)}$ with $H^0(A)$ to get a fibre sequence $V^0 \rightarrow H^0(A) \otimes_{A(0)} \mathbb{L}_{A(0)} \rightarrow V^{-1}[1]$ in which the connecting morphism $V^{-1} \rightarrow V^0$ is as claimed.

Similarly, assuming we have proved the proposition for $\mathbb{L}_{A(n-1)}$, tensor the fibre sequence $A \otimes_{A(n-1)} \mathbb{L}_{A(n-1)} \rightarrow \mathbb{L}_A \rightarrow \mathbb{L}_{A/A(n-1)}$ with $H^0(A)$ to get a fibre sequence $H^0(A) \otimes_{A(n-1)} \mathbb{L}_{A(n-1)} \rightarrow H^0(A) \otimes_A \mathbb{L}_A \rightarrow V^{-n}[n]$ in which the connecting morphism is as claimed. \square

Definition 2.13. Let A be a standard form cdga constructed as in Example 2.8. We call A *minimal* at $p \in \text{Spec } H^0(A)$ if $d^{-k}|_p = 0$ for $k = 1, \dots, n$, for d^{-k} the internal differential in \mathbb{L}_A as in (2.7). That is, the compositions

$$(2.8) \quad H^{-k}(\mathbb{L}_{A(k)/A(k-1)}) \longrightarrow H^{1-k}(\mathbb{L}_{A(k-1)}) \longrightarrow H^{1-k}(\mathbb{L}_{A(k-1)/A(k-2)})$$

in the cotangent complexes restricted to $\text{Spec } H^0(A)$ vanish at p for all k .

3. BACKGROUND FROM DERIVED ALGEBRAIC GEOMETRY

Next we outline the background material from derived algebraic geometry that we need, aiming in particular to explain the key notion from Pantev, Toën, Vaquié, and Vezzosi [24] of a *k-shifted symplectic structure* on a *derived \mathbb{K} -scheme*, which is central to our paper. There are two main frameworks for derived algebraic geometry in the literature, due to Toën and Vezzosi [27–29] and Lurie [22, 23], which are broadly interchangeable in characteristic 0. Following our principal reference [24], we use the Toën–Vezzosi version.

To understand this paper (except for a few technical details), one does not really need to study derived algebraic geometry in any depth or to know what a derived stack is. The main point for us is that a *derived \mathbb{K} -scheme is a geometric space locally modelled on $\mathbf{Spec } A$ for A a cdga over \mathbb{K}* , just as a classical \mathbb{K} -scheme is a space locally modelled on $\text{Spec } A$ for A a commutative \mathbb{K} -algebra; though

the meaning of “locally modelled” is more subtle in the derived than the classical setting. Readers who are comfortable with this description of derived schemes can omit §3.1 and §3.2.

As in §2, all cdgas in this paper are *connective* (graded in nonpositive degrees), as these are the allowed local models in derived algebraic geometry.

3.1. Derived stacks. We will use Toën and Vezzosi’s theory of derived algebraic geometry [27–29]. We give a brief outline to fix notation. Fix a base field \mathbb{K} , of characteristic 0. In [27, §3], [28, §2.1], Toën and Vezzosi define an ∞ -category of (*higher*) \mathbb{K} -stacks $\mathbf{St}_{\mathbb{K}}$. Objects X in $\mathbf{St}_{\mathbb{K}}$ are ∞ -functors

$$X : \{\text{commutative } \mathbb{K}\text{-algebras}\} \longrightarrow \{\text{simplicial sets}\}$$

satisfying sheaf-type conditions. They also define a full ∞ -subcategory $\mathbf{Art}_{\mathbb{K}} \subset \mathbf{St}_{\mathbb{K}}$ of (*higher*) *Artin* \mathbb{K} -stacks, with better geometric properties.

Classical \mathbb{K} -schemes and algebraic \mathbb{K} -spaces may be written as functors

$$X : \{\text{commutative } \mathbb{K}\text{-algebras}\} \longrightarrow \{\text{sets}\},$$

and classical Artin \mathbb{K} -stacks may be written as functors

$$X : \{\text{commutative } \mathbb{K}\text{-algebras}\} \longrightarrow \{\text{groupoids}\}.$$

As simplicial sets (an ∞ -category) generalise both sets (a 1-category) and groupoids (a 2-category), higher \mathbb{K} -stacks generalise \mathbb{K} -schemes, algebraic \mathbb{K} -spaces, and Artin \mathbb{K} -stacks.

Toën and Vezzosi define the ∞ -category $\mathbf{dSt}_{\mathbb{K}}$ of *derived* \mathbb{K} -stacks (or D^- -stacks) [28, Def. 2.2.2.14], [27, Def. 4.2]. Objects \mathbf{X} in $\mathbf{dSt}_{\mathbb{K}}$ are ∞ -functors

$$\mathbf{X} : \{\text{simplicial commutative } \mathbb{K}\text{-algebras}\} \longrightarrow \{\text{simplicial sets}\}$$

satisfying sheaf-type conditions. They also define a full ∞ -subcategory $\mathbf{dArt}_{\mathbb{K}} \subset \mathbf{dSt}_{\mathbb{K}}$ of *derived Artin* \mathbb{K} -stacks, with better geometric properties.

There is a *truncation functor* $t_0 : \mathbf{dSt}_{\mathbb{K}} \rightarrow \mathbf{St}_{\mathbb{K}}$ from derived stacks to (higher) stacks, which maps $t_0 : \mathbf{dArt}_{\mathbb{K}} \rightarrow \mathbf{Art}_{\mathbb{K}}$, and a fully faithful left adjoint *inclusion functor* $i : \mathbf{St}_{\mathbb{K}} \rightarrow \mathbf{dSt}_{\mathbb{K}}$ mapping $\mathbf{Art}_{\mathbb{K}} \rightarrow \mathbf{dArt}_{\mathbb{K}}$. As i is fully faithful, we can regard it as embedding $\mathbf{St}_{\mathbb{K}}, \mathbf{Art}_{\mathbb{K}}$ as full subcategories of $\mathbf{dSt}_{\mathbb{K}}, \mathbf{dArt}_{\mathbb{K}}$. Thus, we can regard classical \mathbb{K} -schemes and Artin \mathbb{K} -stacks as examples of derived \mathbb{K} -stacks.

The adjoint property of i, t_0 implies that for any $X \in \mathbf{St}_{\mathbb{K}}$ there is a natural morphism $X \rightarrow t_0 \circ i(X)$, which is an equivalence as i is fully faithful, and for any $\mathbf{X} \in \mathbf{dSt}_{\mathbb{K}}$ there is a natural morphism $i \circ t_0(\mathbf{X}) \rightarrow \mathbf{X}$, which we may regard as embedding the classical truncation $X = t_0(\mathbf{X})$ of \mathbf{X} as a substack of \mathbf{X} . On notation: we generally write derived schemes and stacks $\mathbf{X}, \mathbf{Y}, \dots$ and their morphisms $\mathbf{f}, \mathbf{g}, \dots$ in bold, and classical schemes, stacks, and higher stacks X, Y, \dots and morphisms f, g, \dots not in bold, and we will write $X = t_0(\mathbf{X}), f = t_0(\mathbf{f})$, and so on, for classical truncations of derived $\mathbf{X}, \mathbf{f}, \dots$.

3.2. Derived schemes and cdgas. Toën and Vezzosi [27–29] base their derived algebraic geometry on *simplicial commutative* \mathbb{K} -algebras, but we prefer to work with *commutative differential graded* \mathbb{K} -algebras (*cdgas*). As in [23, §8.1.4] there is a *normalisation functor*

$$N : \{\text{simplicial commutative } \mathbb{K}\text{-algebras}\} \longrightarrow \{\text{cdgas over } \mathbb{K}\}$$

which is an equivalence of ∞ -categories, since \mathbb{K} has characteristic 0 by our assumption in §1. So, working with simplicial commutative \mathbb{K} -algebras and with cdgas over \mathbb{K} are essentially equivalent.

There is a *spectrum functor*

$$\mathbf{Spec} : \{\text{commutative } \mathbb{K}\text{-algebras}\} \longrightarrow \mathbf{St}_{\mathbb{K}}.$$

An object X in $\mathbf{St}_{\mathbb{K}}$ is called an *affine \mathbb{K} -scheme* if it is equivalent to $\mathbf{Spec} A$ for some commutative \mathbb{K} -algebra A , and a *\mathbb{K} -scheme* if it may be covered by Zariski open $Y \subseteq X$ with Y an affine \mathbb{K} -scheme. Write $\mathbf{Sch}_{\mathbb{K}}$ for the full (∞ -)subcategory of \mathbb{K} -schemes in $\mathbf{St}_{\mathbb{K}}$.

Similarly, there is a *spectrum functor*

$$\mathbf{Spec} : \{\text{commutative differential graded } \mathbb{K}\text{-algebras}\} \longrightarrow \mathbf{dSt}_{\mathbb{K}}.$$

A derived \mathbb{K} -stack \mathbf{X} is called an *affine derived \mathbb{K} -scheme* if \mathbf{X} is equivalent in $\mathbf{dSt}_{\mathbb{K}}$ to $\mathbf{Spec} A$ for some cdga A over \mathbb{K} . (This is true if and only if \mathbf{X} is equivalent to $\mathbf{Spec}^{\Delta}(A^{\Delta})$ for some simplicial commutative \mathbb{K} -algebra A^{Δ} , as the normalisation functor N is an equivalence.) As in Toën [27, §4.2], a derived \mathbb{K} -stack \mathbf{X} is called a *derived \mathbb{K} -scheme* if it may be covered by Zariski open $\mathbf{Y} \subseteq \mathbf{X}$ with \mathbf{Y} an affine derived \mathbb{K} -scheme. Write $\mathbf{dSch}_{\mathbb{K}}$ for the full ∞ -subcategory of derived \mathbb{K} -schemes in $\mathbf{dSt}_{\mathbb{K}}$. Then $\mathbf{dSch}_{\mathbb{K}} \subset \mathbf{dArt}_{\mathbb{K}} \subset \mathbf{dSt}_{\mathbb{K}}$.

If A is a cdga over \mathbb{K} , there is an equivalence $t_0 \circ \mathbf{Spec} A \simeq \mathbf{Spec} H^0(A)$ in $\mathbf{St}_{\mathbb{K}}$. Hence, if \mathbf{X} is an affine derived \mathbb{K} -scheme, then $X = t_0(\mathbf{X})$ is an affine \mathbb{K} -scheme, and if \mathbf{X} is a derived \mathbb{K} -scheme, then $X = t_0(\mathbf{X})$ is a \mathbb{K} -scheme, and the truncation functor maps $t_0 : \mathbf{dSch}_{\mathbb{K}} \rightarrow \mathbf{Sch}_{\mathbb{K}}$. Also the inclusion functor maps $i : \mathbf{Sch}_{\mathbb{K}} \rightarrow \mathbf{dSch}_{\mathbb{K}}$. As in [27, §4.2], one can show that if \mathbf{X} is a derived Artin \mathbb{K} -stack and $t_0(\mathbf{X})$ is a \mathbb{K} -scheme, then \mathbf{X} is a derived \mathbb{K} -scheme.

Let \mathbf{X} be a derived \mathbb{K} -scheme, and let $X = t_0(\mathbf{X})$ be the corresponding classical truncation, a classical \mathbb{K} -scheme. Then as in §3.1 there is a natural inclusion morphism $i_{\mathbf{X}} : X \rightarrow \mathbf{X}$, which embeds X as a derived \mathbb{K} -subscheme in \mathbf{X} . A good analogy in classical algebraic geometry is this: Let Y be a nonreduced scheme, and let Y^{red} be the corresponding reduced scheme. Then there is a natural inclusion $Y^{\text{red}} \hookrightarrow Y$ of Y^{red} as a subscheme of Y , and we can think of Y as an infinitesimal thickening of Y^{red} . In a similar way, we can regard a derived \mathbb{K} -scheme \mathbf{X} as an infinitesimal thickening of its classical \mathbb{K} -scheme $X = t_0(\mathbf{X})$.

We shall assume throughout this paper that any derived \mathbb{K} -scheme \mathbf{X} is *locally finitely presented*, which means that it can be covered by Zariski open affine $\mathbf{Y} \simeq \mathbf{Spec} A$, where A is a cdga over \mathbb{K} of finite presentation.

Points x of a derived \mathbb{K} -scheme \mathbf{X} are the same as points x of $X = t_0(\mathbf{X})$. If A is a standard form cdga, as in Example 2.8, and $A' = A \otimes_{A^0} A^0[f^{-1}]$ is a localisation of A , as in Definition 2.9, then $\mathbf{Spec} A'$ is the Zariski open subset of $\mathbf{Spec} A$ where $f \neq 0$. If A' is a localisation of A at $p \in \mathbf{Spec}(H^0(A)) = t_0(\mathbf{Spec} A)$, then $\mathbf{Spec} A'$ is a Zariski open neighbourhood of p in $\mathbf{Spec} A$.

A morphism $\mathbf{f} : \mathbf{X} \rightarrow \mathbf{Y}$ of derived \mathbb{K} -schemes is called *étale* if it is Zariski locally modelled on $\mathbf{Spec} \phi : \mathbf{Spec} A \rightarrow \mathbf{Spec} B$, for $\phi : B \rightarrow A$ a morphism of cdgas over \mathbb{K} such that $H^0(\phi) : H^0(B) \rightarrow H^0(A)$ is étale in the classical sense and $H^i(\phi) : H^i(B) \rightarrow H^i(A)$ induces an isomorphism $H^i(B) \rightarrow H^i(A) \otimes_{H^0(A)} H^0(B)$ for all $i < 0$. In particular, $t_0(\mathbf{f}) : t_0(\mathbf{X}) \rightarrow t_0(\mathbf{Y})$ is an étale morphism of classical schemes.

Since the ∞ -category of affine derived \mathbb{K} -schemes is a localisation of the ∞ -category of cdgas over \mathbb{K} at quasi-isomorphisms, morphisms of affine derived \mathbb{K} -schemes need not lift to morphisms of the corresponding cdgas. We can ask:

Question 3.1. Suppose A, B are standard form cdgas over \mathbb{K} , as in Example 2.8, and $\mathbf{f} : \mathbf{Spec} A \rightarrow \mathbf{Spec} B$ is a morphism in the ∞ -category $\mathbf{dSch}_{\mathbb{K}}$. Then:

- (a) does there exist a morphism of cdgas $\alpha : B \rightarrow A$ (that is, a strict morphism of cdgas, not a morphism in the ∞ -category) with $\mathbf{f} \simeq \mathbf{Spec} \alpha$?
- (b) for each $p \in \mathbf{Spec} H^0(A)$, does there exist a localisation A' of A at p and a (strict) morphism of cdgas $\alpha : B \rightarrow A'$ with $\mathbf{f}|_{\mathbf{Spec} A'} \simeq \mathbf{Spec} \alpha$?

One can show that:

- (i) For general A, B , the answers to Question 3.1(a),(b) may both be no.
- (ii) If A is general, but for B , the smooth \mathbb{K} -scheme $\mathbf{Spec} B^0$ is isomorphic to an affine space \mathbb{A}^n , then the answers to Questions 3.1(a),(b) are both yes. Indeed, the condition that $\mathbf{Spec} B^0$ be isomorphic to an affine space ensures that B is a cofibrant as a cdga, and therefore every map $B \rightarrow A$ in the ∞ -category of cdgas is represented by a strict map out of B .
- (iii) If A is general, but for B the smooth \mathbb{K} -scheme $\mathbf{Spec} B^0$ is isomorphic to a Zariski open set in an affine space \mathbb{A}^n , then the answer to Question 3.1(a) may be no, but to Question 3.1(b) is yes.
- (iv) For general A, B , the answer to Question 3.1(b) at p is yes if and only if there exists a localisation $A'^0 = A^0[g^{-1}]$ of A^0 at p and a morphism of \mathbb{K} -algebras $\alpha^0 : B^0 \rightarrow A'^0$ such that the following commutes:

$$\begin{array}{ccc}
 \mathbf{Spec}(H^0(A)[g^{-1}]) & \xrightarrow{t_0(\mathbf{f})|_{\mathbf{Spec}(H^0(A)[g^{-1}]}}} & \mathbf{Spec} H^0(B) \\
 \downarrow \text{inc} & & \text{inc} \downarrow \\
 \mathbf{Spec} A'^0 & \xrightarrow{\mathbf{Spec} \alpha^0} & \mathbf{Spec} B^0.
 \end{array}$$

That is, if \mathbf{f} lifts to a morphism $B^0 \rightarrow A^0$, then (possibly after further localisation) it lifts to a morphism $B^k \rightarrow A^k$ for all k .

Remark 3.2. Our notion of *standard form* cdgas is a compromise. We chose to start the induction in Example 2.8 with a smooth \mathbb{K} -algebra A^0 . This ensured that cotangent complexes of \mathbb{L}_A behave well, as in §2.3. We could instead have adopted one of the stronger conditions that $\mathbf{Spec} A^0$ is isomorphic to \mathbb{A}^n or to a Zariski open subset of \mathbb{A}^n . This would give better lifting properties of morphisms of derived \mathbb{K} -schemes, as in (ii) and (iii) above. However, requiring $\mathbf{Spec} A^0 \subseteq \mathbb{A}^n$ to be open does not fit well with the notion of A *minimal* in Definition 2.13, and Theorem 4.1 would be false for these stronger notions of standard form cdga.

3.3. Cotangent complexes of derived schemes and stacks. We discuss *cotangent complexes* of derived schemes and stacks, following Toën and Vezzosi [28, §1.4], [27, §4.2.4–§4.2.5], [31], and Lurie [22, §3.4]. We will restrict our attention to derived Artin \mathbb{K} -stacks \mathbf{X} , rather than general derived \mathbb{K} -stacks, which ensures in particular that cotangent complexes exist.

Let \mathbf{X} be a derived Artin \mathbb{K} -stack. In classical algebraic geometry, cotangent complexes \mathbb{L}_X lie in $D(\text{qcoh}(X))$, but in derived algebraic geometry they lie in a dg category $L_{\text{qcoh}}(\mathbf{X})$ defined by Toën [27, §3.1.7, §4.2.4], which is a generalisation

of $D(\text{qcoh}(X))$. Here are some properties of these:

- (i) If \mathbf{X} is a derived Artin \mathbb{K} -stack, then $L_{\text{qcoh}}(\mathbf{X})$ is a pretriangulated dg category with a t-structure whose heart is the abelian category $\text{qcoh}(t_0(\mathbf{X}))$ of quasi-coherent sheaves on the classical truncation $t_0(\mathbf{X})$ of \mathbf{X} . In particular, if $\mathbf{X} = i(X)$ is a classical Artin \mathbb{K} -stack or \mathbb{K} -scheme, then $L_{\text{qcoh}}(\mathbf{X}) \simeq D(\text{qcoh}(X))$, but in general $L_{\text{qcoh}}(\mathbf{X}) \not\simeq D(\text{qcoh}(t_0(\mathbf{X})))$.
- (ii) If $\mathbf{f} : \mathbf{X} \rightarrow \mathbf{Y}$ is a morphism of derived Artin \mathbb{K} -stacks, it induces a pullback functor $\mathbf{f}^* : L_{\text{qcoh}}(\mathbf{Y}) \rightarrow L_{\text{qcoh}}(\mathbf{X})$, analogous to the left derived pullback functor $f^* : D(\text{qcoh}(Y)) \rightarrow D(\text{qcoh}(X))$ in the classical case.
- (iii) If A is a cdga over \mathbb{K} and $\mathbf{X} = \mathbf{Spec} A$ is the corresponding derived affine \mathbb{K} -scheme, then $L_{\text{qcoh}}(\mathbf{X}) \simeq \text{dgmod-}A$ is the pre-triangulated dg category of dg-modules over A . In contrast, $D(\text{qcoh}(\mathbf{X})) \simeq D(\text{mod-}H^0(A))$, which depends only on the classical truncation $X = t_0(\mathbf{X})$.
- (iv) There is a notion of when a complex \mathcal{E}^\bullet in $L_{\text{qcoh}}(\mathbf{X})$ is *perfect*. When $\mathbf{X} = \mathbf{Spec} A$ is affine, as in (iii), perfect complexes in $L_{\text{qcoh}}(\mathbf{X})$ correspond to *perfect dg-modules* in $\text{dgmod-}A$, that is, to modules which may be built from finitely many copies of $A[k]$ for $k \in \mathbb{Z}$ by repeated extensions and splitting of idempotents.

If \mathbf{X} is a derived Artin \mathbb{K} -stack, then Toën and Vezzosi [27, §4.2.5], [28, §1.4] or Lurie [22, §3.2] define an (*absolute*) *cotangent complex* $\mathbb{L}_{\mathbf{X}}$ in $L_{\text{qcoh}}(\mathbf{X})$. If $\mathbf{f} : \mathbf{X} \rightarrow \mathbf{Y}$ is a morphism of derived Artin stacks, they construct a morphism $\mathbb{L}_{\mathbf{f}} : \mathbf{f}^*(\mathbb{L}_{\mathbf{Y}}) \rightarrow \mathbb{L}_{\mathbf{X}}$ in $L_{\text{qcoh}}(\mathbf{X})$, and the (*relative*) *cotangent complex* $\mathbb{L}_{\mathbf{X}/\mathbf{Y}}$ is defined to be the cone on this, giving the distinguished triangle

$$\mathbf{f}^*(\mathbb{L}_{\mathbf{Y}}) \xrightarrow{\mathbb{L}_{\mathbf{f}}} \mathbb{L}_{\mathbf{X}} \longrightarrow \mathbb{L}_{\mathbf{X}/\mathbf{Y}} \longrightarrow \mathbf{f}^*(\mathbb{L}_{\mathbf{Y}})[1].$$

Here are some properties of these:

- (a) If \mathbf{X} is a derived \mathbb{K} -scheme, then $h^k(\mathbb{L}_{\mathbf{X}}) = 0$ for $k > 0$. Also, if $X = t_0(\mathbf{X})$ is the corresponding classical \mathbb{K} -scheme, then $h^0(\mathbb{L}_{\mathbf{X}}) \cong h^0(\mathbb{L}_X) \cong \Omega_X$ under the equivalence $\text{qcoh}(\mathbf{X}) \simeq \text{qcoh}(X)$.
- (b) Let $\mathbf{X} \xrightarrow{\mathbf{f}} \mathbf{Y} \xrightarrow{\mathbf{g}} \mathbf{Z}$ be morphisms of derived Artin \mathbb{K} -stacks. Then by Lurie [22, Prop. 3.2.12] there is a distinguished triangle in $L_{\text{qcoh}}(\mathbf{X})$,

$$\mathbf{f}^*(\mathbb{L}_{\mathbf{Y}/\mathbf{Z}}) \longrightarrow \mathbb{L}_{\mathbf{X}/\mathbf{Z}} \longrightarrow \mathbb{L}_{\mathbf{X}/\mathbf{Y}} \longrightarrow \mathbf{f}^*(\mathbb{L}_{\mathbf{Y}/\mathbf{Z}})[1].$$

- (c) Suppose we have a Cartesian diagram of derived Artin \mathbb{K} -stacks,

$$\begin{array}{ccc} \mathbf{W} & \xrightarrow{\quad f \quad} & \mathbf{Y} \\ \downarrow e & & \downarrow h \\ \mathbf{X} & \xrightarrow{\quad g \quad} & \mathbf{Z}. \end{array}$$

Then by Toën and Vezzosi [28, Lems. 1.4.1.12 and 1.4.1.16] or Lurie [22, Prop. 3.2.10] we have *base change isomorphisms*

$$(3.1) \quad \mathbb{L}_{\mathbf{W}/\mathbf{Y}} \cong e^*(\mathbb{L}_{\mathbf{X}/\mathbf{Z}}), \quad \mathbb{L}_{\mathbf{W}/\mathbf{X}} \cong \mathbf{f}^*(\mathbb{L}_{\mathbf{Y}/\mathbf{Z}}).$$

Note that the analogous result for classical \mathbb{K} -schemes requires g or h to be flat, but (3.1) holds with no flatness assumption.

- (d) Suppose \mathbf{X} is a *locally finitely presented* derived Artin \mathbb{K} -stack [28, §2.2.3] (also called *locally of finite presentation* in [27, §3.1.1], and *fp-smooth* in [29, §4.4]). Then $\mathbb{L}_{\mathbf{X}}$ is a perfect complex in $L_{\text{qcoh}}(\mathbf{X})$.

If $\mathbf{f} : \mathbf{X} \rightarrow \mathbf{Y}$ is a morphism of locally finitely presented derived Artin \mathbb{K} -stacks, then $\mathbb{L}_{\mathbf{X}/\mathbf{Y}}$ is perfect.

- (e) Suppose \mathbf{X} is a derived \mathbb{K} -scheme, $X = t_0(\mathbf{X})$ is its classical truncation, and $i_{\mathbf{X}} : X \hookrightarrow \mathbf{X}$ is the natural inclusion. Then Schürg, Toën, and Vezzosi [25, Prop. 1.2] show that $h^k(\mathbb{L}_{X/\mathbf{X}}) = 0$ for $k \geq -1$, and they deduce that $\mathbb{L}_{i_{\mathbf{X}}} : i_{\mathbf{X}}^*(\mathbb{L}_{\mathbf{X}}) \rightarrow \mathbb{L}_X$ in $D(\text{qcoh}(X))$ is an *obstruction theory* on X in the sense of Behrend and Fantechi [2].

Now suppose A is a cdga over \mathbb{K} and \mathbf{X} is a derived \mathbb{K} -scheme with $\mathbf{X} \simeq \mathbf{Spec} A$ in $\mathbf{dSch}_{\mathbb{K}}$. Then we have an equivalence of triangulated categories $L_{\text{qcoh}}(\mathbf{X}) \simeq D(\text{mod } A)$, where $D(\text{mod } A)$ is the derived category of dg-modules over A . This equivalence identifies cotangent complexes $\mathbb{L}_{\mathbf{X}} \simeq \mathbb{L}_A$. If A is of standard form, then as in §2.3 the Kähler differentials Ω_A^1 are a model for \mathbb{L}_A in $D(\text{mod } A)$, and Proposition 2.12 gives a simple explicit description of Ω_A^1 . Thus, if \mathbf{X} is a derived \mathbb{K} -scheme with $\mathbf{X} \simeq \mathbf{Spec} A$ for A a standard form cdga, we can understand $\mathbb{L}_{\mathbf{X}}$ well. We will use this to do computations with *k-shifted p-forms* and *k-shifted closed p-forms* on \mathbf{X} , as in §3.4.

3.4. *k*-shifted symplectic structures on derived stacks. We now outline the main ideas of our principal reference Pantev, Toën, Vaquié, and Vezzosi [24]. Let X be a classical smooth \mathbb{K} -scheme. Its tangent and cotangent bundles TX, T^*X are vector bundles on X . A *p*-form on X is a global section ω of $\Lambda^p T^*X$. There is a de Rham differential $d_{dR} : \Lambda^p T^*X \rightarrow \Lambda^{p+1} T^*X$, which is a morphism of sheaves of \mathbb{K} -vector spaces but not of sheaves of \mathcal{O}_X -modules, and induces $d_{dR} : H^0(\Lambda^p T^*X) \rightarrow H^0(\Lambda^{p+1} T^*X)$. A *closed p-form* is $\omega \in H^0(\Lambda^p T^*X)$ with $d_{dR}\omega = 0$. A *symplectic form* on X is a closed 2-form ω such that the induced morphism $\omega : TX \rightarrow T^*X$ is an isomorphism.

The *derived loop stack* $\mathcal{L}X$ of X is the fibre product $X \times_{\Delta_X, X \times X, \Delta_X} X$ in $\mathbf{dSch}_{\mathbb{K}}$, where $\Delta_X : X \rightarrow X \times X$ is the diagonal map. When X is smooth, $\mathcal{L}X$ is a quasi-smooth derived \mathbb{K} -scheme. It is interpreted in derived algebraic geometry as $\mathcal{L}X = \mathbb{R}\mathbf{Map}(\mathcal{S}^1, X)$, the moduli stack of *loops* in X . Here the circle \mathcal{S}^1 may be thought of as the simplicial complex $\bullet \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \bullet$, so a map from \mathcal{S}^1 to X means two points x, y in X , corresponding to the two vertices “ \bullet ” in \mathcal{S}^1 , plus two relations $x = y, x = y$ corresponding to the two edges “ \rightarrow ” in \mathcal{S}^1 . This agrees with $\mathcal{L}X$, as points of $X \times_{\Delta_X, X \times X, \Delta_X} X$ correspond to points x, y in X satisfying the relation $(x, x) = (y, y)$ in $X \times X$. Note that in derived algebraic geometry, imposing a relation twice is not the same as imposing it once, as the derived structure sheaf records the relation in its simplicial structure.

Consider the projection $p : \mathcal{L}X = X \times_{X \times X} X \rightarrow X$, where X is a classical smooth \mathbb{K} -scheme. By the Hochschild–Kostant–Rosenberg theorem, there is a decomposition $p_*(\mathcal{O}_{\mathcal{L}X}) \cong \bigoplus_p \Lambda^p T^*X[p]$ in $D(\text{qcoh}(X))$. Thus, *p*-forms on X may be interpreted as “functions on the loop space $\mathcal{L}X$ of weight *p*”, while closed *p*-forms can be interpreted as “ \mathcal{S}^1 -invariant functions on the loop space $\mathcal{L}X$ of weight *p*” by identifying the \mathcal{S}^1 -action with the de Rham differential. See Toën and Vezzosi [30] and Ben-Zvi and Nadler [4].

Now let \mathbf{X} be a locally finitely presented derived Artin \mathbb{K} -stack. The aim of Pantev et al. [24] is to define good notions of *p-forms*, *closed p-forms*, and

symplectic structures on \mathbf{X} , and show that these occur naturally on certain derived moduli stacks. The rough idea is as above: we form the derived loop stack $\mathcal{L}\mathbf{X} := \mathbf{X} \times_{\Delta_{\mathbf{X}}, \mathbf{X} \times \mathbf{X}, \Delta_{\mathbf{X}}} \mathbf{X}$ in $\mathbf{dSch}_{\mathbb{K}}$ and then (closed) p -forms on \mathbf{X} are called “ $(S^1$ -invariant) functions on $\mathcal{L}\mathbf{X}$ of weight p ”.

However, the problem is more complicated than the classical smooth case in two respects. First, as the p -forms $\Lambda^p \mathbb{L}_{\mathbf{X}}$ are a complex, rather than a vector bundle, one should consider cohomology classes $H^k(\mathbf{X}, \Lambda^p \mathbb{L}_{\mathbf{X}})$ for $k \in \mathbb{Z}$ rather than just global sections $\omega \in H^0(\mathbf{X}, \Lambda^p T^* \mathbf{X})$. This leads to the idea of a p -form of degree k on \mathbf{X} for $p \geq 0$ and $k \in \mathbb{Z}$, which is roughly an element of $H^k(\mathbf{X}, \Lambda^p \mathbb{L}_{\mathbf{X}})$. Essentially, $\mathcal{O}_{\mathcal{L}\mathbf{X}}$ has two gradings, the cohomological grading k and the grading by weight p .

Secondly, as the S^1 -action on $\mathcal{L}\mathbf{X} = \mathbb{R}\mathbf{Map}(S^1, \mathbf{X})$ by rotating the domain S^1 is only up to homotopy, to be S^1 -invariant is not a property of a function on $\mathcal{L}\mathbf{X}$, but an extra structure. As an analogy, if an algebraic \mathbb{K} -group G acts on a \mathbb{K} -scheme V , then for a vector bundle $E \rightarrow V$ to be G -invariant is not really a property of E : the right question to ask is whether the G -action on V lifts to a G -action on E , and there can be many such lifts, each a different way for E to be G -invariant. Similarly, the definition of “closed p -form on \mathbf{X} ” in [24] is not just a p -form satisfying an extra condition, but a p -form with extra data, a *closing structure*, satisfying some conditions.

In fact, for technical reasons, for a general locally finitely presented derived Artin \mathbb{K} -stack \mathbf{X} , Pantev et al. [24] do not define p -forms as elements of $H^{-p}(\mathcal{O}_{\mathcal{L}\mathbf{X}})$, and so on, as we have sketched above. Instead, they first define explicit notions of (closed) p -forms on affine derived \mathbb{K} -schemes $\mathbf{Y} = \mathbf{Spec} A$ which are spectra of cdgas A , and show these satisfy étale descent. Then for general \mathbf{X} , k -shifted (closed) p -forms are defined as a mapping stack; basically, a k -shifted (closed) p -form ω on \mathbf{X} is the functorial choice for all \mathbf{Y}, \mathbf{f} of a k -shifted (closed) p -form $\mathbf{f}^*(\omega)$ on \mathbf{Y} whenever $\mathbf{Y} = \mathbf{Spec} A$ is affine and $\mathbf{f} : \mathbf{Y} \rightarrow \mathbf{X}$ is a morphism.

The families (simplicial sets) of p -forms and of closed p -forms of degree k on \mathbf{X} are written $\mathcal{A}_{\mathbb{K}}^p(\mathbf{X}, k)$ and $\mathcal{A}_{\mathbb{K}}^{p, \text{cl}}(\mathbf{X}, k)$, respectively. There is a morphism $\pi : \mathcal{A}_{\mathbb{K}}^{p, \text{cl}}(\mathbf{X}, k) \rightarrow \mathcal{A}_{\mathbb{K}}^p(\mathbf{X}, k)$, which is in general neither injective nor surjective.

A 2-form ω^0 of degree k on \mathbf{X} induces a morphism $\omega^0 : \mathbb{T}_{\mathbf{X}} \rightarrow \mathbb{L}_{\mathbf{X}}[k]$ in $L_{\text{qcoh}}(\mathbf{X})$. We call ω^0 *nondegenerate* if $\omega^0 : \mathbb{T}_{\mathbf{X}} \rightarrow \mathbb{L}_{\mathbf{X}}[k]$ is an equivalence. As in [24, Def. 0.2], a closed 2-form ω of degree k on \mathbf{X} for $k \in \mathbb{Z}$ is called a *k -shifted symplectic structure* if the corresponding 2-form $\omega^0 = \pi(\omega)$ is nondegenerate.

A 0-shifted symplectic structure on a classical \mathbb{K} -scheme is a equivalent to a classical symplectic structure. Pantev et al. [24] construct k -shifted symplectic structures on several classes of derived moduli stacks. In particular, if Y is a Calabi–Yau m -fold and \mathcal{M} a derived moduli stack of coherent sheaves or perfect complexes on Y , then \mathcal{M} has a $(2 - m)$ -shifted symplectic structure.

4. LOCAL MODELS FOR DERIVED SCHEMES

The next theorem, based on Lurie [23, Th. 8.4.3.18] and proved in §4.1, says every derived \mathbb{K} -scheme \mathbf{X} is Zariski locally modelled on $\mathbf{Spec} A$ for A a minimal standard form cdga. Recall that all derived \mathbb{K} -schemes in this paper are assumed *locally finitely presented*. A morphism $\mathbf{f} : \mathbf{X} \rightarrow \mathbf{Y}$ in $\mathbf{dSch}_{\mathbb{K}}$ is called a *Zariski open inclusion* if \mathbf{f} is an equivalence from \mathbf{X} to a Zariski open derived \mathbb{K} -subscheme

$Y' \subseteq Y$. A morphism of cdgas $\alpha : B \rightarrow A$ is a *Zariski open inclusion* if $\mathbf{Spec} \alpha : \mathbf{Spec} A \rightarrow \mathbf{Spec} B$ is a Zariski open inclusion.

Theorem 4.1. *Let \mathbf{X} be a locally finitely presented derived \mathbb{K} -scheme, and let $x \in \mathbf{X}$. Then there exist a standard form cdga A over \mathbb{K} which is minimal at a point $p \in \mathbf{Spec} H^0(A)$, in the sense of Example 2.8 and Definitions 2.9 and 2.13, and a morphism $\mathbf{f} : \mathbf{Spec} A \rightarrow \mathbf{X}$ in $\mathbf{dSch}_{\mathbb{K}}$ which is a Zariski open inclusion with $\mathbf{f}(p) = x$.*

We think of A, \mathbf{f} in Theorem 4.1 as being like a coordinate system on \mathbf{X} near x . As well as being able to choose coordinates near any point, we want to be able to compare different coordinate systems on their overlaps. That is, given local equivalences $\mathbf{f} : \mathbf{Spec} A \rightarrow \mathbf{X}$, $\mathbf{g} : \mathbf{Spec} B \rightarrow \mathbf{X}$, we would like to compare the cdgas A, B on the overlap of their images in \mathbf{X} .

This is related to the discussion after Question 3.1 in §3.2: for general A, B we cannot (even locally) find a cdga morphism $\alpha : B \rightarrow A$ with $\mathbf{f} \simeq \mathbf{g} \circ \mathbf{Spec} \alpha$. However, the next theorem, proved in §4.2, shows we can find a third cdga C and open inclusions $\alpha : A \rightarrow C$, $\beta : B \rightarrow C$ with $\mathbf{f} \circ \mathbf{Spec} \alpha \simeq \mathbf{g} \circ \mathbf{Spec} \beta$.

Theorem 4.2. *Let \mathbf{X} be a locally finitely presented derived \mathbb{K} -scheme, let A, B be standard form cdgas over \mathbb{K} , and let $\mathbf{f} : \mathbf{Spec} A \rightarrow \mathbf{X}$, $\mathbf{g} : \mathbf{Spec} B \rightarrow \mathbf{X}$ be Zariski open inclusions in $\mathbf{dSch}_{\mathbb{K}}$. Suppose $p \in \mathbf{Spec} H^0(A)$ and $q \in \mathbf{Spec} H^0(B)$ with $\mathbf{f}(p) = \mathbf{g}(q)$ in \mathbf{X} . Then there exist a standard form cdga C over \mathbb{K} which is minimal at $r \in \mathbf{Spec} H^0(C)$ and morphisms of cdgas $\alpha : A \rightarrow C$, $\beta : B \rightarrow C$ which are Zariski open inclusions, such that $\mathbf{Spec} \alpha : r \mapsto p$, $\mathbf{Spec} \beta : r \mapsto q$, and $\mathbf{f} \circ \mathbf{Spec} \alpha \simeq \mathbf{g} \circ \mathbf{Spec} \beta$ as morphisms $\mathbf{Spec} C \rightarrow \mathbf{X}$ in $\mathbf{dSch}_{\mathbb{K}}$.*

If instead \mathbf{f}, \mathbf{g} are étale rather than Zariski open inclusions, the same holds with α, β étale rather than Zariski open inclusions.

4.1. Proof of Theorem 4.1. The proof is a variation on Lurie [23, Th. 8.4.3.18]. We give an outline, referring the reader to [23] for more details.

As \mathbf{X} is a locally finitely presented derived \mathbb{K} -scheme, it is covered by affine finitely presented derived \mathbb{K} -subschemes. So we can choose an open neighbourhood \mathbf{Y} of x in \mathbf{X} , a cdga B of finite presentation over \mathbb{K} , and an equivalence $\mathbf{g} : \mathbf{Spec} B \rightarrow \mathbf{Y} \subseteq \mathbf{X}$. There is then a unique $q \in \mathbf{Spec} H^0(B)$ with $\mathbf{g}(q) = x$.

Since B is of finite presentation, the cotangent complex \mathbb{L}_B has finite Tor-amplitude, say in the interval $[-n, 0]$. We will show that a localisation B' of B at q is equivalent to a standard form cdga $A = A(n)$ over \mathbb{K} , constructed inductively in a sequence $A(0), A(1), \dots, A(n)$ as in Example 2.8, with A minimal at the point $p \in \mathbf{Spec} H^0(A)$ corresponding to $q \in \mathbf{Spec} H^0(B') \subseteq \mathbf{Spec} H^0(B)$. For the inductive construction, we do not fix a particular model for the cdga B , but understand it as an object in the ∞ -category of cdgas. Similarly, when we assert the existence of a map $A(k) \rightarrow B$ from some level of the inductive construction to B , this map should be understood as a map in the ∞ -category of cdgas.

Let $m_0 = \dim \Omega_{H^0(B)}^1|_q$, the embedding dimension of $\mathbf{Spec} H^0(B)$ at q . Localising $H^0(B)$ at q if necessary, we can find a smooth algebra $A(0)$ of dimension m_0 , an ideal $I \subset A(0)$, and an isomorphism of algebras $A(0)/I \rightarrow H^0(B)$ such that the induced surjection of modules $\Omega_{A(0)}^1 \otimes_{A(0)} H^0(B) \rightarrow \Omega_{H^0(B)}^1$ is an isomorphism at q . Geometrically, we have chosen an embedding of $\mathbf{Spec} H^0(B)$ into a smooth scheme $\mathbf{Spec} A(0)$ that is minimal at q . Let $p \in \mathbf{Spec} A(0)$ be the image of $q \in \mathbf{Spec} H^0(B)$.

Since B is the homotopy limit of its Postnikov tower $\cdots \rightarrow \tau_{\geq -1}B \rightarrow \tau_{\geq 0}B \simeq H^0(B)$ in which each map is a square-zero extension of cdgas [23, Prop. 7.1.3.19], and since $A(0)$ is smooth and hence maps out of it can be lifted along square-zero extensions, we can lift the surjection $A(0) \rightarrow H^0(B)$ along the canonical map $B \rightarrow H^0(B)$ to obtain a map $\alpha^0 : A(0) \rightarrow B$ which is a surjection on H^0 . Consider the fibre sequence $\text{fib}(\alpha^0) \rightarrow A(0) \rightarrow B$. One can show that there is an isomorphism $H^0(B) \otimes_{A(0)} H^0(\text{fib}(\alpha^0)) \cong H^{-1}(\mathbb{L}_{B/A(0)})$. Furthermore, we see that there is a surjection $H^0(\text{fib}(\alpha^0)) \rightarrow I$ and hence a surjection $H^{-1}(\mathbb{L}_{B/A(0)}) = H^0(B) \otimes_{A(0)} H^0(\text{fib}(\alpha^0)) \rightarrow I/I^2$.

Localising if necessary and using Nakayama's lemma, we may choose a free finite rank $A(0)$ -module M^{-1} together with a surjection $M^{-1} \rightarrow H^{-1}(\mathbb{L}_{B/A(0)})$ that is an isomorphism at p . We therefore obtain a surjection $M^{-1} \rightarrow I/I^2$ which can be lifted through the map $I \rightarrow I/I^2$. Localising again if necessary and using Nakayama's lemma, we may assume the lift $M^{-1} \rightarrow I$ is surjective.

Using this choice of M^{-1} together with the map $M^{-1} \rightarrow I \subset A(0)$, we define a cdga $A(1)$ as in Example 2.8. Note that by construction, the induced map $\alpha^{-1} : A(1) \rightarrow B$ induces an isomorphism $H^0(A(1)) \rightarrow H^0(B)$.

Now consider the fibre sequence $B \otimes_{A(1)} \mathbb{L}_{A(1)/A(0)} \rightarrow \mathbb{L}_{B/A(0)} \rightarrow \mathbb{L}_{B/A(1)}$. By construction, $\mathbb{L}_{A(1)/A(0)} \simeq A(1) \otimes_{A(0)} M^{-1}[1]$, and the map $H^0(B) \otimes_{A(0)} M^{-1} = H^{-1}(\mathbb{L}_{A(1)/A(0)}) \rightarrow H^{-1}(\mathbb{L}_{B/A(0)})$ is surjective, so $H^i(\mathbb{L}_{B/A(1)})$ vanishes for $i = 0, -1$. One can then deduce that there are isomorphisms

$$H^{-1}(\text{fib}(\alpha^{-1})) \cong H^{-2}(\text{cof}(\alpha^{-1})) \cong H^{-2}(\mathbb{L}_{B/A(1)}).$$

Localising if necessary, we can choose a free $A(1)$ -module M^{-2} of finite rank with a surjection $H^0(B) \otimes_{H^0(A(1))} H^0(M^{-2}) \rightarrow H^{-1}(\text{fib}(\alpha^{-1})) \cong H^{-2}(\mathbb{L}_{B/A(1)})$ that is an isomorphism at p . Choosing a lift $M^{-2}[1] \rightarrow \text{fib}(\alpha^{-1}) \rightarrow A(1)$, we construct $A(2)$ from $A(1)$ as in Example 2.8.

Continuing in this manner, we construct a sequence of cdgas

$$A(0) \longrightarrow A(1) \longrightarrow \cdots \longrightarrow A(k-1) \longrightarrow A(k) \longrightarrow \cdots \longrightarrow A(n-1)$$

so that $\mathbb{L}_{A(k)/A(k-1)} \simeq A(k) \otimes_{A(k-1)} M^{-k}[k]$ is a free $A(k)$ -module and the natural map $H^{-k}(\mathbb{L}_{A(k)/A(k-1)}) \rightarrow H^{-k}(\mathbb{L}_{B/A(k-1)})$ is surjective and is an isomorphism at p .

By induction on k , we see that $\mathbb{L}_{A(k)}$ has Tor-amplitude in $[-k, 0]$. Considering the fibre sequence $B \otimes_{A(n-1)} \mathbb{L}_{A(n-1)} \rightarrow \mathbb{L}_B \rightarrow \mathbb{L}_{B/A(n-1)}$ and bearing in mind that \mathbb{L}_B has Tor-amplitude in $[-n, 0]$ by assumption, we see that $\mathbb{L}_{B/A(n-1)}$ has Tor-amplitude in $[-n, 0]$. But by (2.5), $H^i(\mathbb{L}_{B/A(n-1)}) = 0$ for $i > -n$, so we see that $\mathbb{L}_{B/A(n-1)}[-n]$ is a projective B -module of finite rank. Localising if necessary, we may assume it is free.

Finally, choose a free $A(n-1)$ -module M^{-n} with $\mathbb{L}_{B/A(n-1)} \simeq B \otimes_{A(n-1)} M^{-n}[n]$. Then $H^0(M^{-n}) \cong H^{-n}(\mathbb{L}_{B/A(n-1)}) \cong H^{1-n}(\text{fib}(\alpha^{1-n}))$, so using $M^{-n}[n-1] \rightarrow A(n-1)$ to build $A(n)$, we have a fibre sequence $\mathbb{L}_{A(n)/A(n-1)} \rightarrow \mathbb{L}_{B/A(n-1)} \rightarrow \mathbb{L}_{B/A(n)}$ in which the first arrow is an equivalence. Thus $\mathbb{L}_{B/A(n)} \simeq 0$, and since the map $\alpha^{-n} : A(n) \rightarrow B$ induces an isomorphism on H^0 , it must be an equivalence. Set $A = A(n)$, so that A is a standard form cdga over \mathbb{K} . As $\mathbf{Spec} \alpha^{-n} : \mathbf{Spec} B \rightarrow \mathbf{Spec} A$ is an equivalence in $\mathbf{dSch}_{\mathbb{K}}$ with $\mathbf{Spec} \alpha^{-n} : q \mapsto p$, there exists a

quasi-inverse $h : \mathbf{Spec} A \rightarrow \mathbf{Spec} B$ for $\mathbf{Spec} \alpha^{-n}$, with $h(p) = q$. Then $f = g \circ h : \mathbf{Spec} A \rightarrow \mathbf{X}$ is a Zariski open inclusion with $f(p) = g \circ h(p) = g(q) = x$, as we have to prove.

It remains to show that A is minimal at p . For this we must check that for each k , the composition (2.8) is zero at p . Using the commutative diagram (2.6), and (2.6) with $k - 1$ in place of k , we have a commutative diagram

$$\begin{array}{ccc}
 H^{-k}(\mathbb{L}_{A(k)/A(k-1)}) & \xlongequal{\quad\quad\quad} & H^{-k}(\mathbb{L}_{A(k)/A(k-1)}) \\
 \downarrow & & \downarrow \\
 H^{-k}(\mathbb{L}_{B/A(k-1)}) & \xrightarrow{\quad\quad\quad} & H^{1-k}(\mathbb{L}_{A(k-1)}) \\
 \downarrow & & \downarrow \\
 H^{1-k}(\mathbb{L}_{A(k-1)/A(k-2)}) & \xlongequal{\quad\quad\quad} & H^{1-k}(\mathbb{L}_{A(k-1)/A(k-2)}),
 \end{array}$$

where the right-hand column is (2.8). It is therefore enough to see that composition in the left-hand column is zero at p .

But by construction, the first map in (2.8) is an isomorphism at p , so we need the second map to be zero at p . But again by construction, we have an exact sequence $H^{-k}(\mathbb{L}_{B/A(k-1)}) \rightarrow H^{1-k}(\mathbb{L}_{A(k-1)/A(k-2)}) \rightarrow H^{1-k}(\mathbb{L}_{B/A(k-2)})$ in which the second map is an isomorphism at p , and hence $H^{-k}(\mathbb{L}_{B/A(k-1)}) \rightarrow H^{1-k}(\mathbb{L}_{A(k-1)/A(k-2)})$ is indeed zero at p . This completes the proof.

4.2. Proof of Theorem 4.2. Given a derived scheme \mathbf{X} with a point $x \in \mathbf{X}$ and Zariski open neighbourhoods $f : U = \mathbf{Spec} A \rightarrow \mathbf{X}$ and $g : V = \mathbf{Spec} B \rightarrow \mathbf{X}$ of x , we may assume that A and B are given as standard form cdgas. We want to show that we can find a Zariski open neighbourhood $h : W = \mathbf{Spec} C \rightarrow \mathbf{X}$ of x contained in U and V , where C is a standard form cdga minimal at $r \in \mathbf{Spec} H^0(C)$ with $h(r) = x$ and such that in the homotopy commutative diagram

$$\begin{array}{ccccc}
 & & \mathbf{W} = \mathbf{Spec} C & & \\
 & \swarrow & \downarrow h & \searrow & \\
 (4.1) & & U = \mathbf{Spec} A & & V = \mathbf{Spec} B \\
 & \swarrow & \downarrow f & \searrow & \downarrow g \\
 & \mathbf{Spec} A(0) & & \mathbf{X} & & \mathbf{Spec} B(0),
 \end{array}$$

the maps c and d are induced by maps of cdgas $\alpha : A \rightarrow C$ and $\beta : B \rightarrow C$.

To begin with, choose a Zariski open immersion of an affine derived scheme $\mathbf{W} \rightarrow U \times_{\mathbf{X}} V \rightarrow \mathbf{X}$ whose image contains x . From the above commutative diagram, we have maps $\mathbf{W} \rightarrow \mathbf{Spec} A \rightarrow \mathbf{Spec} A(0)$ and $\mathbf{W} \rightarrow \mathbf{Spec} B \rightarrow \mathbf{Spec} B(0)$ in which the first arrows in each sequence are open immersions and the second arrows are closed immersions on the underlying classical schemes. The induced map $\mathbf{W} \rightarrow \mathbf{Spec} A(0) \times \mathbf{Spec} B(0)$ is therefore a locally closed immersion on the underlying classical scheme $W = t_0(\mathbf{W})$. Localising if necessary, we may assume that in fact $\mathbf{W} \rightarrow \mathbf{Spec} A(0) \times \mathbf{Spec} B(0)$ factors via a closed immersion $W \hookrightarrow \mathbf{Spec} C(0)$ into a smooth, affine, locally closed \mathbb{K} -subscheme $\mathbf{Spec} C(0) \subseteq \mathbf{Spec} A(0) \times \mathbf{Spec} B(0)$ of minimal dimension at $r := (p, q)$.

Proceeding as in the proof of Theorem 4.1, we may build a standard form model C for $\mathbf{W} = \mathbf{Spec} C$ that is free over $C(0)$ and minimal at x . Since A is free over $A(0)$ the map $\mathbf{W} = \mathbf{Spec} C \rightarrow \mathbf{Spec} C(0) \rightarrow \mathbf{Spec} A(0)$ factors through $c : \mathbf{Spec} C \rightarrow \mathbf{Spec} A$, and the latter is induced by an actual map of cdgas $\alpha : A \rightarrow C$. Similarly,

the map $\mathbf{d} : \mathbf{Spec} C \rightarrow \mathbf{Spec} B$ is induced by an actual map of cdgas $\beta : B \rightarrow C$, since B is free over $B(0)$. This proves the first part of Theorem 4.2.

For the second part, if instead $\mathbf{f} : \mathbf{U} = \mathbf{Spec} A \rightarrow \mathbf{X}$ and $\mathbf{g} : \mathbf{V} = \mathbf{Spec} B \rightarrow \mathbf{X}$ are étale neighbourhoods of x , we apply the same fibre product construction. Since étale maps are stable under pullbacks, \mathbf{c} and \mathbf{d} are now étale, rather than Zariski open inclusions. However, the induced map $\mathbf{W} \rightarrow \mathbf{Spec} A(0) \times \mathbf{Spec} B(0)$ is still a locally closed immersion on $W = t_0(\mathbf{W})$. So in the same way we obtain a standard form model C for $\mathbf{W} = \mathbf{Spec} C$ that is free over $C(0)$ and minimal at $r = (p, q)$ with étale maps \mathbf{c}, \mathbf{d} that are induced by actual maps of cdgas.

5. A DERIVED DARBOUX THEOREM

Section 5.1 explains what is meant by a k -shifted symplectic structure $\omega = (\omega^0, \omega^1, \omega^2, \dots)$ on an affine derived \mathbb{K} -scheme $\mathbf{Spec} A$ for A of standard form and $k < 0$, expanding on §3.4, and §5.2 proves that up to equivalence we can take $\omega = (d_{dR}\phi, 0, 0, \dots)$. Sections 5.3 and 5.4 define standard models for k -shifted symplectic structures on $\mathbf{Spec} A$, which we call Darboux form.

Our main result Theorem 5.18, stated in §5.5 and proved in §5.6, says that every k -shifted symplectic derived \mathbb{K} -scheme $(\mathbf{X}, \tilde{\omega})$ for $k < 0$ is locally equivalent to $(\mathbf{Spec} A, \omega)$ for A, ω in Darboux form. Section 5.7 explains how to compare different Darboux form presentations of $(\mathbf{X}, \tilde{\omega})$ on their overlaps.

5.1. k -shifted symplectic structures on $\mathbf{Spec} A$. Let $A = A(n)$ be a standard form cdba over \mathbb{K} , as in Example 2.8, constructed from a smooth \mathbb{K} -algebra $A(0)$ and free finite rank modules $M^{-1}, M^{-2}, \dots, M^{-n}$ over $A(0), \dots, A(n-1)$. Write $\mathbf{X} = \mathbf{Spec} A$ for the corresponding affine derived \mathbb{K} -scheme. We will explain in more detail how the material of §3.4 works out for $\mathbf{X} = \mathbf{Spec} A$.

As in §3.3 there is an equivalence $L_{\text{qcoh}}(\mathbf{X}) \simeq D(\text{mod } A)$ which identifies $\mathbb{L}_{\mathbf{X}} \simeq \mathbb{L}_A$, and as A is of standard form, as in §2.3 the Kähler differentials (Ω_A^1, d) are a model for \mathbb{L}_A in $D(\text{mod } A)$, and Proposition 2.12 gives an explicit description of (Ω_A^1, d) . Write $\Omega_A^1 = \bigoplus_{k=-\infty}^0 (\Omega_A^1)^k$ for the decomposition of Ω_A^1 into graded pieces, so that the differential maps $d : (\Omega_A^1)^k \rightarrow (\Omega_A^1)^{k+1}$.

As in §2.1 and §2.3, the *de Rham algebra* of A is

$$\text{DR}(A) := \text{Sym}_A(\Omega_A^1[1]) = \bigoplus_{p=0}^{\infty} \Lambda^p \Omega_A^1[p] = \bigoplus_{p=0}^{\infty} \bigoplus_{k=-\infty}^0 (\Lambda^p \Omega_A^1)^k[p].$$

It has two gradings, *degree* and *weight*, where the component $(\Lambda^p \Omega_A^1)^k[p]$ has degree $k - p$ and weight p . It has two differentials, the *internal differential* $d : \text{DR}(A) \rightarrow \text{DR}(A)$ of degree 1 and weight 0, so that d maps

$$d : (\Lambda^p \Omega_A^1)^k[p] \longrightarrow (\Lambda^p \Omega_A^1)^{k+1}[p],$$

and the *de Rham differential* $d_{dR} : \text{DR}(A) \rightarrow \text{DR}(A)$ of degree -1 and weight 1, so that d_{dR} maps

$$d_{dR} : (\Lambda^p \Omega_A^1)^k[p] \longrightarrow (\Lambda^{p+1} \Omega_A^1)^k[p+1].$$

They satisfy $d \circ d = d_{dR} \circ d_{dR} = d \circ d_{dR} + d_{dR} \circ d = 0$. The multiplication “ \cdot ” on $\text{DR}(A)$ maps

$$(\Lambda^p \Omega_A^1)^k[p] \times (\Lambda^q \Omega_A^1)^l[q] \longrightarrow (\Lambda^{p+q} \Omega_A^1)^{k+l}[p+q].$$

In [24, Def. 1.7], Pantev et al. define a simplicial set $\mathcal{A}_{\mathbb{K}}^p(\mathbf{X}, k)$ of p -forms of degree $k \in \mathbb{Z}$ on the derived \mathbb{K} -scheme $\mathbf{X} = \mathbf{Spec} A$ by

$$(5.1) \quad \mathcal{A}_{\mathbf{X}}^p(Y, k) = |\Lambda^p \mathbb{L}_A[k]|.$$

In our case we may take $\Lambda^p \mathbb{L}_A = \Lambda^p \Omega_A^1$. If E^\bullet is a complex of \mathbb{K} -modules, then $|E^\bullet|$ means: take the truncation $\tau_{\leq 0}(E^\bullet)$, and turn it into a simplicial set via the Dold–Kan correspondence. To avoid dealing with simplicial sets, note that this implies that $\pi_0(|E^\bullet|) \cong H^0(\tau_{\leq 0}(E^\bullet)) \cong H^0(E^\bullet)$. Thus (5.1) yields

$$(5.2) \quad \pi_0(\mathcal{A}_{\mathbb{K}}^p(\mathbf{X}, k)) \cong H^k(\Lambda^p \Omega_A^1, d) = H^{k-p}(\Lambda^p \Omega_A^1[p], d).$$

So, (connected components of the simplicial set of) p -forms of degree k on \mathbf{X} are just k -cohomology classes of the complex $(\Lambda^p \Omega_A^1, d)$. We prefer to deal with explicit representatives, rather than cohomology classes. So we define:

Definition 5.1. In the situation above with $\mathbf{X} = \mathbf{Spec} A$, a p -form of degree k on \mathbf{X} for $p \geq 0$ and $k \leq 0$ is $\omega^0 \in (\Lambda^p \Omega_A^1)^k$ with $d\omega^0 = 0$ in $(\Lambda^p \Omega_A^1)^{k+1}$. Two p -forms $\omega^0, \tilde{\omega}^0$ of degree k are *equivalent*, written $\omega^0 \sim \tilde{\omega}^0$, if there exists $\alpha^0 \in (\Lambda^p \Omega_A^1)^{k-1}$ with $\omega^0 - \tilde{\omega}^0 = d\alpha^0$.

Then equivalence classes $[\omega^0]$ of p -forms ω^0 of degree k on \mathbf{X} in the sense of Definition 5.1 correspond to connected components of the simplicial set of p -forms of degree k on \mathbf{X} in the sense of Pantev et al. [24]. The reason for including the superscripts 0 in ω^0, α^0 will become clear in Definition 5.2.

Letting $\mathbb{T}_A = \underline{\mathbf{Hom}}_A(\Omega_A^1, A) = \underline{\mathbf{Der}}(A)$ denote the *tangent complex* of A , a 2-form ω^0 of degree k on \mathbf{X} defines an antisymmetric morphism

$$(5.3) \quad \omega^0 : \mathbb{T}_A \longrightarrow \Omega_A^1[k],$$

via $X \mapsto \iota_X \omega^0$ for $X \in \underline{\mathbf{Der}}(A)$. The 2-form ω^0 is said to be *nondegenerate* if this induced map is a quasi-isomorphism.

The definition of the simplicial set $\mathcal{A}_{\mathbb{K}}^{p,\text{cl}}(\mathbf{X}, k)$ of *closed* p -forms of degree $k \in \mathbb{Z}$ on $\mathbf{X} = \mathbf{Spec} A$ in Pantev et al. [24, Def. 1.7] yields

$$(5.4) \quad \mathcal{A}_{\mathbb{K}}^{p,\text{cl}}(\mathbf{X}, k) = |\prod_{i \geq 0} \Lambda^{p+i} \mathbb{L}_A[k-i]|.$$

In our case we may take $\Lambda^{p+i} \mathbb{L}_A[k-i] = \Lambda^{p+i} \Omega_A^1[k-i]$. Then $\prod_{i \geq 0} \Lambda^{p+i} \Omega_A^1[k-i]$ means the complex which as a graded vector space is the product over all $i \geq 0$ of the graded vector spaces $\Lambda^{p+i} \Omega_A^1[k-i]$, with differential $d + d_{dR}$.

The difference between $\prod_{i \geq 0} \Lambda^{p+i} \mathbb{L}_A[k-i]$ and $\bigoplus_{i \geq 0} \Lambda^{p+i} \mathbb{L}_A[k-i]$ is that elements $(\omega^i)_{i \geq 0} \in \bigoplus_{i \geq 0} \Lambda^{p+i} \mathbb{L}_A[k-i]$ have $\omega^i \neq 0$ for only finitely many i , but elements $(\omega^i)_{i \geq 0} \in \prod_{i \geq 0} \Lambda^{p+i} \mathbb{L}_A[k-i]$ can have $\omega^i \neq 0$ for infinitely many i .

Thus, as for (5.1)–(5.2), equation (5.4) implies that

$$\begin{aligned} \pi_0(\mathcal{A}_{\mathbb{K}}^{p,\text{cl}}(\mathbf{X}, k)) &\cong H^0(\prod_{i \geq 0} \Lambda^{p+i} \Omega_A^1[k-i], d + d_{dR}) \\ &= H^k(\prod_{i \geq 0} \Lambda^{p+i} \Omega_A^1[-i], d + d_{dR}). \end{aligned}$$

As for Definition 5.1, we define:

Definition 5.2. In the situation above with $\mathbf{X} = \mathbf{Spec} A$, a *closed* p -form of degree k on \mathbf{X} for $p \geq 0$ and $k \leq 0$ is $\omega = (\omega^0, \omega^1, \omega^2, \dots)$ with $\omega^i \in (\Lambda^{p+i} \Omega_A^1)^{k-i}$ for $i = 0, 1, 2, \dots$, satisfying the equations

$$(5.5) \quad \begin{aligned} d\omega^0 &= 0 \quad \text{in } (\Lambda^p \Omega_A^1)^{k+1}, \text{ and} \\ d_{dR} \omega^i + d\omega^{i+1} &= 0 \quad \text{in } (\Lambda^{p+i+1} \Omega_A^1)^{k-i}, \text{ for all } i \geq 0. \end{aligned}$$

We call two closed p -forms $\omega = (\omega^0, \dots)$, $\tilde{\omega} = (\tilde{\omega}^0, \dots)$ of degree k *equivalent*, written $\omega \sim \tilde{\omega}$, if there exists $\alpha = (\alpha^0, \alpha^1, \dots)$ with $\alpha^i \in (\Lambda^{p+i}\Omega_A^1)^{k-i-1}$ for $i = 0, 1, \dots$, satisfying

$$\begin{aligned} \omega^0 - \tilde{\omega}^0 &= d\alpha^0 && \text{in } (\Lambda^p\Omega_A^1)^k, \text{ and} \\ \omega^{i+1} - \tilde{\omega}^{i+1} &= d_{dR}\alpha^i + d\alpha^{i+1} && \text{in } (\Lambda^{p+i+1}\Omega_A^1)^{k-i-1}, \text{ for all } i \geq 0. \end{aligned}$$

The morphism $\pi : \mathcal{A}_{\mathbb{K}}^{p,\text{cl}}(\mathbf{X}, k) \rightarrow \mathcal{A}_{\mathbb{K}}^p(\mathbf{X}, k)$ from closed p -forms of degree k to p -forms of degree k in §3.4 corresponds to $\pi : \omega = (\omega^0, \dots) \mapsto \omega^0$.

A closed 2-form $\omega = (\omega^0, \omega^1, \dots)$ of degree k on \mathbf{X} is called a *k -shifted symplectic form* if $\omega^0 = \pi(\omega)$ is a nondegenerate 2-form of degree k .

Then equivalence classes $[\omega]$ of closed p -forms ω of degree k on \mathbf{X} in the sense of Definition 5.1 correspond to connected components of the simplicial set $\mathcal{A}_{\mathbb{K}}^{p,\text{cl}}(\mathbf{X}, k)$ of closed p -forms of degree k on \mathbf{X} in the sense of Pantev et al. [24].

5.2. Closed forms and cyclic homology of mixed complexes. In order to work effectively with such symplectic forms, it is very useful to interpret them in the context of cyclic homology of mixed complexes. The following definitions are essentially as in Loday [21, §2.5.13], except that we use cohomological grading and take into account an extra weight grading in our mixed complexes, as in Pantev et al. [24, §1.1].

Definition 5.3. A *mixed complex* E is a complex over \mathbb{K} with a differential b of degree 1 together with an additional square-zero operator B of degree -1 anti-commuting with b . A *graded mixed complex* has in addition a *weight grading* giving by a decomposition $E = \bigoplus_p E(p)$, where b has degree 0 and B has degree 1 with respect to the weight grading. A morphism between graded mixed complexes $E = \bigoplus_p E(p)$ and $F = \bigoplus_p F(p)$ is a \mathbb{K} -linear map $\varphi : E \rightarrow F$ of degree 0 with respect to both the cohomological and weight grading that commutes with b and B . The morphism $\varphi : E \rightarrow F$ is a *weak equivalence* if it is a quasi-isomorphism for the cohomology taken with respect to the differential b . For simplicity and since it is sufficient for our applications, we shall consider only graded mixed complexes that are bounded above at 0 with respect to the cohomological grading and bounded below at 0 with respect to the weight grading.

Example 5.4. For us, the main example of a graded mixed complex is $E = \text{DR}(A) = \bigoplus_p \Lambda^p\Omega_A^1[p]$ with $b = d$ and $B = d_{dR}$, for A a standard form cdga.

Definition 5.5. Given a graded mixed complex E , for each p we define three complexes, the *negative cyclic complex of weight p* , denoted $\text{NC}(E)(p)$, the *periodic cyclic complex of weight p* , denoted $\text{PC}(E)(p)$, and the *cyclic complex of weight p* , denoted $\text{CC}(E)(p)$. The degree k terms of these complexes are:

$$\begin{aligned} \text{NC}^k(E)(p) &= \prod_{i \geq 0} E^{k-2i}(p+i), & \text{PC}^k(E)(p) &= \prod_i E^{k-2i}(p+i), \\ \text{CC}^k(E)(p) &= \prod_{i \leq 0} E^{k-2i}(p+i). \end{aligned}$$

In each case, the differential is simply $b + B$, and the complexes can be constructed as Tot^Π of appropriate double complexes. The k th cohomology of the complexes

$\mathrm{NC}(E)(p)$, $\mathrm{PC}(E)(p)$, and $\mathrm{CC}(E)(p)$ are denoted $\mathrm{HN}^k(E)(p)$, $\mathrm{HP}^k(E)(p)$, and $\mathrm{HC}^k(E)(p)$, respectively.

There is an evident short exact sequence of complexes

$$0 \longrightarrow \mathrm{NC}(E)(p) \longrightarrow \mathrm{PC}(E)(p) \longrightarrow \mathrm{CC}(E)(p-1)[2] \longrightarrow 0,$$

and an induced long exact sequence of cohomology groups

$$(5.6) \quad \begin{array}{ccccccc} \cdots & \longrightarrow & \mathrm{HC}^{k+1}(E)(p-1) & \longrightarrow & \mathrm{HN}^k(E)(p) & \longrightarrow & \mathrm{HP}^k(E)(p) \\ & & \longrightarrow & & \longrightarrow & & \cdots \end{array}$$

When $E = \mathrm{DR}(A)$ for A a standard form cdga, we denote the corresponding cochain groups by $\mathrm{NC}^k(A)(p)$, $\mathrm{PC}^k(A)(p)$, $\mathrm{CC}^k(A)(p)$, and the cohomology groups by $\mathrm{HN}^k(A)(p)$, $\mathrm{HP}^k(A)(p)$, $\mathrm{HC}^k(A)(p)$. As in Loday [21, Ch. 5], these groups are known to be compatible with other definitions that the reader may have seen. The connection of all this with the material of §5.1 is that closed p -forms $\omega = (\omega^0, \omega^1, \omega^2, \dots)$ of degree k in Definition 5.2 are cocycles in $\mathrm{NC}^{k-p}(A)(p)$, and equivalence classes $[\omega]$ of closed p -forms $\omega = (\omega^0, \omega^1, \dots)$ of degree k on $\mathbf{X} = \mathbf{Spec} A$ are elements of $\mathrm{HN}^{k-p}(A)(p)$, by the Hochschild–Kostant–Rosenberg theorem.

Here is a useful vanishing result. (Note that the group $\mathrm{HN}^{k-2}(A)(2)$ in (5.7) classifies closed 2-forms ω of degree k on $\mathbf{X} = \mathbf{Spec} A$ up to equivalence.)

Proposition 5.6.

- (a) *Let A be a connective cdga over \mathbb{K} , with $H^0(A)$ of finite type over \mathbb{K} . If $p+k \leq 2$, then in the sequence (5.6) the map $\mathrm{HP}^{k-3}(A)(p) \rightarrow \mathrm{HC}^{k-1}(A)(p-1)$ is an injection, hence the map $\mathrm{HN}^{k-3}(A)(p) \rightarrow \mathrm{HP}^{k-3}(A)(p)$ is zero and the map $\mathrm{HC}^{k-2}(A)(p-1) \rightarrow \mathrm{HN}^{k-3}(A)(p)$ is a surjection.*

In particular, for $k < 0$, we have a short exact sequence

$$(5.7) \quad 0 \longrightarrow \mathrm{HP}^{k-3}(A)(2) \longrightarrow \mathrm{HC}^{k-1}(A)(1) \longrightarrow \mathrm{HN}^{k-2}(A)(2) \longrightarrow 0.$$

- (b) *For $k = -1$, the left-hand group $\mathrm{HP}^{-4}(A)(2)$ in (5.7) has $\mathrm{HP}^{-4}(A)(2) \cong \mathrm{H}_{\mathrm{inf}}^0(\mathrm{Spec} H^0(A))$, where $\mathrm{H}_{\mathrm{inf}}^*(X)$ is the algebraic de Rham cohomology of a \mathbb{K} -scheme X . Thus, if $\mathrm{Spec} H^0(A)$ is connected, then $\mathrm{HP}^{-4}(A)(2) = \mathbb{K}$.*
- (c) *For $k \leq -2$, we have $\mathrm{HP}^{k-3}(A)(2) = 0$.*

Proof. When A is a cdga concentrated in degree 0, the fact that $\mathrm{HP}^{k-3}(A)(p) \rightarrow \mathrm{HC}^{k-1}(A)(p-1)$ is an injection is essentially Emmanouil [10, Prop. 2.6], noting that $\mathrm{HP}^{k-3}(A)(p) \cong \mathrm{H}_{\mathrm{inf}}^{2p+k-3}(A)$ by [10, Th. 2.2]. We therefore have a short exact sequence as claimed. Here $\mathrm{H}_{\mathrm{inf}}^{2p+k-3}(A)$ is *infinitesimal* or algebraic de Rham cohomology. In particular, for $k = -1$, $\mathrm{HP}^{-4}(A)(2) \cong \mathrm{H}_{\mathrm{inf}}^0(A) \cong \mathbb{K}$ when $\mathrm{Spec} A$ is connected, and for $k \leq -2$, $\mathrm{HP}^{k-3}(A)(2) \cong \mathrm{H}_{\mathrm{inf}}^{k+1}(A) = 0$.

Now let A be a connective cdga with $H^0(A)$ of finite type over \mathbb{K} and consider the natural map $A \rightarrow H^0(A)$. By Goodwillie [13, Th. IV.2.1], the induced map $\mathrm{HP}^{k-3}(A)(p) \rightarrow \mathrm{HP}^{k-3}(H^0(A))(p)$ is an isomorphism, so that (b) and (c) follow from the above. The injectivity of the map $\mathrm{HP}^{k-3}(A)(p) \rightarrow \mathrm{HC}^{k-1}(A)(p-1)$ follows from the injectivity of the map $\mathrm{HP}^{k-3}(H^0(A))(p) \rightarrow \mathrm{HC}^{k-1}(H^0(A))(p-1)$ by functoriality of the long exact sequence (5.6), and this proves (a). \square

Using Proposition 5.6 we show that if $\omega = (\omega^0, \omega^1, \dots)$ is a k -shifted symplectic structure on $\mathbf{X} = \mathbf{Spec} A$ for $k < 0$, then up to equivalence we can take ω^0 to be exact and $\omega^i = 0$ for $i = 1, 2, \dots$, which is a considerable simplification. Also we parametrise how to write ω this way in its equivalence class.

Proposition 5.7.

- (a) Let $\omega = (\omega^0, \omega^1, \omega^2, \dots)$ be a closed 2-form of degree $k < 0$ on $\mathbf{X} = \mathbf{Spec} A$, for A a standard form cdga over \mathbb{K} . Then there exist $\Phi \in A^{k+1}$ and $\phi \in (\Omega_A^1)^k$ such that $d\Phi = 0$ in A^{k+2} and $d_{dR}\Phi + d\phi = 0$ in $(\Omega_A^1)^{k+1}$ and $\omega \sim (d_{dR}\phi, 0, 0, \dots)$, in the sense of Definition 5.2.
- (b) In the case $k = -1$ in (a), we have $\Phi \in A^0 = A(0)$, so we can consider the restriction $\Phi|_{X^{\text{red}}}$ of Φ to the reduced \mathbb{K} -subscheme X^{red} of $X = t_0(\mathbf{X}) = \mathbf{Spec} H^0(A)$. Then $\Phi|_{X^{\text{red}}}$ is locally constant on X^{red} , and we may choose (Φ, ϕ) in (a) such that $\Phi|_{X^{\text{red}}} = 0$.
- (c) Suppose (Φ, ϕ) and (Φ', ϕ') are alternative choices in (a) for fixed ω, k, \mathbf{X}, A , where if $k = -1$ we suppose $\Phi|_{X^{\text{red}}} = 0 = \Phi'|_{X^{\text{red}}}$ as in (b). Then there exist $\Psi \in A^k$ and $\psi \in (\Omega_A^1)^{k-1}$ with $\Phi - \Phi' = d\Psi$ and $\phi - \phi' = d_{dR}\Psi + d\psi$.

Proof. As in Definition 5.5, the \sim -equivalence class $[\omega]$ of ω lies in $\text{HN}^{k-2}(A)(2)$. Thus by (5.7) in Proposition 5.6, $[\omega]$ lies in the image of the map $\text{HC}^{k-1}(A)(1) \rightarrow \text{HN}^{k-2}(A)(2)$. A class in $\text{HC}^{k-1}(A)(1)$ is represented by $(\Phi, \phi) \in \text{CC}^{k-1}(A)(1) = A^{k+1} \times (\Omega_A^1)^k$ with $d\Phi = 0 = d_{dR}\Phi + d\phi$, and the map $\text{CC}^{k-1}(A)(1) \rightarrow \text{NC}^{k-2}(A)(2)$ takes $(\Phi, \phi) \mapsto (d_{dR}\phi, 0, 0, \dots)$. Note that

$$d(d_{dR}\phi) = -d_{dR} \circ d\phi = -d_{dR} \circ d_{dR}\Phi - d_{dR} \circ d\phi = -d_{dR}(d_{dR}\Phi + d\phi) = 0,$$

so that $(d_{dR}\phi, 0, 0, \dots)$ is a closed 2-form of degree k on \mathbf{X} . This proves (a).

For (b), $\Phi|_X : X \rightarrow \mathbb{A}^1$ is a regular function on X when $k = -1$. We have $H^0(T^*X) \cong H^0(\Omega_A^1)$. As $d_{dR}\Phi + d\phi = 0$, we have $[d_{dR}\Phi] = 0$ in $H^0(\Omega_A^1)$, so $d_{dR}(\Phi|_X) = 0$ in $H^0(T^*X)$. Therefore $\Phi|_{X^{\text{red}}} : X^{\text{red}} \rightarrow \mathbb{A}^1$ is locally constant.

This is related to the isomorphism $\text{HP}^{-4}(A)(2) \cong \text{H}_{\text{inf}}^0(\text{Spec} H^0(A))$ in Proposition 5.6(b). There is a natural isomorphism from $\text{H}_{\text{inf}}^0(\text{Spec} H^0(A))$ to the \mathbb{K} -vector space of locally constant maps $\Phi|_{X^{\text{red}}} : X^{\text{red}} \rightarrow \mathbb{A}^1$. When we lift $[\omega] \in \text{HN}^{-3}(A)(2)$ to $[\Phi, \phi] \in \text{HC}^{-2}(A)(1)$ in (5.7) for $k = -1$, the possible choice in the lift $[\Phi, \phi]$ is $\text{HP}^{-4}(A)(2) \cong \text{H}_{\text{inf}}^0(\text{Spec} H^0(A))$, which corresponds exactly to the space of locally constant maps $\Phi|_{X^{\text{red}}} : X^{\text{red}} \rightarrow \mathbb{A}^1$.

So, by adjusting the choice of lift $[\Phi, \phi]$ of $[\omega]$ to $\text{HC}^{-2}(A)(1)$, we can take $\Phi|_{X^{\text{red}}} = 0$, proving (b). Note that this determines the class $[\Phi, \phi]$ in $\text{HC}^{-2}(A)(1)$ lifting $[\omega]$ uniquely. That is, we have constructed a canonical splitting of the exact sequence (5.7) when $k = -1$.

For (c), note that for all $k < 0$ the class $[\Phi, \phi] \in \text{HC}^{k-1}(A)(1)$ in (a) lifting $[\omega] \in \text{HN}^{k-2}(A)(2)$ is uniquely determined, requiring $\Phi|_{X^{\text{red}}} = 0$ when $k = -1$ as above, and since $\text{HP}^{k-3}(A)(2) = 0$ in (5.7) for $k \leq -2$ by Proposition 5.6(c). Hence, if (Φ, ϕ) and (Φ', ϕ') are alternative choices in (a), then $[\Phi, \phi] = [\Phi', \phi'] \in \text{HC}^{k-1}(A)(1)$. Thus, there exists $(\Psi, \psi) \in \text{CC}^{k-2}(A)(1)$ with $d(\Psi, \psi) = (\Phi, \phi) - (\Phi', \phi')$. From the definitions, this means that $\Psi \in A^k$ and $\psi \in (\Omega_A^1)^{k-1}$ with $\Phi - \Phi' = d\Psi$ and $\phi - \phi' = d_{dR}\Psi + d\psi$. □

When we apply Proposition 5.7 in §5.4–§5.7, we will do it with $k\omega$ in place of ω , yielding Φ, ϕ with $d\Phi = 0$, $d_{dR}\Phi + d\phi = 0$, and $k\omega \sim (d_{dR}\phi, 0, 0, \dots)$. This will give simpler formulae, eliminating factors of $k, 1/k$.

5.3. Darboux forms for k -shifted symplectic structures. The next four examples give standard models for k -shifted symplectic affine derived \mathbb{K} -schemes for $k < 0$, which we will call “in Darboux form”. Theorem 5.18 will prove that every k -shifted symplectic derived scheme (\mathbf{X}, ω) is Zariski/étale locally equivalent to one in Darboux form. We divide into three cases:

- (a) k is odd, so that $k = -2d - 1$ for $d = 0, 1, 2, \dots$;
- (b) $k \equiv 0 \pmod{4}$, so that $k = -4d$ for $d = 1, 2, \dots$; and
- (c) $k \equiv 2 \pmod{4}$, so that $k = -4d - 2$ for $d = 0, 1, \dots$.

The difference is in the behaviour of 2-forms in the variables of *middle degree* $k/2$. In case (a) $k/2 \notin \mathbb{Z}$, so there is no middle degree, and this is the simplest case, which we handle in Example 5.8. In (b) $k/2$ is even, so 2-forms in the middle degree variables are antisymmetric; we discuss this in Example 5.9. In (c) $k/2$ is odd, so 2-forms in the middle degree variables are symmetric. For this case we give both a *strong Darboux form* in Example 5.10, to which k -shifted symplectic derived \mathbb{K} -schemes are equivalent étale locally, and a *weak Darboux form* in Example 5.12, to which they are equivalent Zariski locally.

For A as in Example 5.8 below, as in Bouaziz and Grojnowski [6] we can regard $\mathbf{X} = \mathbf{Spec} A$ as a *twisted shifted cotangent bundle* $T_\sigma^*[k]\mathbf{Y}$ for $\mathbf{Y} = \mathbf{Spec} B$, where $B \subset A$ is the sub-cdga generated by the variables x_j^{-i} . Then in (5.8) the x_j^{-i} are coordinates on the base \mathbf{Y} , and y_j^{k-i} are the dual coordinates on the fibres of $T_\sigma^*[k]\mathbf{Y} \rightarrow \mathbf{Y}$.

Example 5.8. Fix $d = 0, 1, \dots$. We will explain how to define a class of explicit standard form cdgas $(A, d) = A(n)$ for $n = 2d + 1$ with a very simple, explicit k -shifted symplectic form $\omega = (\omega^0, 0, 0, \dots)$ for $k = -2d - 1$.

First choose a smooth \mathbb{K} -algebra $A(0)$ of dimension m_0 . Localising $A(0)$ if necessary, we may assume there exist $x_1^0, \dots, x_{m_0}^0 \in A(0)$ such that $d_{dR}x_1^0, \dots, d_{dR}x_{m_0}^0$ form a basis of $\Omega_{A(0)}^1$ over $A(0)$. Geometrically, $\mathbf{Spec} A(0)$ is a smooth \mathbb{K} -scheme of dimension m_0 , and $(x_1^0, \dots, x_{m_0}^0) : \mathbf{Spec} A(0) \rightarrow \mathbb{A}^{m_0}$ are global étale coordinates on $\mathbf{Spec} A(0)$.

Next, choose $m_1, \dots, m_d \in \mathbb{N} = \{0, 1, \dots\}$. Define A as a commutative graded algebra to be the free algebra over $A(0)$ generated by variables

$$(5.8) \quad \begin{array}{ll} x_1^{-i}, \dots, x_{m_i}^{-i} & \text{in degree } -i \text{ for } i = 1, \dots, d, \text{ and} \\ y_1^{i-2d-1}, \dots, y_{m_i}^{i-2d-1} & \text{in degree } i - 2d - 1 \text{ for } i = 0, 1, \dots, d. \end{array}$$

So the upper index i in x_j^i, y_j^i always indicates the degree. We will define the differential d in the cdga (A, d) later.

As in §2.1 and §2.3, the spaces $(\Lambda^p \Omega_A^1)^k$ and the de Rham differential d_{dR} upon them depend only on the commutative graded algebra A , not on the (not yet defined) differential d . Note that Ω_A^1 is the free A -module with basis $d_{dR}x_j^{-i}$,

$d_{dR}y_j^{i-2d-1}$ for $i = 0, \dots, d$ and $j = 1, \dots, m_i$. Define an element

$$(5.9) \quad \omega^0 = \sum_{i=0}^d \sum_{j=1}^{m_i} d_{dR}x_j^{-i} d_{dR}y_j^{i-2d-1} \quad \text{in } (\Lambda^2\Omega_A^1)^{-2d-1}.$$

Clearly, $d_{dR}\omega^0 = 0$ in $(\Lambda^3\Omega_A^1)^{-2d-1}$.

Now choose a superpotential Φ in A^{-2d} , which we will call the *Hamiltonian*, and which we require to satisfy the *classical master equation*

$$(5.10) \quad \sum_{i=1}^d \sum_{j=1}^{m_i} \frac{\partial\Phi}{\partial x_j^{-i}} \frac{\partial\Phi}{\partial y_j^{i-2d-1}} = 0 \quad \text{in } A^{1-2d}.$$

Note that (5.10) is trivial when $d = 0$, so that $k = -1$, as $A^1 = 0$. We have

$$(5.11) \quad d_{dR}\Phi = \sum_{i=0}^d \sum_{j=1}^{m_i} \frac{\partial\Phi}{\partial x_j^{-i}} d_{dR}x_j^{-i} + \sum_{i=1}^d \sum_{j=1}^{m_i} \frac{\partial\Phi}{\partial y_j^{i-2d-1}} d_{dR}y_j^{i-2d-1},$$

with $\frac{\partial\Phi}{\partial x_j^{-i}} \in A^{i-2d}$ and $\frac{\partial\Phi}{\partial y_j^{i-2d-1}} \in A^{1-i}$. Note that when $i = 0$, $\frac{\partial\Phi}{\partial y_j^{-2d-1}} = 0$ for degree reasons. Define the differential d on A by $d = 0$ on $A(0)$, and

$$(5.12) \quad dx_j^{-i} = \frac{\partial\Phi}{\partial y_j^{i-2d-1}}, \quad dy_j^{i-2d-1} = \frac{\partial\Phi}{\partial x_j^{-i}}, \quad \begin{matrix} i = 0, \dots, d, \\ j = 1, \dots, m_i. \end{matrix}$$

When $i = 0$ this gives $dx_j^0 = \frac{\partial\Phi}{\partial y_j^{-2d-1}} = 0$, consistent with $d = 0$ on $A(0)$. To show that $d \circ d = 0$, note that

$$(5.13) \quad \begin{aligned} d \circ dx_{j'}^{-i'} &= d \left[\frac{\partial\Phi}{\partial y_{j'}^{i'-2d-1}} \right] \\ &= \sum_{i=0}^d \sum_{j=1}^{m_i} \left[dy_j^{i-2d-1} \cdot \frac{\partial^2\Phi}{\partial y_j^{i-2d-1} \partial y_{j'}^{i'-2d-1}} + dx_j^{-i} \cdot \frac{\partial^2\Phi}{\partial x_j^{-i} \partial y_{j'}^{i'-2d-1}} \right] \\ &= \sum_{i=0}^d \sum_{j=1}^{m_i} \left[\frac{\partial\Phi}{\partial x_j^{-i}} \cdot \frac{\partial^2\Phi}{\partial y_j^{i-2d-1} \partial y_{j'}^{i'-2d-1}} + \frac{\partial\Phi}{\partial y_j^{i-2d-1}} \cdot \frac{\partial^2\Phi}{\partial x_j^{-i} \partial y_{j'}^{i'-2d-1}} \right] \\ &= (-1)^{i'-2d-1} \frac{\partial}{\partial y_{j'}^{i'-2d-1}} \left[\sum_{i=0}^d \sum_{j=1}^{m_i} \frac{\partial\Phi}{\partial x_j^{-i}} \cdot \frac{\partial\Phi}{\partial y_j^{i-2d-1}} \right] = 0, \end{aligned}$$

using (5.12) in the first and third steps and (5.10) in the last. In the same way we show that $d \circ dy_{j'}^{i'-2d-1} = 0$.

Observe that (A, d) is a standard form cdga $A = A(n)$ as in Example 2.8 for $n = 2d + 1$, defined inductively using free modules $M^{-i} = \langle x_1^{-i}, \dots, x_{m_i}^{-i} \rangle_{A(i-1)}$ for $i = 1, \dots, d$ and $M^{i-2d-1} = \langle y_1^{i-2d-1}, \dots, y_{m_i}^{i-2d-1} \rangle_{A(i-2d-2)}$ for $i = 0, \dots, d$.

We claim that $\omega := (\omega^0, 0, 0, \dots)$ is a k -shifted symplectic structure on $\mathbf{X} = \text{Spec } A$ for $k = -2d - 1$. To show that ω is a k -shifted closed 2-form, since

$d_{dR}\omega^0 = 0$, by (5.5) we have to show that $d\omega^0 = 0$. This follows from

$$\begin{aligned}
 d\omega^0 &= \sum_{i=0}^d \sum_{j=1}^{m_i} \left[(d \circ d_{dR}y_j^{i-2d-1}) d_{dR}x_j^{-i} + (d \circ d_{dR}x_j^{-i}) d_{dR}y_j^{i-2d-1} \right] \\
 &= - \sum_{i=0}^d \sum_{j=1}^{m_i} \left[(d_{dR} \circ dy_j^{i-2d-1}) d_{dR}x_j^{-i} + (d_{dR} \circ dx_j^{-i}) d_{dR}y_j^{i-2d-1} \right] \\
 &= - \sum_{i=0}^d \sum_{j=1}^{m_i} \left[d_{dR} \left[\frac{\partial \Phi}{\partial x_j^{-i}} \right] d_{dR}x_j^{-i} + d_{dR} \left[\frac{\partial \Phi}{\partial y_j^{i-2d-1}} \right] d_{dR}y_j^{i-2d-1} \right] \\
 &= -d_{dR} \left[\sum_{i=0}^d \sum_{j=1}^{m_i} \frac{\partial \Phi}{\partial x_j^{-i}} d_{dR}x_j^{-i} + \frac{\partial \Phi}{\partial y_j^{i-2d-1}} d_{dR}y_j^{i-2d-1} \right] \\
 (5.14) \quad &= -d_{dR} \circ d_{dR}\Phi = 0,
 \end{aligned}$$

using (5.9) in the first step, $d \circ d_{dR} = -d_{dR} \circ d$ in the second, (5.12) in the third, $d_{dR} \circ d_{dR} = 0$ in the fourth and sixth, and (5.11) in the fifth.

To show ω^0 is nondegenerate, we must check (5.3) is a quasi-isomorphism. It is enough to show that $\omega^0 \otimes \text{id}_{A(0)} : \mathbb{T}_A \otimes_A A(0) \rightarrow \Omega_A^1[k] \otimes_A A(0)$ is one. Using Proposition 2.12, we see that $\omega^0 \otimes \text{id}_{A(0)}$ may be written

$$\begin{array}{ccc}
 \left\langle \frac{\partial}{\partial x_1^0}, \dots, \frac{\partial}{\partial x_{m_0}^0} \right\rangle_{A(0)} & \xrightarrow{\omega^0} & \left\langle d_{dR}y_1^{-2d-1}, \dots, d_{dR}y_{m_0}^{-2d-1} \right\rangle_{A(0)} \\
 \downarrow d & & \downarrow d \\
 \vdots & & \vdots \\
 \downarrow d & & \downarrow d \\
 \left\langle \frac{\partial}{\partial x_1^{-d}}, \dots, \frac{\partial}{\partial x_{m_d}^{-d}} \right\rangle_{A(0)} & \xrightarrow{\omega^0} & \left\langle d_{dR}y_1^{-d-1}, \dots, d_{dR}y_{m_d}^{-d-1} \right\rangle_{A(0)} \\
 \downarrow d & & \downarrow d \\
 (5.15) \quad \left\langle \frac{\partial}{\partial y_1^{-d-1}}, \dots, \frac{\partial}{\partial y_{m_d}^{-d-1}} \right\rangle_{A(0)} & \xrightarrow{\omega^0} & \left\langle d_{dR}x_1^{-d}, \dots, d_{dR}x_{m_d}^{-d} \right\rangle_{A(0)} \\
 \downarrow d & & \downarrow d \\
 \vdots & & \vdots \\
 \downarrow d & & \downarrow d \\
 \left\langle \frac{\partial}{\partial y_1^{-2d-1}}, \dots, \frac{\partial}{\partial y_{m_0}^{-2d-1}} \right\rangle_{A(0)} & \xrightarrow{\omega^0} & \left\langle d_{dR}x_1^0, \dots, d_{dR}x_{m_0}^0 \right\rangle_{A(0)}.
 \end{array}$$

But by (5.9), the rows of (5.15) are isomorphisms, so $\omega^0 \otimes \text{id}_{A(0)}$ is an isomorphism of complexes, and thus a quasi-isomorphism. Hence $\omega = (\omega^0, 0, 0, \dots)$ is a k -shifted symplectic structure on $\mathbf{X} = \text{Spec } A$ for $k = -2d - 1$.

Finally, define $\phi \in (\Omega_A^1)^{-2d-1}$ by

$$(5.16) \quad \phi = - \sum_{i=0}^d \sum_{j=1}^{m_i} [i x_j^{-i} d_{dR}y_j^{i-2d-1} + (2d + 1 - i)y_j^{i-2d-1} d_{dR}x_j^{-i}].$$

Then calculating using (5.9)–(5.12) shows that $d\Phi = 0$ in A^{1-2d} , and $d_{dR}\Phi + d\phi = 0$ in $(\Omega_A^1)^{-2d}$, and $k\omega^0 = d_{dR}\phi$ in $(\Lambda^2\Omega_A^1)^{-2d-1}$. Here to prove $d_{dR}\Phi + d\phi = 0$, we

use that $\frac{\partial\Phi}{\partial x_j^{-i}}$ lies in A^{i-2d} so that

$$(i - 2d) \frac{\partial\Phi}{\partial x_j^{-i}} = \sum_{i'=0}^d \sum_{j'=1}^{m_{i'}} \left[-i' x_{j'}^{-i'} \frac{\partial^2\Phi}{\partial x_{j'}^{-i'} \partial x_j^{-i}} + (i' - 2d - 1) y_{j'}^{i'-2d-1} \frac{\partial^2\Phi}{\partial y_{j'}^{i'-2d-1} \partial x_j^{-i}} \right],$$

and a similar equation for $\frac{\partial\Phi}{\partial y_j^{i-2d-1}}$. Thus (Φ, ϕ) satisfy Proposition 5.7(a) with $k\omega$ in place of ω .

Example 5.9. Fix $d = 1, 2, \dots$. We will define a class of explicit standard form cdgas $(A, d) = A(n)$ for $n = 4d$ with an explicit k -shifted symplectic form $\omega = (\omega^0, 0, 0, \dots)$ for $k = -4d$.

As in Example 5.8, choose a smooth \mathbb{K} -algebra $A(0)$ of dimension m_0 , and $x_1^0, \dots, x_{m_0}^0 \in A(0)$ with $d_{dR}x_1^0, \dots, d_{dR}x_{m_0}^0$ a basis of $\Omega_{A(0)}^1$ over $A(0)$. Choose $m_1, \dots, m_{2d} \in \mathbb{N}$, and define A as a commutative graded algebra to be the free algebra over $A(0)$ generated by variables, as for (5.8)

$$(5.17) \quad \begin{array}{ll} x_1^{-i}, \dots, x_{m_i}^{-i} & \text{in degree } -i \text{ for } i = 1, \dots, 2d - 1, \\ x_1^{-2d}, \dots, x_{m_{2d}}^{-2d}, y_1^{-2d}, \dots, y_{m_{2d}}^{-2d} & \text{in degree } -2d, \text{ and} \\ y_1^{i-4d}, \dots, y_{m_i}^{i-4d} & \text{in degree } i - 4d \text{ for } i = 0, 1, \dots, 2d - 1. \end{array}$$

Note that there are $2m_{2d}$ rather than m_{2d} generating variables in degree $-2d$.

As for (5.9), define $\omega^0 \in (\Lambda^2\Omega_A^1)^{-4d}$ with $d_{dR}\omega^0 = 0$ in $(\Lambda^3\Omega_A^1)^{-4d}$ by

$$(5.18) \quad \omega^0 = \sum_{i=0}^{2d} \sum_{j=1}^{m_i} d_{dR}x_j^{-i} d_{dR}y_j^{i-4d} \quad \text{in } (\Lambda^2\Omega_A^1)^{-4d}.$$

Choose a Hamiltonian Φ in A^{1-4d} , which we require to satisfy the analogue of (5.10). The analogue of (5.11) holds. As for (5.12), define the differential d on A by $d = 0$ on $A(0)$, and

$$(5.19) \quad dx_j^{-i} = (-1)^{i+1} \frac{\partial\Phi}{\partial y_j^{i-4d}}, \quad dy_j^{i-4d} = \frac{\partial\Phi}{\partial x_j^{-i}}, \quad \begin{array}{l} i = 0, \dots, 2d, \\ j = 1, \dots, m_i. \end{array}$$

We prove that $d \circ d = 0$ as in (5.13). The analogue of (5.14) holds with signs inserted, and the analogue of (5.15) has rows isomorphisms. Hence $\omega := (\omega^0, 0, 0, \dots)$ is a k -shifted symplectic structure on $\mathbf{X} = \mathbf{Spec} A$ for $k = -4d$.

Finally, defining $\phi \in (\Omega_A^1)^{-4d}$ by the analogue of (5.16), we find that (Φ, ϕ) satisfy Proposition 5.7(a) with $k\omega$ in place of ω .

Example 5.10. Fix $d = 0, 1, 2, \dots$. We will define a class of explicit standard form cdgas $(A, d) = A(n)$ for $n = 4d + 2$ with an explicit k -shifted symplectic form $\omega = (\omega^0, 0, 0, \dots)$ for $k = -4d - 2$.

As in Examples 5.8 and 5.9, choose a smooth \mathbb{K} -algebra $A(0)$ of dimension m_0 , and $x_1^0, \dots, x_{m_0}^0 \in A(0)$ with $d_{dR}x_1^0, \dots, d_{dR}x_{m_0}^0$ a basis of $\Omega_{A(0)}^1$ over $A(0)$. Choose $m_1, \dots, m_{2d+1} \in \mathbb{N}$, and define A as a commutative graded algebra to be the free algebra over $A(0)$ generated by variables, as for (5.8) and (5.17)

$$(5.20) \quad \begin{array}{ll} x_1^{-i}, \dots, x_{m_i}^{-i} & \text{in degree } -i \text{ for } i = 1, \dots, 2d, \\ z_1^{-2d-1}, \dots, z_{m_{2d+1}}^{-2d-1} & \text{in degree } -2d - 1, \text{ and} \\ y_1^{i-4d-2}, \dots, y_{m_i}^{i-4d-2} & \text{in degree } i - 4d - 2 \text{ for } i = 0, 1, \dots, 2d. \end{array}$$

As for (5.9) and (5.18), define $\omega^0 \in (\Lambda^2 \Omega_A^1)^{-4d-2}$ with $d_{dR}\omega^0 = 0$ by

$$(5.21) \quad \omega^0 = \sum_{i=0}^{2d} \sum_{j=1}^{m_i} d_{dR}x_j^{-i} d_{dR}y_j^{i-4d-2} + \sum_{j=1}^{m_{2d+1}} d_{dR}z_j^{-2d-1} d_{dR}z_j^{-2d-1}.$$

Choose a Hamiltonian Φ in A^{-4d-1} , which we require to satisfy the *classical master equation*

$$(5.22) \quad \sum_{i=1}^{2d} \sum_{j=1}^{m_i} \frac{\partial \Phi}{\partial x_j^{-i}} \frac{\partial \Phi}{\partial y_j^{i-4d-2}} + \frac{1}{4} \sum_{j=1}^{m_{2d+1}} \left(\frac{\partial \Phi}{\partial z_j^{-2d-1}} \right)^2 = 0 \quad \text{in } A^{-4d}.$$

The analogue of (5.11) holds, with extra terms $\frac{\partial \Phi}{\partial z_j^{-2d-1}} d_{dR}z_j^{-2d-1}$. As for (5.12) and (5.19), define the differential d on A by $d = 0$ on $A(0)$, and

$$(5.23) \quad \begin{aligned} dx_j^{-i} &= (-1)^{i+1} \frac{\partial \Phi}{\partial y_j^{i-4d-2}}, & dy_j^{i-4d-2} &= \frac{\partial \Phi}{\partial x_j^{-i}}, & i &= 0, \dots, 2d, \\ \text{and } dz_j^{-2d-1} &= \frac{1}{2} \frac{\partial \Phi}{\partial z_j^{-2d-1}}, & j &= 1, \dots, m_{2d+1}. \end{aligned}$$

The analogue of (5.13) shows that $d \circ d = 0$. The analogue of (5.14) holds with extra signs and terms from the z_j^{-2d-1} , and the analogue of (5.15) has rows isomorphisms. Thus $\omega := (\omega^0, 0, 0, \dots)$ is a k -shifted symplectic structure on $\mathbf{X} = \mathbf{Spec} A$ for $k = -4d - 2$. Defining

$$(5.24) \quad \begin{aligned} \phi &= - \sum_{i=0}^{2d} \sum_{j=1}^{m_i} [i x_j^{-i} d_{dR}y_j^{i-4d-2} + (-1)^{i+1} (4d+2-i) y_j^{i-4d-2} d_{dR}x_j^{-i}] \\ &\quad - (4d+2) \sum_{j=1}^{m_{2d+1}} z_j^{-2d-1} d_{dR}z_j^{-2d-1} \quad \text{in } (\Omega_A^1)^{-4d-2}, \end{aligned}$$

the analogue of (5.16), we find that (Φ, ϕ) satisfy Proposition 5.7(a) with $k\omega$ in place of ω .

Remark 5.11. In Example 5.10, if m_{2d+1} is even, or equivalently if the *virtual dimension* $\text{vdim } \mathbf{X}$ of \mathbf{X} is even, given by

$$\text{vdim } \mathbf{X} = -m_{2d+1} + 2 \sum_{i=0}^{2d} (-1)^i m_i,$$

then we may change variables from $z_1^{-2d-1}, \dots, z_{m_{2d+1}}^{-2d-1}$ to $x_1^{-2d-1}, \dots, x_{m_{2d+1}/2}^{-2d-1}, y_1^{-2d-1}, \dots, y_{m_{2d+1}/2}^{-2d-1}$ defined for $j = 1, \dots, m_{2d+1}/2$ by

$$x_j^{-2d-1} = z_j^{-2d-1} + \sqrt{-1} z_{j+m_{2d+1}/2}^{-2d-1}, \quad y_j^{-2d-1} = z_j^{-2d-1} - \sqrt{-1} z_{j+m_{2d+1}/2}^{-2d-1}.$$

Then replacing m_{2d+1} by $m_{2d+1}/2$, we find that A is freely generated over $A(0)$ by variables

$$\begin{array}{ll} x_1^{-i}, \dots, x_{m_i}^{-i} & \text{in degree } -i \text{ for } i = 1, \dots, 2d, \\ x_1^{-2d-1}, \dots, x_{m_{2d+1}}^{-2d-1}, y_1^{-2d-1}, \dots, y_{m_{2d+1}}^{-2d-1} & \text{in degree } -2d - 1, \text{ and} \\ y_1^{i-4d-2}, \dots, y_{m_i}^{i-4d-2} & \text{in degree } i - 4d - 2 \text{ for } i = 0, 1, \dots, 2d, \end{array}$$

and the symplectic form is given by

$$\omega^0 = \sum_{i=0}^{2d+1} \sum_{j=1}^{m_i} d_{dR} x_j^{-i} d_{dR} y_j^{i-4d-2},$$

and everything works out as in Examples 5.8 and 5.9.

Example 5.12. Here is a variation on Example 5.10. Choose $d, A(0), m_0, \dots, m_{2d+1}, x_j^i, y_j^i, z_j^i$ and the commutative graded algebra A as in Example 5.10. Let $q_1, \dots, q_{m_{2d+1}}$ be invertible elements of $A(0)$ and, generalising (5.21), define

$$(5.25) \quad \omega^0 = \sum_{i=0}^{2d} \sum_{j=1}^{m_i} d_{dR} x_j^{-i} d_{dR} y_j^{i-4d-2} + \sum_{j=1}^{m_{2d+1}} d_{dR} (q_j z_j^{-2d-1}) d_{dR} z_j^{-2d-1}.$$

Choose a Hamiltonian Φ in A^{-4d-1} , which we require to satisfy the *classical master equation*

$$(5.26) \quad \sum_{i=1}^{2d} \sum_{j=1}^{m_i} \frac{\partial \Phi}{\partial x_j^{-i}} \frac{\partial \Phi}{\partial y_j^{i-4d-2}} + \frac{1}{4} \sum_{j=1}^{m_{2d+1}} \frac{1}{q_j} \left(\frac{\partial \Phi}{\partial z_j^{-2d-1}} \right)^2 = 0 \quad \text{in } A^{-4d}.$$

As for (5.23), define the differential d on A by $d = 0$ on $A(0)$, and

$$(5.27) \quad \begin{aligned} dx_j^0 &= 0, & dy_j^{-4d-2} &= \frac{\partial \Phi}{\partial x_j^0} - \sum_{j'=1}^{m_{2d+1}} \frac{z_j'^{-2d-1}}{2q_{j'}} \frac{\partial q_{j'}}{\partial x_j^0} \frac{\partial \Phi}{\partial z_j'^{-2d-1}}, & j &= 1, \dots, m_0, \\ dx_j^{-i} &= (-1)^{i+1} \frac{\partial \Phi}{\partial y_j^{i-4d-2}}, & dy_j^{i-4d-2} &= \frac{\partial \Phi}{\partial x_j^{-i}}, & i &= 1, \dots, 2d, \\ & & \text{and} & & dz_j^{-2d-1} &= \frac{1}{2q_j} \frac{\partial \Phi}{\partial z_j^{-2d-1}}, & j &= 1, \dots, m_{2d+1}. \end{aligned}$$

Similar proofs to (5.13)–(5.15) show that $d \circ d = 0$, so that $(A, d) = A(n)$ for $n = 4d + 2$ is a standard form cdga over \mathbb{K} , as in Example 2.8, and $\omega := (\omega^0, 0, 0, \dots)$ is a k -shifted symplectic structure on $\mathbf{X} = \mathbf{Spec} A$ for $k = -4d - 2$.

Defining

$$\begin{aligned} \phi &= - \sum_{i=0}^{2d} \sum_{j=1}^{m_i} [i x_j^{-i} d_{dR} y_j^{i-4d-2} + (-1)^{i+1} (4d + 2 - i) y_j^{i-4d-2} d_{dR} x_j^{-i}] \\ &\quad - (4d + 2) \sum_{j=1}^{m_{2d+1}} q_j z_j^{-2d-1} d_{dR} z_j^{-2d-1} \quad \text{in } (\Omega_A^1)^{-4d-2}, \end{aligned}$$

the analogue of (5.24), we find that (Φ, ϕ) satisfy Proposition 5.7(a) with $k\omega$ in place of ω .

Remark 5.13. To relate Examples 5.10 and 5.12, note that if $q_1 = \dots = q_{m_{2d+1}} = 1$, then Example 5.12 reduces immediately to Example 5.10. More generally, if $q_1, \dots, q_{m_{2d+1}}$ admit square roots $q_i^{1/2}$ in $A(0)$, then changing variables in A from $x_j^i, y_j^i, z_j^{-2d-1}$ to x_j^i, y_j^i and $\tilde{z}_j^{-2d-1} = q_j^{1/2} z_j^{-2d-1}$, we find that Example 5.12 reduces to Example 5.10 with \tilde{z}_j^{-2d-1} in place of z_j^{-2d-1} .

Definition 5.14. We say that the standard form cdga A over \mathbb{K} and k -shifted symplectic structure ω on $\mathbf{X} = \mathbf{Spec} A$ in Examples 5.8 and 5.9 are in Darboux

form. We say that A, ω in Example 5.10 are in strong Darboux form, and A, ω in Example 5.12 are in weak Darboux form.

5.4. Darboux forms when $k = -1, -2$ and -3 . We now write out the Darboux form k -shifted symplectic cdgas (A, ω) of Examples 5.8–5.10, and 5.12 more explicitly in the first three cases $k = -1, k = -2$, and $k = -3$. These correspond to the geometric structures on moduli schemes of coherent sheaves on Calabi–Yau m -folds for $m = 3, 4, 5$.

Example 5.15. Choose a smooth \mathbb{K} -algebra $A(0)$ of dimension m_0 and elements $x_1^0, \dots, x_{m_0}^0 \in A(0)$ such that $d_{dR}x_1^0, \dots, d_{dR}x_{m_0}^0$ form a basis of $\Omega_{A(0)}^1$ over $A(0)$. Choose an arbitrary Hamiltonian $\Phi \in A(0)$.

Example 5.8 with $d = 0$ defines $A = A(0)[y_1^{-1}, \dots, y_{m_0}^{-1}]$, where $y_1^{-1}, \dots, y_{m_0}^{-1}$ are variables of degree -1 , with differential

$$dx_i^0 = 0, \quad dy_i^{-1} = \frac{\partial \Phi}{\partial x_i^0}, \quad i = 1, \dots, m_0,$$

and -1 -shifted 2-form

$$\omega^0 = d_{dR}x_1^0 d_{dR}y_1^{-1} + \dots + d_{dR}x_{m_0}^0 d_{dR}y_{m_0}^{-1}.$$

Then $\omega = (\omega^0, 0, 0, \dots)$ is a -1 -shifted symplectic structure on $\mathbf{X} = \mathbf{Spec} A$. We have $H^0(A) = A(0)/(\frac{\partial \Phi}{\partial x_1^0}, \dots, \frac{\partial \Phi}{\partial x_{m_0}^0}) = A(0)/(d_{dR}\Phi)$.

Geometrically, $U = \mathbf{Spec} A(0)$ is a smooth classical \mathbb{K} -scheme with étale coordinates $(x_1^0, \dots, x_{m_0}^0) : U \rightarrow \mathbb{A}^{m_0}$, and $\Phi : U \rightarrow \mathbb{A}^1$ is regular, and \mathbf{X} is the derived critical locus of Φ , with $X = t_0(\mathbf{X})$ the classical critical locus of Φ . As in Proposition 5.7(b), $\Phi|_{X^{\text{red}}} : X^{\text{red}} \rightarrow \mathbb{A}^1$ is locally constant, and we generally suppose Φ is chosen so that $\Phi|_{X^{\text{red}}} = 0$. If X is connected this can be achieved by adding a constant to Φ .

Thus, *the important geometric data in writing a -1 -shifted symplectic derived \mathbb{K} -scheme (\mathbf{X}, ω) in Darboux form is a smooth affine \mathbb{K} -scheme U and a regular function $\Phi : U \rightarrow \mathbb{A}^1$, which we may take to satisfy $\Phi|_{\text{Crit}(\Phi)^{\text{red}}} = 0$, such that $X = t_0(\mathbf{X}) \cong \text{Crit}(\Phi)$.* The remaining data is a choice of étale coordinates $(x_1^0, \dots, x_{m_0}^0) : U \rightarrow \mathbb{A}^{m_0}$, but this is not very interesting geometrically.

Example 5.16. We will work out Example 5.12 with $d = 0$ and $k = -2$ in detail. For Example 5.10 with $d = 0$, set $q_1 = \dots = q_{m_1} = 1$ in what follows.

Choose a smooth \mathbb{K} -algebra $A(0)$ of dimension m_0 and elements $x_1^0, \dots, x_{m_0}^0$ in $A(0)$ such that $d_{dR}x_1^0, \dots, d_{dR}x_{m_0}^0$ form a basis of $\Omega_{A(0)}^1$ over $A(0)$. Fix $m_1 \geq 0$, and as a commutative graded algebra set $A = A(0)[z_1^{-1}, \dots, z_{m_1}^{-1}, y_1^{-2}, \dots, y_{m_0}^{-2}]$, where z_j^{-1} has degree -1 and y_j^{-2} degree -2 .

Choose invertible functions q_1, \dots, q_{m_1} in $A(0)$. Define

$$\begin{aligned} \omega^0 &= d_{dR}x_1^0 d_{dR}y_1^{-2} + \dots + d_{dR}x_{m_0}^0 d_{dR}y_{m_0}^{-2} \\ &\quad + d_{dR}(q_1 z_1^{-1}) d_{dR}z_1^{-1} + \dots + d_{dR}(q_{m_1} z_{m_1}^{-1}) d_{dR}z_{m_1}^{-1} \end{aligned}$$

in $(\Lambda^2 \Omega_A^1)^{-2}$, as in (5.25). A general element Φ in A^{-1} may be written

$$\Phi = z_1^{-1} s_1 + \dots + z_{m_1}^{-1} s_{m_1},$$

for $s_1, \dots, s_{m_1} \in A(0)$. Then the classical master equation (5.26) reduces to

$$(5.28) \quad \frac{(s_1)^2}{q_1} + \dots + \frac{(s_{m_1})^2}{q_{m_1}} = 0 \quad \text{in } A(0).$$

By (5.27), the differential d on A is given by

$$dx_i^0 = 0, \quad dz_j^{-1} = \frac{s_j}{2q_j}, \quad dy_i^{-2} = \sum_{j=1}^{m_1} z_j^{-1} \left(\frac{\partial s_j}{\partial x_i^0} - \frac{s_j}{2q_j} \frac{\partial q_j}{\partial x_i^0} \right),$$

and $d \circ dy_i^{-2} = 0$ follows from applying $\frac{1}{4} \frac{\partial}{\partial x_i^0}$ to (5.28). We have

$$H^0(A) = A(0)/(s_1/2q_1, \dots, s_{m_1}/2q_{m_1}) = A(0)/(s_1, \dots, s_{m_1}),$$

as q_1, \dots, q_{m_1} are invertible.

Geometrically, we have a smooth classical \mathbb{K} -scheme $U = \text{Spec } A(0)$ with étale coordinates $(x_1^0, \dots, x_{m_0}^0) : U \rightarrow \mathbb{A}^{m_0}$, a trivial vector bundle $E \rightarrow U$ with fibre \mathbb{K}^{m_1} , a nondegenerate quadratic form Q on E given by $Q(e_1, \dots, e_{m_1}) = q_1^{-1}e_1^2 + \dots + q_{m_1}^{-1}e_{m_1}^2$ for all regular functions $e_1, \dots, e_{m_1} : U \rightarrow \mathbb{A}^1$, and a section $s = (s_1, \dots, s_{m_1})$ in $H^0(E)$ with $Q(s, s) = 0$ by (5.28). The underlying classical \mathbb{K} -scheme $X = t_0(\mathbf{X}) = \text{Spec } H^0(A)$ is the \mathbb{K} -subscheme $s^{-1}(0)$ in U .

Thus, *the important geometric data in writing a -2 -shifted symplectic derived \mathbb{K} -scheme (\mathbf{X}, ω) in Darboux form is a smooth affine \mathbb{K} -scheme U , a vector bundle $E \rightarrow U$, a nondegenerate quadratic form Q on E , and a section $s \in H^0(E)$ with $Q(s, s) = 0$, such that $X = t_0(\mathbf{X}) \cong s^{-1}(0) \subseteq U$.* The remaining data is a choice of étale coordinates $(x_1^0, \dots, x_{m_0}^0) : U \rightarrow \mathbb{A}^{m_0}$ and a trivialisation $E \cong U \times \mathbb{A}^{m_1}$, but these are not very interesting geometrically.

Example 5.17. We will work out Example 5.8 with $d = 1$ and $k = -3$ in detail. Choose a smooth \mathbb{K} -algebra $A(0)$ of dimension m_0 and elements $x_1^0, \dots, x_{m_0}^0$ in $A(0)$ such that $d_{dR}x_1^0, \dots, d_{dR}x_{m_0}^0$ form a basis of $\Omega_{A(0)}^1$ over $A(0)$. Fix $m_1 \geq 0$, and as a commutative graded algebra set

$$A = A(0)[x_1^{-1}, \dots, x_{m_1}^{-1}, y_1^{-2}, \dots, y_{m_1}^{-2}, y_1^{-3}, \dots, y_{m_0}^{-3}],$$

where $x_i^{-1}, y_i^{-2}, y_i^{-3}$ have degrees $-1, -2, -3$, respectively. Define

$$\begin{aligned} \omega^0 &= d_{dR}x_1^0 d_{dR}y_1^{-3} + \dots + d_{dR}x_{m_0}^0 d_{dR}y_{m_0}^{-3} \\ &\quad + d_{dR}x_1^{-1} d_{dR}y_1^{-2} + \dots + d_{dR}x_{m_1}^{-1} d_{dR}y_{m_1}^{-2} \end{aligned}$$

in $(\Lambda^2 \Omega_A^1)^{-3}$, as in (5.9). A general element Φ in A^{-2} may be written

$$\Phi = \sum_{i=1}^{m_1} y_i^{-2} s_i + \sum_{i,j=1}^{m_1} x_i^{-1} x_j^{-1} t_{ij},$$

for $s_i, t_{ij} \in A(0)$ with $t_{ij} = -t_{ji}$. Then the classical master equation (5.10) reduces to $2 \sum_{i,j=1}^{m_1} x_i^{-1} t_{ij} s_j = 0$ in A^{-1} , or equivalently the m_1 equations

$$(5.29) \quad \sum_{j=1}^{m_1} t_{ij} s_j = 0 \quad \text{in } A(0) \text{ for all } i = 1, \dots, m_1.$$

By (5.12), the differential d on A is given by

$$\begin{aligned} dx_i^0 &= 0, \quad dx_i^{-1} = s_i, \quad dy_i^{-2} = 2 \sum_{j=1}^{m_1} x_j^{-1} t_{ij}, \\ dy_{i'}^{-3} &= \sum_{i=1}^{m_1} y_i^{-2} \frac{\partial s_i}{\partial x_{i'}^0} + \sum_{i,j=1}^{m_1} x_i^{-1} x_j^{-1} \frac{\partial t_{ij}}{\partial x_{i'}^0}. \end{aligned}$$

Thus $H^0(A) = A(0)/(s_1, \dots, s_{m_1})$.

Geometrically, we have a smooth classical \mathbb{K} -scheme $U = \text{Spec } A(0)$ equipped with étale coordinates $(x_1^0, \dots, x_{m_0}^0) : U \rightarrow \mathbb{A}^{m_0}$, a trivial vector bundle $E \rightarrow U$ over U with fibre \mathbb{K}^{m_1} , and sections $s = (s_1, \dots, s_{m_1})$ in $H^0(E)$ and $t = (t_{ij})_{i,j=1}^{m_1}$ in $H^0(\Lambda^2 E^*)$, where t need not be nondegenerate, such that regarding t as a morphism

$E \rightarrow E^*$ we have $t \circ s = 0$ by (5.29). The underlying classical \mathbb{K} -scheme $X = t_0(\mathbf{X}) = \text{Spec } H^0(A)$ is the \mathbb{K} -subscheme $s = 0$ in U .

Thus, *the important geometric data in writing a -3 -shifted symplectic derived \mathbb{K} -scheme (\mathbf{X}, ω) in Darboux form is a smooth affine \mathbb{K} -scheme U , a vector bundle $E \rightarrow U$, sections $s \in H^0(E)$ and $t \in H^0(\Lambda^2 E^*)$ such that $t \circ s = 0 \in H^0(E^*)$ regarding t as morphism $E \rightarrow E^*$, and $X = t_0(\mathbf{X}) \cong s^{-1}(0) \subseteq U$. The remaining data is a choice of étale coordinates $(x_1^0, \dots, x_{m_0}^0) : U \rightarrow \mathbb{A}^{m_0}$ and a trivialisation $E \cong U \times \mathbb{A}^{m_1}$, but these are not very interesting geometrically.*

5.5. A Darboux-type theorem for derived schemes. Here is the main theorem of this paper, a k -shifted analogue of Darboux’s theorem in symplectic geometry, which will be proved in §5.6.

Theorem 5.18. *Let \mathbf{X} be a derived \mathbb{K} -scheme with k -shifted symplectic form $\tilde{\omega}$ for $k < 0$, and let $x \in \mathbf{X}$. Then there exists a standard form cdga A over \mathbb{K} which is minimal at $p \in \text{Spec } H^0(A)$, a k -shifted symplectic form ω on $\mathbf{Spec } A$, a morphism $\mathbf{f} : \mathbf{Spec } A \rightarrow \mathbf{X}$ with $\mathbf{f}(p) = x$, and a path $\mathbf{f}^*(\tilde{\omega}) \sim \omega$ in the space of k -shifted closed 2-forms on $\mathbf{Spec } A$ such that the following hold.*

- (i) *If k is odd or divisible by 4, then \mathbf{f} is a Zariski open inclusion and A, ω are in Darboux form, as in Examples 5.8 and 5.9.*
- (ii) *If $k \equiv 2 \pmod{4}$, then \mathbf{f} is a Zariski open inclusion and A, ω are in weak Darboux form, as in Example 5.12.*
- (iii) *Alternatively, if $k \equiv 2 \pmod{4}$, then we may instead take \mathbf{f} to be étale and A, ω to be in strong Darboux form, as in Example 5.10.*

The theorem has interesting consequences even in classical algebraic geometry. For example, let Y be a Calabi–Yau m -fold over \mathbb{K} , that is, a smooth projective \mathbb{K} -scheme with trivial canonical bundle. Suppose \mathcal{M} is a classical moduli \mathbb{K} -scheme of simple coherent sheaves in $\text{coh}(Y)$, where we call $F \in \text{coh}(Y)$ *simple* if $\text{Hom}(F, F) = \mathbb{K}$. More generally, suppose \mathcal{M} is a moduli \mathbb{K} -scheme of simple complexes of coherent sheaves in $D^b \text{coh}(Y)$, where we call $F^\bullet \in D^b \text{coh}(Y)$ *simple* if $\text{Hom}(F^\bullet, F^\bullet) = \mathbb{K}$ and $\text{Ext}^{<0}(F^\bullet, F^\bullet) = 0$. (Such moduli spaces \mathcal{M} are only known to be algebraic \mathbb{K} -spaces in general, but we assume \mathcal{M} is a \mathbb{K} -scheme. The case of algebraic spaces can be treated by using étale in place of Zariski neighbourhoods.)

Then $\mathcal{M} = t_0(\mathcal{M})$, for \mathcal{M} the corresponding derived moduli \mathbb{K} -scheme. To make \mathcal{M}, \mathcal{M} into schemes rather than stacks, we consider moduli of sheaves or complexes with fixed determinant. Then Pantev et al. [24, §2.1] prove \mathcal{M} has a $(2 - m)$ -shifted symplectic structure ω , so Theorem 5.18 shows that (\mathcal{M}, ω) is Zariski locally modelled on $(\mathbf{Spec } A, \omega)$ as in Examples 5.8–5.10 and 5.12, and \mathcal{M} is Zariski locally modelled on $\text{Spec } H^0(A)$. In the case $m = 3$, so that $k = -1$, from Example 5.15 we deduce:

Corollary 5.19. *Suppose Y is a Calabi–Yau 3-fold over a field \mathbb{K} , and suppose \mathcal{M} is a classical moduli \mathbb{K} -scheme of simple coherent sheaves, or simple complexes of coherent sheaves, on Y . Then for each $[F] \in \mathcal{M}$, there exist a smooth \mathbb{K} -scheme U with $\dim U = \dim \text{Ext}^1(F, F)$, a regular function $f : U \rightarrow \mathbb{A}^1$, and an isomorphism from $\text{Crit}(f) \subseteq U$ to a Zariski open neighbourhood of $[F]$ in \mathcal{M} .*

Here $\dim U = \dim \text{Ext}^1(F, F)$ comes from A minimal at p and $\mathbf{f}(p) = [F]$ in Theorem 5.18. Related results are important in Donaldson–Thomas theory [18–20]. When $\mathbb{K} = \mathbb{C}$ and \mathcal{M} is a moduli space of simple coherent sheaves on Y , using

gauge theory and transcendental complex methods, Joyce and Song [18, Th. 5.4] prove that the underlying complex analytic space \mathcal{M}^{an} of \mathcal{M} is locally of the form $\text{Crit}(f)$ for U a complex manifold and $f : U \rightarrow \mathbb{C}$ a holomorphic function. Behrend and Getzler announced the analogue of [18, Th. 5.4] for moduli of complexes in $D^b \text{coh}(Y)$, but the proof has not yet appeared. Over general \mathbb{K} , as in Kontsevich and Soibelman [19, §3.3] the formal neighbourhood $\hat{\mathcal{M}}_{[F]}$ of \mathcal{M} at any $[F] \in \mathcal{M}$ is isomorphic to the critical locus $\text{Crit}(\hat{f})$ of a formal power series \hat{f} on $\text{Ext}^1(F, F)$ with only cubic and higher terms.

In the case $m = 4$, so that $k = -2$, from Example 5.16 we deduce a local description of Calabi–Yau 4-fold moduli schemes, which may be new:

Corollary 5.20. *Suppose Y is a Calabi–Yau 4-fold over a field \mathbb{K} , and suppose \mathcal{M} is a classical moduli \mathbb{K} -scheme of simple coherent sheaves, or simple complexes of coherent sheaves, on Y . Then for each $[F] \in \mathcal{M}$, there exist a smooth \mathbb{K} -scheme U with $\dim U = \dim \text{Ext}^1(F, F)$, a vector bundle $E \rightarrow U$ with $\text{rank } E = \dim \text{Ext}^2(F, F)$, a nondegenerate quadratic form Q on E , a section $s \in H^0(E)$ with $Q(s, s) = 0$, and an isomorphism from $s^{-1}(0) \subseteq U$ to a Zariski open neighbourhood of $[F]$ in \mathcal{M} .*

If (S, ω) is an algebraic symplectic manifold over \mathbb{K} (that is, a 0-shifted symplectic derived \mathbb{K} -scheme in the language of [24]) and $L, M \subseteq S$ are Lagrangians, then Pantev et al. [24, Th. 2.10] show that the derived intersection $\mathbf{X} = L \times_S M$ has a -1 -shifted symplectic structure. So Theorem 5.18 and Example 5.15 imply:

Corollary 5.21. *Suppose (S, ω) is an algebraic symplectic manifold, and suppose L, M are algebraic Lagrangian submanifolds in S . Then the intersection $X = L \cap M$, as a classical \mathbb{K} -subscheme of S , is Zariski locally modelled on the critical locus $\text{Crit}(f)$ of a regular function $f : U \rightarrow \mathbb{A}^1$ on a smooth \mathbb{K} -scheme U .*

In real or complex symplectic geometry, it is easy to prove analogues of Corollary 5.21 using Darboux’s theorem or the Lagrangian neighbourhood theorem. However, these do not hold for algebraic symplectic manifolds, so it is not obvious how to prove Corollary 5.21 using classical techniques.

5.6. Proof of Theorem 5.18. We divide the proof into four steps.

Step 1. Locally represent \mathbf{X} by a minimal standard form cdga A . For all $k < 0$, first apply Theorem 4.1 to get a standard form cdga A over \mathbb{K} minimal at $p \in \text{Spec } H^0(A)$, and a Zariski open inclusion $\mathbf{f} : \mathbf{Spec } A \rightarrow \mathbf{X}$ with $\mathbf{f}(p) = x$. Then $\mathbf{f}^*(\tilde{\omega})$ is a closed 2-form of degree k on $\mathbf{Spec } A$. So Proposition 5.7(a) with $k\mathbf{f}^*(\tilde{\omega})$ in place of ω gives $\Phi \in A^{k+1}$ and $\phi \in (\Omega_A^1)^k$ such that $d\Phi = 0$ in A^{k+2} , and $d_{dR}\Phi + d\phi = 0$ in $(\Omega_A^1)^{k+1}$, and

$$\mathbf{f}^*(\tilde{\omega}) \sim \frac{1}{k}(d_{dR}\phi, 0, 0, \dots) =: (\omega^0, 0, 0, \dots) =: \omega.$$

By Proposition 2.12, $\Omega_A^1 \otimes_A H^0(A)$ is a complex of free $H^0(A)$ -modules

$$0 \longrightarrow V^k \xrightarrow{d^k} V^{k+1} \xrightarrow{d^{k+2}} \dots \xrightarrow{d^{-2}} V^{-1} \xrightarrow{d^{-1}} V^0 \longrightarrow 0,$$

with $d^i|_p = 0$ for $i = k, k + 1, \dots, -1$ by Definition 2.13, as A is minimal at p . Since ω is a k -shifted symplectic form on $\mathbf{Spec } A$, the morphism $\omega^0 : \mathbb{T}_A \rightarrow \Omega_A^1[k]$ in (5.3)

is a quasi-isomorphism. Hence $\omega^0 \otimes \text{id}_{H^0(A)} : \mathbb{T}_A \otimes_A H^0(A) \rightarrow \Omega_A^1 \otimes_A H^0(A)[k]$ is a quasi-isomorphism. That is, in the commutative diagram

$$(5.30) \quad \begin{array}{ccccccccccc} 0 & \rightarrow & (V^0)^* & \xrightarrow{\quad} & (V^{-1})^* & \xrightarrow{\quad} & \dots & \xrightarrow{\quad} & (V^{k+1})^* & \xrightarrow{\quad} & (V^k)^* & \rightarrow & 0 \\ & & \downarrow \omega^0 & & \downarrow \omega^0 & & & & \downarrow \omega^0 & & \downarrow \omega^0 & & \\ 0 & \rightarrow & V^k & \xrightarrow{\quad d^k \quad} & V^{k+1} & \xrightarrow{\quad d^{k+2} \quad} & \dots & \xrightarrow{\quad d^{-2} \quad} & V^{-1} & \xrightarrow{\quad d^{-1} \quad} & V^0 & \rightarrow & 0, \end{array}$$

the columns are a quasi-isomorphism. As the horizontal differentials $d^i, (d^i)^*$ are zero at p , so the vertical maps are isomorphisms at p , and hence are isomorphisms in a neighbourhood of p . Localising A at p if necessary, we may therefore assume the vertical maps $\omega^0 : (V^{k-i})^* \rightarrow V^i$ in (5.30) are isomorphisms.

Step 2. Proof of Theorem 5.18(i) when k is odd. For the next part of the proof we first suppose k is odd, so that $k = -2d - 1$ for $d = 0, 1, \dots$. Localising A at p if necessary, choose $x_1^0, \dots, x_{m_0}^0 \in A(0)$ such that $d_{dR}x_1^0, \dots, d_{dR}x_{m_0}^0$ form a basis of $\Omega_{A(0)}^1 \cong V^0$ over $A(0)$. Next, for $i = 1, \dots, d$, choose $x_1^{-i}, \dots, x_{m_i}^{-i}$ in A^{-i} such that $d_{dR}x_1^{-i}, \dots, d_{dR}x_{m_i}^{-i}$ form a basis for V^{-i} over $A(0)$. Then, for $i = 0, \dots, d$, choose $y_1^{i-2d-1}, \dots, y_{m_i}^{i-2d-1}$ in A^{i-2d-1} such that $d_{dR}y_1^{i-2d-1}, \dots, d_{dR}y_{m_i}^{i-2d-1}$ are the basis of V^{i-2d-1} over $A(0)$ which is dual to the basis $d_{dR}x_1^{-i}, \dots, d_{dR}x_{m_i}^{-i}$ for V^{-i} under the isomorphism $\omega^0 : (V^{i-2d-1})^* \rightarrow V^{-i}$.

Since A is a standard form cdga, these variables x_j^i for $i < 0$ and y_j^i generate A freely over $A(0)$ as a commutative graded algebra. That is, as a commutative graded algebra, A is freely generated over $A(0)$ by the graded variables (5.8), exactly as in Example 5.8. Then the condition on ω^0 sending the dual basis of $d_{dR}x_1^{-i}, \dots, d_{dR}x_{m_i}^{-i}$ to $d_{dR}y_1^{i-2d-1}, \dots, d_{dR}y_{m_i}^{i-2d-1}$ implies that

$$(5.31) \quad \omega^0 \otimes \text{id}_{A(0)} = \sum_{i=0}^d \sum_{j=1}^{m_i} d_{dR}x_j^{-i} d_{dR}y_j^{i-2d-1} \quad \text{in } (\Lambda^2 \Omega_A^1)^{-2d-1} \otimes_A A(0).$$

As above we have Φ, ϕ with $d_{dR}\Phi + d\phi = 0$ and $d_{dR}\phi = k\omega^0$. Using the coordinates x_j^i, y_j^i , we may write

$$\phi = \sum_{i=0}^d \sum_{j=1}^{m_i} [a_j^{i-2d-1} d_{dR}x_j^{-i} + b_j^{-i} d_{dR}y_j^{i-2d-1}],$$

with $a_j^l, b_j^l \in A^l$. For degree reasons, the b_j^l depend on $A(0)$ and the x_j^{-i} , but they do not involve the y_j^{i-2d-1} . By leaving ω^0 unchanged but replacing Φ, ϕ by

$$\tilde{\Phi} = \Phi - d \left[\sum_{i=0}^d \sum_{j=1}^{m_i} (-1)^i b_j^{-i} y_j^{i-2d-1} \right], \quad \tilde{\phi} = \phi - d_{dR} \left[\sum_{i=0}^d \sum_{j=1}^{m_i} (-1)^i b_j^{-i} y_j^{i-2d-1} \right]$$

and noting that $d_{dR}b_j^l$ includes no terms in $d_{dR}y_j^{i-2d-1}$, we may assume that $b_j^{-i} = 0$ for all i, j . Then $d_{dR}\phi = k\omega^0$ gives

$$(5.32) \quad k\omega^0 = \sum_{i=0}^d \sum_{j=1}^{m_i} d_{dR}a_j^{i-2d-1} d_{dR}x_j^{-i}.$$

Comparing (5.31) and (5.32) shows that

$$a_j^{i-2d-1} = ky_j^{i-2d-1} + \text{degree } \geq 2 \text{ terms in } x_{j'}^{-i'} \text{ for } i > 0 \text{ and } y_{j''}^{i''-2d-1}.$$

Thus the $\frac{1}{k}a_j^{i-2d-1}$ are alternative choices for the y_j^{i-2d-1} above. Hence, replacing y_j^{i-2d-1} by $\frac{1}{k}a_j^{i-2d-1}$ for all i, j , we see that ω^0 is given by (5.9), and

$$\phi = k \sum_{i=0}^d \sum_{j=1}^{m_i} y_j^{i-2d-1} d_{dR} x_j^{-i}.$$

Leaving ω^0 unchanged but replacing Φ, ϕ by

$$\tilde{\Phi} = \Phi - d \left[\sum_{i=0}^d \sum_{j=1}^{m_i} (-1)^i i x_j^{-i} y_j^{i-2d-1} \right], \quad \tilde{\phi} = \phi - d_{dR} \left[\sum_{i=0}^d \sum_{j=1}^{m_i} (-1)^i i x_j^{-i} y_j^{i-2d-1} \right],$$

we find that ϕ is given by (5.16).

Let us summarise our progress so far. In the case $k = -2d - 1$ for $d = 0, 1, \dots$, we have shown that we can identify A as a commutative graded algebra with the commutative graded algebra A in Example 5.8 generated over $A(0)$ by the graded variables (5.8), we have a k -shifted symplectic form $\omega = (\omega^0, 0, 0, \dots)$ on $\mathbf{Spec} A$ with ω^0 given by (5.9), we have $\Phi \in A^{k+1}$ and $\phi \in (\Omega_A^1)^k$ such that $d\Phi = 0$, $d_{dR}\Phi + d\phi = 0$, $d_{dR}\phi = k\omega^0$, and ϕ is as in (5.16).

It remains to show that Φ satisfies the classical master equation (5.10), and the differential d on A is given by (5.12). Expanding $d_{dR}\Phi + d\phi = 0$ using (5.16), comparing coefficients of $d_{dR}y_j^{i-2d-1}, d_{dR}x_j^{-i}$, and rearranging gives

$$(5.33) \quad (2d + 1)dx_j^{-i} = \frac{\partial}{\partial y_j^{i-2d-1}} \left[\Phi - \sum_{i'=0}^d \sum_{j'=1}^{m_{i'}} (i' - 2d - 1)y_{j'}^{i'-2d-1} dx_{j'}^{-i'} - i' x_{j'}^{-i'} dy_{j'}^{i'-2d-1} \right],$$

$$(5.34) \quad (2d + 1)dy_j^{i-2d-1} = \frac{\partial}{\partial x_j^{-i}} \left[\Phi - \sum_{i'=0}^d \sum_{j'=1}^{m_{i'}} (i' - 2d - 1)y_{j'}^{i'-2d-1} dx_{j'}^{-i'} - i' x_{j'}^{-i'} dy_{j'}^{i'-2d-1} \right].$$

Write F for the function in brackets $[\dots]$ on the right-hand sides of (5.33)–(5.34). Using (5.33)–(5.34) to substitute for $dx_{j'}^{-i'}, dy_{j'}^{i'-2d-1}$ gives

$$\begin{aligned} F &= \Phi - \frac{1}{2d + 1} \sum_{i'=0}^d \sum_{j'=1}^{m_{i'}} (i' - 2d - 1)y_{j'}^{i'-2d-1} \frac{\partial F}{\partial y_{j'}^{i'-2d-1}} - i' x_{j'}^{-i'} \frac{\partial F}{\partial x_{j'}^{-i'}} \\ &= \Phi - \frac{1}{2d + 1} \cdot (-2d) F, \end{aligned}$$

where the second line holds as F has degree $-2d$. Therefore $F = (2d + 1)\Phi$, so (5.33)–(5.34) prove (5.12). Then expanding $d\Phi = 0$ using (5.12) yields (5.10). This completes the proof of Theorem 5.18(i) when k is odd.

Step 3. Proof of Theorem 5.18(i) and (ii) when k is even. The remaining cases in Theorem 5.18(i) and (ii) are fairly similar, so we explain the differences with the case of k odd. If k is divisible by 4, so that $k = -4d$ for $d = 1, 2, \dots$, choose $x_1^0, \dots, x_{m_0}^0 \in A^0 = A(0)$ and $x_1^{-i}, \dots, x_{m_i}^{-i} \in A^{-i}$ for $i = 1, \dots, 2d - 1$ as above such that $d_{dR}x_1^{-i}, \dots, d_{dR}x_{m_i}^{-i}$ form a basis for V^{-i} over $A(0)$ for $i = 0, \dots, 2d - 1$, and for $i = 0, \dots, 2d - 1$, choose $y_1^{i-4d}, \dots, y_{m_i}^{i-4d}$ in A^{i-4d} as above such that $d_{dR}y_1^{i-4d}, \dots, d_{dR}y_{m_i}^{i-4d}$ are the basis of V^{i-4d} over $A(0)$ which is dual to the basis $d_{dR}x_1^{-i}, \dots, d_{dR}x_{m_i}^{-i}$ for V^{-i} under the isomorphism $\omega^0: (V^{i-4d})^* \rightarrow V^{-i}$.

We have chosen the variables x_j^{-i}, y_j^{i-4d} except in the middle degree $-2d$. The isomorphism $\omega^0 : (V^{-2d})^* \rightarrow V^{-2d}$ is antisymmetric, so we can choose $x_1^{-2d}, \dots, x_{m_{2d}}^{-2d}, y_1^{-2d}, \dots, y_{m_{2d}}^{-2d} \in A^{-2d}$ with $d_{dR}x_1^{-2d}, \dots, d_{dR}x_{m_{2d}}^{-2d}, d_{dR}y_1^{-2d}, \dots, d_{dR}y_{m_{2d}}^{-2d}$ a standard symplectic basis of V^{-2d} over $A(0)$. The rest of the proof goes through with only cosmetic changes, following Example 5.9 rather than Example 5.8, to prove Theorem 5.18(i) when k is divisible by 4.

Next suppose that $k \equiv 2 \pmod 4$, so that $k = -4d - 2$ for $d = 0, 1, \dots$. Choose $x_1^0, \dots, x_{m_0}^0 \in A^0 = A(0)$ and $x_1^{-i}, \dots, x_{m_i}^{-i} \in A^{-i}$ for $i = 1, \dots, 2d$ as above such that $d_{dR}x_1^{-i}, \dots, d_{dR}x_{m_i}^{-i}$ form a basis for V^{-i} over $A(0)$ for $i = 0, \dots, 2d$, and for $i = 0, \dots, 2d$, choose $y_1^{i-4d-2}, \dots, y_{m_i}^{i-4d-2}$ in A^{i-4d-2} as above such that $d_{dR}y_1^{i-4d-2}, \dots, d_{dR}y_{m_i}^{i-4d-2}$ are the basis of V^{i-4d-2} over $A(0)$ dual to the basis $d_{dR}x_1^{-i}, \dots, d_{dR}x_{m_i}^{-i}$ for V^{-i} under the isomorphism $\omega^0 : (V^{i-4d})^* \rightarrow V^{-i}$.

We have chosen the x_j^{-i}, y_j^{i-4d-2} except in the middle degree $-2d - 1$. In this case the isomorphism $\omega^0 : (V^{-2d-1})^* \rightarrow V^{-2d-1}$ is symmetric, that is, ω^0 is a nondegenerate quadratic form on $(V^{-2d-1})^*$. Now in general, nondegenerate quadratic forms cannot be trivialised Zariski locally, but they can at least be diagonalised. That is, in general we cannot choose a basis of V^{-2d-1} over $A(0)$ in which ω^0 is represented by a constant matrix, but we can choose one in which it is represented by a diagonal matrix.

Thus, we may choose $z_1^{-2d-1}, \dots, z_{m_{2d+1}}^{-2d-1}$ in A^{-2d-1} such that $d_{dR}z_1^{-2d-1}, \dots, d_{dR}z_{m_{2d+1}}^{-2d-1}$ form a basis for V^{-2d-1} over $A(0)$, and there exist invertible elements $q_1, \dots, q_{m_{2d+1}}$ in $A(0)$ such that $\omega^0 : (V^{-2d-1})^* \rightarrow V^{-2d-1}$ has matrix $\text{diag}(q_1, \dots, q_{m_{2d+1}})$ with respect to $d_{dR}z_1^{-2d-1}, \dots, d_{dR}z_{m_{2d+1}}^{-2d-1}$ and its dual basis. Then the standard form cdga A is generated over $A(0)$ by the variables (5.20), so as a commutative graded algebra A is as in Examples 5.10 and 5.12. Also the analogue of (5.25) is

$$\omega^0 \otimes \text{id}_{A(0)} = \sum_{i=0}^{2d} \sum_{j=1}^{m_i} d_{dR}x_j^{-i} d_{dR}y_j^{i-4d-2} + \sum_{j=1}^{m_{2d+1}} q_j d_{dR}z_j^{-2d-1} d_{dR}z_j^{-2d-1}$$

in $(\Lambda^2 \Omega_A^1)^{-4d-2} \otimes_A A(0)$, which is a truncation of (5.25). The rest of the proof then works as above, but with extra terms in z_j^{-2d-1}, q_j , and $\frac{\partial q_j}{\partial x_j^0}$ inserted as in Example 5.12. This proves Theorem 5.18(ii).

Step 4. Proof of Theorem 5.18(iii). Finally, for Theorem 5.18(iii) we first apply part (ii) to get $\check{A}, \check{\omega}$ in weak Darboux form, as in Example 5.12, with \check{A} minimal at $\check{p} \in \text{Spec } H^0(\check{A})$, and a Zariski open inclusion $\check{f} : \mathbf{Spec } \check{A} \rightarrow \mathbf{X}$ with $\check{f}(p) = x$. Then we have invertible elements $q_1, \dots, q_{m_{2d+1}}$ in $\check{A}(0)$. Define $A(0) = \check{A}(0)[q_1^{1/2}, \dots, q_{m_{2d+1}}^{1/2}]$ to be the \mathbb{K} -algebra obtained by adjoining square roots of $q_1, \dots, q_{m_{2d+1}}$ to $\check{A}(0)$. Then $A(0)$ is a smooth \mathbb{K} -algebra with inclusion morphism $i_{A(0)} : \check{A}(0) \rightarrow A(0)$, such that $\text{Spec } i_{A(0)} : \text{Spec } A(0) \rightarrow \text{Spec } \check{A}(0)$ is an étale cover of degree $2^{m_{2d+1}}$. Set $A = \check{A} \otimes_{\check{A}(0)} A(0)$, with inclusion morphism $i_A : \check{A} \rightarrow A$. Then $\mathbf{Spec } i_A : \mathbf{Spec } A \rightarrow \mathbf{Spec } \check{A}$ is an étale cover of derived \mathbb{K} -schemes of degree $2^{m_{2d+1}}$. Let $p \in \mathbf{Spec } A$ be one of the $2^{m_{2d+1}}$ preimages of \check{p} , and set $f = \check{f} \circ \mathbf{Spec } i_A$. Then $f : \mathbf{Spec } A \rightarrow \mathbf{X}$ is étale, with $f(p) = x$, as we want.

In \check{A} we have variables $\check{x}_j^{-i}, \check{y}_j^{i-4d-2}, \check{z}_j^{-2d-1}$, and a Hamiltonian $\check{\Phi}$. Set

$$x_j^{-i} = i_A(\check{x}_j^{-i}), y_j^{i-4d-2} = i_A(\check{y}_j^{i-4d-2}), z_j^{-2d-1} = q_j^{1/2} \cdot i_A(\check{z}_j^{-2d-1}), \Phi = i_A(\check{\Phi})$$

in A , for all i, j . As in Remark 5.13, changing coordinates from \check{z}_j^{-2d-1} to $z_j^{-2d-1} = q_j^{1/2} \cdot \check{z}_j^{-2d-1}$ has the effect of setting $q_j = 1$, and it transforms the weak Darboux form in Example 5.12 to the strong Darboux form in Example 5.10. One can check that these $x_j^{-i}, y_j^{i-4d-2}, z_j^{-2d-1}, \Phi$ satisfy the conditions of Example 5.10, and Theorem 5.18(iii) follows.

5.7. Comparing Darboux form presentations on overlaps. Let $(\mathbf{X}, \tilde{\omega})$ be a k -shifted symplectic derived \mathbb{K} -scheme for $k < 0$. Then for $x \in \mathbf{X}$, Theorem 5.18 gives (Zariski or étale local) presentations $\mathbf{f} : \mathbf{Spec} A \rightarrow \mathbf{X}$ near x with $\mathbf{f}^*(\tilde{\omega}) \sim \omega = (\omega^0, 0, \dots)$ for (A, ω) of (weak or strong) Darboux form, so we have a Hamiltonian $\Phi \in A^{k+1}$, and $\phi \in (\Omega_A^1)^k$ with $d_{dR}\Phi + d\phi = 0$ and $d_{dR}\phi = k\omega^0$ as in Proposition 5.7(a). In the case $k = -1$ we also suppose that $\Phi|_{\mathbf{Spec} H^0(A)^{\text{red}}} = 0$, as in Proposition 5.7(b). We think of $A, \omega, \mathbf{f}, \Phi, \phi$ as like coordinates on \mathbf{X} near x which write $\mathbf{X}, \tilde{\omega}$ in a nice way.

It is often important in geometric problems to compare different choices of coordinates on the overlap of their domains. So suppose $A, \omega, \mathbf{f}, \Phi, \phi$ and $B, \tilde{\omega}, \mathbf{g}, \check{\Phi}, \check{\phi}$ are two choices as above, and $p \in \mathbf{Spec} H^0(A), q \in \mathbf{Spec} H^0(B)$ with $\mathbf{f}(p) = \mathbf{g}(q) = x$ in \mathbf{X} . We would like to compare the presentations $A, \omega, \mathbf{f}, \Phi, \phi$ and $B, \tilde{\omega}, \mathbf{g}, \check{\Phi}, \check{\phi}$ for \mathbf{X} near x .

Here is a general method for doing this:

- (i) Apply Theorem 4.2 to $\mathbf{f} : \mathbf{Spec} A \rightarrow \mathbf{X}$ and $\mathbf{g} : \mathbf{Spec} B \rightarrow \mathbf{X}$ at $p \in \mathbf{Spec} H^0(A)$ and $q \in \mathbf{Spec} H^0(B)$. This gives a standard form cdga C over \mathbb{K} minimal at $r \in \mathbf{Spec} H^0(C)$, and cdga morphisms $\alpha : A \rightarrow C, \beta : B \rightarrow C$ with $\mathbf{Spec} \alpha : r \mapsto p, \mathbf{Spec} \beta : r \mapsto q$, with $\mathbf{f} \circ \mathbf{Spec} \alpha \simeq \mathbf{g} \circ \mathbf{Spec} \beta$ as morphisms $\mathbf{Spec} C \rightarrow \mathbf{X}$ in $\mathbf{dSch}_{\mathbb{K}}$.

Here α, β are Zariski local inclusions if \mathbf{f}, \mathbf{g} are (if A, \dots, ϕ and $B, \dots, \check{\phi}$ come from Theorem 5.18(i) or (ii)) and are étale if \mathbf{f}, \mathbf{g} are (if A, \dots, ϕ and $B, \dots, \check{\phi}$ come from Theorem 5.18(iii)).

- (ii) As α, β are Zariski local inclusions or étale, the pushforwards $\alpha_*(\omega), \beta_*(\tilde{\omega})$ are k -shifted symplectic structures on $\mathbf{Spec} C$, which are equivalent as

$$\begin{aligned} \alpha_*(\omega) &\simeq (\mathbf{Spec} \alpha)^* \circ \mathbf{f}^*(\tilde{\omega}) \simeq (\mathbf{f} \circ \mathbf{Spec} \alpha)^*(\tilde{\omega}) \\ &\simeq (\mathbf{g} \circ \mathbf{Spec} \beta)^*(\tilde{\omega}) \simeq (\mathbf{Spec} \beta)^* \circ \mathbf{g}^*(\tilde{\omega}) \simeq \beta_*(\tilde{\omega}), \end{aligned}$$

using $\mathbf{f}^*(\tilde{\omega}) \sim \omega, \mathbf{g}^*(\tilde{\omega}) \sim \tilde{\omega}$, and $\mathbf{f} \circ \mathbf{Spec} \alpha \simeq \mathbf{g} \circ \mathbf{Spec} \beta$. Also $\alpha(\Phi), \alpha_*(\phi)$ and $\beta(\check{\Phi}), \beta_*(\check{\phi})$ satisfy Proposition 5.7(a) for $C, \alpha_*(\omega)$ and $C, \beta_*(\tilde{\omega})$, respectively, as Φ, ϕ and $\check{\Phi}, \check{\phi}$ do for A, ω and $B, \tilde{\omega}$.

Noting that $\alpha(\Phi)|_{\mathbf{Spec} H^0(C)^{\text{red}}} = 0 = \beta(\check{\Phi})|_{\mathbf{Spec} H^0(C)^{\text{red}}}$ when $k = -1$, Proposition 5.7(c) applies, yielding $\Psi \in C^k$ and $\psi \in (\Omega_C^1)^{k-1}$ with

$$(5.35) \quad \begin{aligned} \alpha(\Phi) - \beta(\check{\Phi}) &= d\Psi && \text{in } C^{k+1}, \text{ and} \\ \alpha_*(\phi) - \beta_*(\check{\phi}) &= d_{dR}\Psi + d\psi && \text{in } (\Omega_C^1)^k. \end{aligned}$$

The data $C, \alpha, \beta, \Psi, \psi$ compare the Darboux presentations $A, \omega, \mathbf{f}, \Phi, \phi$ and $B, \tilde{\omega}, \mathbf{g}, \check{\Phi}, \check{\phi}$ for \mathbf{X} near x .

We work out this comparison more explicitly in the case $k = -1$:

Example 5.22. Let $(X, \tilde{\omega})$ be a -1 -shifted symplectic derived \mathbb{K} -scheme, and suppose $A, \omega, \mathbf{f}, \Phi, \phi$ and $B, \tilde{\omega}, \mathbf{g}, \check{\Phi}, \check{\phi}$ and p, q with $\mathbf{f}(p) = \mathbf{g}(q)$ are as above. Follow (i)–(ii) above to get $C, r, \alpha, \beta, \Psi, \psi$ satisfying (5.35). Set $U = \text{Spec } A(0)$, $V = \text{Spec } B(0)$, and $W = \text{Spec } C(0)$, so that U, V, W are smooth \mathbb{K} -schemes with $p \in U, q \in V, r \in W$, and write $a = \text{Spec } \alpha(0) : W \rightarrow U$ and $b = \text{Spec } \beta(0) : W \rightarrow V$. Then $a(r) = p$ and $b(r) = q$.

Since A, ω, Φ, ϕ are as in Examples 5.8 and 5.15, we have étale coordinates $(x_1^0, \dots, x_{m_0}^0) : U \rightarrow \mathbb{A}^{m_0}$ on U such that $A = A(0)[y_1^{-1}, \dots, y_{m_0}^{-1}]$ with $dx_i^0 = 0, dy_i^{-1} = \frac{\partial \Phi}{\partial x_i^0}$ for all i , and

$$(5.36) \quad \begin{aligned} \omega^0 &= d_{dR}x_1^0 d_{dR}y_1^{-1} + \dots + d_{dR}x_{m_0}^0 d_{dR}y_{m_0}^{-1}, \\ \phi &= -y_1^{-1} d_{dR}x_1^0 - \dots - y_{m_0}^{-1} d_{dR}x_{m_0}^0. \end{aligned}$$

Similarly, for $B, \tilde{\omega}, \check{\Phi}, \check{\phi}$ we have étale coordinates $(\tilde{x}_1^0, \dots, \tilde{x}_{\tilde{m}_0}^0) : V \rightarrow \mathbb{A}^{\tilde{m}_0}$ on V such that $B = B(0)[\tilde{y}_1^{-1}, \dots, \tilde{y}_{\tilde{m}_0}^{-1}]$ with $d\tilde{x}_i^0 = 0, d\tilde{y}_i^{-1} = \frac{\partial \check{\Phi}}{\partial \tilde{x}_i^0}$ for all i , and $\tilde{\omega}^0, \check{\phi}$ are given by the analogue of (5.36).

Localising C at r if necessary, choose étale coordinates $(z_1^0, \dots, z_m^0) : W \rightarrow \mathbb{A}^m$, for $m = \dim U$. Since C is minimal at r and $\mathbf{Spec } C$ is -1 -shifted symplectic, we may choose $e_1^{-1}, \dots, e_m^{-1}$ in C^{-1} such that $d_{dR}e_1^{-1}, \dots, d_{dR}e_m^{-1}$ form a basis for V^{-1} over $C(0)$ in Proposition 2.12 for the same $m = \dim U$, and then as a commutative graded algebra $C = C(0)[e_1^{-1}, \dots, e_m^{-1}]$ is freely generated over $C(0)$ by degree -1 variables $e_1^{-1}, \dots, e_m^{-1}$.

For all $i = 1, \dots, m_0, \tilde{i} = 1, \dots, \tilde{m}_0$, and $j, j', j'' = 1, \dots, m$, define functions $I_j, J_{ij}, K_{ij}, L_j, M_{jj'}, N_{jj'j''}$ in $C(0)$ with $N_{jj'j''} = -N_{j'jj''}$ by

$$\begin{aligned} de_j^{-1} &= I_j, \quad a^*(y_i^{-1}) = \sum_{j=1}^m J_{ij} e_j^{-1}, \quad b^*(\tilde{y}_{\tilde{i}}^{-1}) = \sum_{j=1}^m K_{i\tilde{j}} e_j^{-1}, \quad \Psi = \sum_{j=1}^m L_j e_j^{-1}, \\ \text{and } \psi &= \sum_{j,j'=1}^m M_{jj'} e_j^{-1} d_{dR}e_{j'}^{-1} + \sum_{j,j',j''=1}^m N_{jj'j''} e_j^{-1} e_{j'}^{-1} d_{dR}z_{j''}^0. \end{aligned}$$

Then $H^0(C) = C(0)/dC^{-1} = C(0)/(I_1, \dots, I_m)$, for $(I_1, \dots, I_m) \subset C(0)$ the ideal generated by I_1, \dots, I_m . Since $\alpha : A \rightarrow C$ induces $\alpha : H^0(A) \rightarrow H^0(C)$ and both are Zariski open inclusions, with $H^0(A) = A(0)/(d_{dR}\Phi)$ for $(d_{dR}\Phi) \subset A(0)$ the ideal generated by $d_{dR}\Phi$, and similarly for β , we have

$$(5.37) \quad (I_1, \dots, I_m) = (a^*(d_{dR}\Phi)) = (b^*(d_{dR}\check{\Phi})).$$

Equation (5.35) becomes the equations

$$(5.38) \quad \begin{aligned} a^*(\Phi) - b^*(\check{\Phi}) &= \sum_{j=1}^m I_j L_j, \quad 0 = L_j + \sum_{j'=1}^m I_{j'} M_{jj'}, \\ \sum_{i=1}^{m_0} J_{ij} \frac{\partial \alpha(x_i^0)}{\partial z_{j'}^0} - \sum_{\tilde{i}=1}^{\tilde{m}_0} K_{i\tilde{j}} \frac{\partial \beta(\tilde{x}_{\tilde{i}}^0)}{\partial z_{j'}^0} &= -\frac{\partial L_j}{\partial z_{j'}^0} + \sum_{j''=1}^m M_{jj''} \frac{\partial I_{j''}}{\partial z_{j'}^0} + 2 \sum_{j''=1}^m I_{j''} N_{jj''j'}, \end{aligned}$$

where the second and third equations are the coefficients of $d_{dR}e_j^{-1}, e_j^{-1} d_{dR}z_{j'}^0$ in the second equation of (5.35).

The first two equations of (5.38) imply that

$$(5.39) \quad a^*(\Phi) - b^*(\check{\Phi}) \in (I_1, \dots, I_m)^2 = (a^*(d_{dR}\Phi))^2 = (b^*(d_{dR}\check{\Phi}))^2 \subset C(0),$$

by (5.37). This will be important in the proof of Theorem 6.6 in §6.3.

6. -1-SHIFTED SYMPLECTIC DERIVED SCHEMES AND D-CRITICAL LOCI

In §6.1 we introduce *d-critical loci* from Joyce [16]. Our second main result Theorem 6.6 is stated and discussed in §6.2, and proved in §6.3.

6.1. Background material on d-critical loci. Here are some of the main definitions and results on d-critical loci, from Joyce [16, Thms. 2.1, 2.20, 2.28 and Defs, 2.5, 2.18, 2.31]. In fact [16] develops two versions of the theory, *algebraic d-critical loci* on \mathbb{K} -schemes and *complex analytic d-critical loci* on complex analytic spaces, but we discuss only the former.

Theorem 6.1. *Let X be a \mathbb{K} -scheme. Then there exists a sheaf \mathcal{S}_X of \mathbb{K} -vector spaces on X , unique up to canonical isomorphism, which is uniquely characterised by the following two properties:*

- (i) *Suppose $R \subseteq X$ is Zariski open, U is a smooth \mathbb{K} -scheme, and $i : R \hookrightarrow U$ is a closed embedding. Then we have an exact sequence of sheaves of \mathbb{K} -vector spaces on R ,*

$$0 \longrightarrow I_{R,U} \longrightarrow i^{-1}(\mathcal{O}_U) \xrightarrow{i^\#} \mathcal{O}_X|_R \longrightarrow 0,$$

where $\mathcal{O}_X, \mathcal{O}_U$ are the sheaves of regular functions on X, U , and $i^\#$ is the morphism of sheaves of \mathbb{K} -algebras on R induced by i .

There is an exact sequence of sheaves of \mathbb{K} -vector spaces on R ,

$$0 \longrightarrow \mathcal{S}_X|_R \xrightarrow{\iota_{R,U}} \frac{i^{-1}(\mathcal{O}_U)}{I_{R,U}^2} \xrightarrow{d} \frac{i^{-1}(T^*U)}{I_{R,U} \cdot i^{-1}(T^*U)},$$

where d maps $f + I_{R,U}^2 \mapsto df + I_{R,U} \cdot i^{-1}(T^*U)$.

- (ii) *Let $R \subseteq S \subseteq X$ be Zariski open, let U, V be smooth \mathbb{K} -schemes, let $i : R \hookrightarrow U$, $j : S \hookrightarrow V$ be closed embeddings, and let $\Psi : U \rightarrow V$ be a morphism with $\Psi \circ i = j|_R : R \rightarrow V$. Then the following diagram of sheaves on R commutes.*

$$(6.1) \quad \begin{array}{ccccc} 0 \longrightarrow \mathcal{S}_X|_R & \xrightarrow{\iota_{S,V}|_R} & \frac{j^{-1}(\mathcal{O}_V)}{I_{S,V}^2}|_R & \xrightarrow{d} & \frac{j^{-1}(T^*V)}{I_{S,V} \cdot j^{-1}(T^*V)}|_R \\ & \downarrow \text{id} & \downarrow i^{-1}(\Psi^\#) & & \downarrow i^{-1}(d\Psi) \\ 0 \longrightarrow \mathcal{S}_X|_R & \xrightarrow{\iota_{R,U}} & \frac{i^{-1}(\mathcal{O}_U)}{I_{R,U}^2} & \xrightarrow{d} & \frac{i^{-1}(T^*U)}{I_{R,U} \cdot i^{-1}(T^*U)} \end{array}$$

Here $\Psi : U \rightarrow V$ induces $\Psi^\# : \Psi^{-1}(\mathcal{O}_V) \rightarrow \mathcal{O}_U$ on U , so we have

$$(6.2) \quad i^{-1}(\Psi^\#) : j^{-1}(\mathcal{O}_V)|_R = i^{-1} \circ \Psi^{-1}(\mathcal{O}_V) \longrightarrow i^{-1}(\mathcal{O}_U),$$

a morphism of sheaves of \mathbb{K} -algebras on R . As $\Psi \circ i = j|_R$, equation (6.2) maps $I_{S,V}|_R \rightarrow I_{R,U}$, and so maps $I_{S,V}^2|_R \rightarrow I_{R,U}^2$. Thus (6.2) induces the morphism in the second column of (6.1). Similarly, $d\Psi : \Psi^{-1}(T^*V) \rightarrow T^*U$ induces the third column of (6.1).

There is a natural decomposition $\mathcal{S}_X = \mathcal{S}_X^0 \oplus \mathbb{K}_X$, where \mathbb{K}_X is the constant sheaf on X with fibre \mathbb{K} , and $\mathcal{S}_X^0 \subset \mathcal{S}_X$ is the kernel of the composition

$$\mathcal{S}_X \longrightarrow \mathcal{O}_X \xrightarrow{i_X^\#} \mathcal{O}_{X^{\text{red}}},$$

with X^{red} the reduced \mathbb{K} -subscheme of X and $i_X : X^{\text{red}} \hookrightarrow X$ the inclusion.

Definition 6.2. An algebraic d -critical locus over a field \mathbb{K} is a pair (X, s) , where X is a \mathbb{K} -scheme and $s \in H^0(\mathcal{S}_X^0)$ for \mathcal{S}_X^0 as in Theorem 6.1, such that for each $x \in X$, there exists a Zariski open neighbourhood R of x in X , a smooth \mathbb{K} -scheme U , a regular function $f : U \rightarrow \mathbb{A}^1 = \mathbb{K}$, and a closed embedding $i : R \hookrightarrow U$, such that $i(R) = \text{Crit}(f)$ as \mathbb{K} -subschemes of U , and $\iota_{R,U}(s|_R) = i^{-1}(f) + I_{R,U}^2$. We call the quadruple (R, U, f, i) a *critical chart* on (X, s) .

Let (X, s) be an algebraic d -critical locus, and let (R, U, f, i) be a critical chart on (X, s) . Let $U' \subseteq U$ be Zariski open, and set $R' = i^{-1}(U') \subseteq R$, $i' = i|_{R'} : R' \hookrightarrow U'$, and set $f' = f|_{U'}$. Then (R', U', f', i') is a critical chart on (X, s) , and we call it a *subchart* of (R, U, f, i) . As a shorthand we write $(R', U', f', i') \subseteq (R, U, f, i)$.

Let $(R, U, f, i), (S, V, g, j)$ be critical charts on (X, s) , with $R \subseteq S \subseteq X$. An *embedding* of (R, U, f, i) in (S, V, g, j) is a locally closed embedding $\Psi : U \hookrightarrow V$ such that $\Psi \circ i = j|_R$ and $f = g \circ \Psi$. As a shorthand we write $\Psi : (R, U, f, i) \hookrightarrow (S, V, g, j)$. If $\Psi : (R, U, f, i) \hookrightarrow (S, V, g, j)$ and $\Xi : (S, V, g, j) \hookrightarrow (T, W, h, k)$ are embeddings, then $\Xi \circ \Psi : (R, U, f, i) \hookrightarrow (T, W, h, k)$ is also an embedding.

Theorem 6.3. Let (X, s) be an algebraic d -critical locus, and let $(R, U, f, i), (S, V, g, j)$ be critical charts on (X, s) . Then for each $x \in R \cap S \subseteq X$ there exist subcharts $(R', U', f', i') \subseteq (R, U, f, i), (S', V', g', j') \subseteq (S, V, g, j)$ with $x \in R' \cap S' \subseteq X$, a critical chart (T, W, h, k) on (X, s) , and embeddings $\Psi : (R', U', f', i') \hookrightarrow (T, W, h, k), \Xi : (S', V', g', j') \hookrightarrow (T, W, h, k)$.

Theorem 6.4. Let (X, s) be an algebraic d -critical locus, and let $X^{\text{red}} \subseteq X$ be the associated reduced \mathbb{K} -scheme. Then there exists a line bundle $K_{X,s}$ on X^{red} which we call the “canonical bundle” of (X, s) , which is natural up to canonical isomorphism, and is characterised by the following properties.

- (i) If (R, U, f, i) is a critical chart on (X, s) , there is a natural isomorphism

$$\iota_{R,U,f,i} : K_{X,s}|_{R^{\text{red}}} \longrightarrow i^*(K_U^{\otimes 2})|_{R^{\text{red}}},$$

where $K_U = \Lambda^{\dim U} T^*U$ is the canonical bundle of U in the usual sense.

- (ii) Let $\Psi : (R, U, f, i) \hookrightarrow (S, V, g, j)$ be an embedding of critical charts on (X, s) . Then [16, Def. 2.26] defines an isomorphism of line bundles

$$J_\Psi : i^*(K_U^{\otimes 2})|_{R^{\text{red}}} \xrightarrow{\cong} j^*(K_V^{\otimes 2})|_{R^{\text{red}}}$$

on R^{red} , and we must have

$$\iota_{S,V,g,j}|_{R^{\text{red}}} = J_\Psi \circ \iota_{R,U,f,i} : K_{X,s}|_{R^{\text{red}}} \longrightarrow j^*(K_V^{\otimes 2})|_{R^{\text{red}}}.$$

Definition 6.5. Let (X, s) be an algebraic d -critical locus, and $K_{X,s}$ its canonical bundle from Theorem 6.4. An *orientation* on (X, s) is a choice of square root line bundle $K_{X,s}^{1/2}$ for $K_{X,s}$ on X^{red} . That is, an orientation is a line bundle L on X^{red} , together with an isomorphism $L^{\otimes 2} = L \otimes L \cong K_{X,s}$. A d -critical locus with an orientation will be called an *oriented d -critical locus*.

6.2. The second main result, and applications. Here is the second main result of this paper, which will be proved in §6.3:

Theorem 6.6. *Suppose $(\mathbf{X}, \tilde{\omega})$ is a -1 -shifted symplectic derived \mathbb{K} -scheme, and let $X = t_0(\mathbf{X})$ be the associated classical \mathbb{K} -scheme of \mathbf{X} . Then X extends uniquely to an algebraic d -critical locus (X, s) , with the property that whenever $(\mathbf{Spec} A, \omega)$ is a -1 -shifted symplectic derived \mathbb{K} -scheme in Darboux form with Hamiltonian $\Phi \in A(0)$, as in Examples 5.8 and 5.15, and $\mathbf{f} : \mathbf{Spec} A \rightarrow \mathbf{X}$ is an equivalence in $\mathbf{dSch}_{\mathbb{K}}$ with a Zariski open derived \mathbb{K} -subscheme $\mathbf{R} \subseteq \mathbf{X}$ with $\mathbf{f}^*(\tilde{\omega}) \sim \omega$, writing $U = \mathbf{Spec} A(0)$, $R = t_0(\mathbf{R})$, $f = t_0(\mathbf{f})$ so that $\Phi : U \rightarrow \mathbb{A}^1$ is regular and $f : \text{Crit}(\Phi) \rightarrow R$ is an isomorphism, for $\text{Crit}(\Phi) \subseteq U$ the classical critical locus of Φ , then (R, U, Φ, f^{-1}) is a critical chart on (X, s) .*

The canonical bundle $K_{X,s}$ from Theorem 6.4 is naturally isomorphic to the determinant line bundle $\det(\mathbb{L}_{\mathbf{X}})|_{X^{\text{red}}}$ of the cotangent complex $\mathbb{L}_{\mathbf{X}}$ of \mathbf{X} .

We can think of Theorem 6.6 as defining a *truncation functor*

$$(6.3) \quad \begin{aligned} F : \{ &\text{category of } -1\text{-shifted symplectic derived } \mathbb{K}\text{-schemes } (\mathbf{X}, \omega) \} \\ &\longrightarrow \{ \text{category of algebraic } d\text{-critical loci } (X, s) \text{ over } \mathbb{K} \}, \end{aligned}$$

where the morphisms $\mathbf{f} : (\mathbf{X}, \omega) \rightarrow (\mathbf{Y}, \omega')$ in the first line are (homotopy classes of) étale maps $\mathbf{f} : \mathbf{X} \rightarrow \mathbf{Y}$ with $\mathbf{f}^*(\omega') \sim \omega$, and the morphisms $f : (X, s) \rightarrow (Y, t)$ in the second line are étale maps $f : X \rightarrow Y$ with $f^*(t) = s$.

In [16, Ex. 2.17] we give an example of -1 -shifted symplectic derived schemes $(\mathbf{X}, \omega), (\mathbf{Y}, \omega')$, both global critical loci, such that \mathbf{X}, \mathbf{Y} are not equivalent as derived \mathbb{K} -schemes, but their truncations $F(\mathbf{X}, \omega), F(\mathbf{Y}, \omega')$ are isomorphic as algebraic d -critical loci. Thus, the functor F in (6.3) is not full.

Suppose Y is a Calabi–Yau 3-fold over \mathbb{K} and \mathcal{M} a classical moduli \mathbb{K} -scheme of simple coherent sheaves in $\text{coh}(Y)$. Then Thomas [26] defined a natural *perfect obstruction theory* $\phi : \mathcal{E}^\bullet \rightarrow \mathbb{L}_{\mathcal{M}}$ on \mathcal{M} in the sense of Behrend and Fantechi [2], and Behrend [1] showed that $\phi : \mathcal{E}^\bullet \rightarrow \mathbb{L}_{\mathcal{M}}$ can be made into a *symmetric obstruction theory*. More generally, if \mathcal{M} is a moduli \mathbb{K} -scheme of simple complexes of coherent sheaves in $D^b \text{coh}(Y)$, then Huybrechts and Thomas [15] defined a natural symmetric obstruction theory on \mathcal{M} .

Now in derived algebraic geometry $\mathcal{M} = t_0(\mathcal{M})$ for \mathcal{M} the corresponding derived moduli \mathbb{K} -scheme, and the obstruction theory $\phi : \mathcal{E}^\bullet \rightarrow \mathbb{L}_{\mathcal{M}}$ from [15, 26] is $\mathbb{L}_{t_0} : \mathbb{L}_{\mathcal{M}}|_{\mathcal{M}} \rightarrow \mathbb{L}_{\mathcal{M}}$. Pantev et al. [24, §2.1] prove \mathcal{M} has a -1 -shifted symplectic structure ω , and the symmetric structure on $\phi : \mathcal{E}^\bullet \rightarrow \mathbb{L}_{\mathcal{M}}$ from [1] is $\omega^0|_{\mathcal{M}}$. So as for Corollary 5.19, Theorem 6.6 implies:

Corollary 6.7. *Suppose Y is a Calabi–Yau 3-fold over \mathbb{K} , and suppose \mathcal{M} is a classical moduli \mathbb{K} -scheme of simple coherent sheaves in $\text{coh}(Y)$, or simple complexes of coherent sheaves in $D^b \text{coh}(Y)$, with perfect obstruction theory $\phi : \mathcal{E}^\bullet \rightarrow \mathbb{L}_{\mathcal{M}}$ as in Thomas [26] or Huybrechts and Thomas [15]. Then \mathcal{M} extends naturally to an algebraic d -critical locus (\mathcal{M}, s) . The canonical bundle $K_{\mathcal{M},s}$ from Theorem 6.4 is naturally isomorphic to $\det(\mathcal{E}^\bullet)|_{\mathcal{M}^{\text{red}}}$.*

If (S, ω) is an algebraic symplectic manifold over \mathbb{K} and $L, M \subseteq S$ are Lagrangians, then Pantev et al. [24, Th. 2.10] show that the derived intersection $\mathbf{X} = L \times_S M$ has a -1 -shifted symplectic structure. If $X = t_0(\mathbf{X})$, then $\mathbb{L}_{\mathbf{X}}|_X \simeq [T^*S|_X \rightarrow T^*L|_X \oplus T^*M|_X]$ with $T^*S|_X$ in degree -1 and $T^*L|_X \oplus T^*M|_X$ in

degree 0. Hence

$$\det(\mathbb{L}_{\mathbf{X}}|_X) \cong K_S|_X^{-1} \otimes K_L|_X \otimes K_M|_X \cong K_L|_X \otimes K_M|_X,$$

since $K_S \cong \mathcal{O}_S$. So as for Corollary 5.21, Theorem 6.6 implies:

Corollary 6.8. *Suppose (S, ω) is an algebraic symplectic manifold over \mathbb{K} , and suppose L, M are algebraic Lagrangians in S . Then the intersection $X = L \cap M$, as a \mathbb{K} -subscheme of S , extends naturally to an algebraic d-critical locus (X, s) . The canonical bundle $K_{X,s}$ from Theorem 6.4 is isomorphic to $K_L|_{X^{\text{red}}} \otimes K_M|_{X^{\text{red}}}$.*

Bussi [8, §3] proves a complex analytic analogue of Corollary 6.8.

We will take these ideas further in Brav, Bussi, Dupont, Joyce, and Szendrői [7] and Bussi, Joyce and Meinhardt [9]. In [7, §6], given an algebraic d-critical locus (X, s) with an orientation $K_{X,s}^{1/2}$, we construct a natural perverse sheaf $P_{X,s}^\bullet$ on X such that if (X, s) is locally modelled on $\text{Crit}(f : U \rightarrow \mathbb{A}^1)$ for U a smooth \mathbb{K} -scheme, then $P_{X,s}^\bullet$ is locally modelled on the perverse sheaf of vanishing cycles $\mathcal{P}\mathcal{V}_{U,f}^\bullet$. This has applications to the categorification of Donaldson–Thomas theory of Calabi–Yau 3-folds, and to constructing a Fukaya category of algebraic Lagrangians in an algebraic symplectic manifold.

In [9, §5], given an algebraic d-critical locus (X, s) with an orientation $K_{X,s}^{1/2}$, we construct a natural motive $MF_{X,s}$ in a certain ring of motives $\overline{\mathcal{M}}_X^\mu$ over X , such that if (X, s) is locally modelled on $\text{Crit}(f : U \rightarrow \mathbb{A}^1)$, then $MF_{X,s}$ is locally modelled on $\mathbb{L}^{-\dim U/2}([X] - MF_{U,f}^{\text{mot}})$, where $MF_{U,f}^{\text{mot}}$ is the motivic Milnor fibre of f . This has applications to motivic Donaldson–Thomas invariants. We refer the reader to [7, 9, 16] for more details.

6.3. Proof of Theorem 6.6. Let $(\mathbf{X}, \tilde{\omega})$ be a -1 -shifted symplectic derived \mathbb{K} -scheme, and let $X = t_0(\mathbf{X})$. For the first part of Theorem 6.6, we must construct a section $s \in H^0(\mathcal{S}_X^0)$ such that (X, s) is a d-critical locus, and if $A, \omega, \Phi, \mathbf{f}, \mathbf{R}, U, R, f$ are as in Theorem 6.6, then (R, U, Φ, f^{-1}) is a critical chart on (X, s) . The condition that (R, U, Φ, f^{-1}) is a critical chart determines $s|_R$ uniquely, as in Definition 6.2.

Theorem 5.18(i) implies that for any $x \in \mathbf{X}$, we can find such $A, \omega, \Phi, \mathbf{f}, \mathbf{R}, U, R, f$ with $x \in R \subseteq X$. So the condition in Theorem 6.6 determines $s|_R$ for Zariski open $R \subseteq X$ in an open cover of X . Thus $s \in H^0(\mathcal{S}_X^0)$ satisfying the conditions of the theorem is unique if it exists, and it exists if and only if the prescribed values $s|_R, s|_S$ agree on overlaps $R \cap S$ between open sets $R, S \subseteq X$.

So suppose $A, \omega, \Phi, \mathbf{f}, \mathbf{R}, U, R, f$ and $B, \tilde{\omega}, \check{\Phi}, \mathbf{g}, \mathbf{S}, V, S, g$ are two choices above. Write s_R and s_S for the sections of \mathcal{S}_X^0 on $R, S \subseteq X$ determined by the critical charts (R, U, Φ, f^{-1}) and $(S, V, \check{\Phi}, g^{-1})$, so that by Definition 6.2

$$(6.4) \quad \iota_{R,U}(s_R) = (f^{-1})^{-1}(\Phi) + I_{R,U}^2, \quad \iota_{S,V}(s_S) = (g^{-1})^{-1}(\check{\Phi}) + I_{S,V}^2.$$

We must show that $s_R|_{R \cap S} = s_S|_{R \cap S}$.

Let $x \in R \cap S$, so that $x = f(p) = g(q)$ for unique $p \in \text{Crit } \Phi \subseteq U$ and $q \in \text{Crit}(\check{\Phi}) \subseteq V$. Then using the method of §5.7, Example 5.22 constructs a standard form cdga C minimal at $r \in \text{Spec } H^0(C)$, and Zariski open inclusions $\alpha : A \rightarrow C, \beta : B \rightarrow C$ with $\mathbf{f} \circ \mathbf{Spec} \alpha \simeq \mathbf{g} \circ \mathbf{Spec} \beta$, such that the smooth \mathbb{K} -scheme $W = \text{Spec } C(0)$ and \mathbb{K} -scheme morphisms $a = \text{Spec } \alpha(0) : W \rightarrow U, b = \text{Spec } \beta(0) : W \rightarrow V$ satisfy, by (5.39),

$$(6.5) \quad a^*(\Phi) - b^*(\check{\Phi}) \in (dC^{-1})^2 = (a^*(d_{dR}\Phi))^2 = (b^*(d_{dR}\check{\Phi}))^2 \subset C(0).$$

Write $Z = \text{Spec } H^0(C)$, regarded as a closed \mathbb{K} -subscheme of $W = \text{Spec } C(0)$. Then $f \circ a|_Z = g \circ b|_Z : Z \rightarrow X$ is an isomorphism with a Zariski open \mathbb{K} -subscheme $T \subseteq R \cap S \subseteq X$ with $x \in T$. Define $s_T \in H^0(\mathcal{S}_X^0|_T)$ by

$$(6.6) \quad \begin{aligned} \iota_{T,W}(s_T) &= ((f \circ a|_Z)^{-1})^{-1}(a^*(\Phi)) + I_{T,W}^2 \\ &= ((g \circ b|_Z)^{-1})^{-1}(b^*(\check{\Phi})) + I_{T,W}^2, \end{aligned}$$

using the notation of Theorem 6.1(i) for the embedding $(f \circ a|_Z)^{-1} = (g \circ b|_Z)^{-1} : T \hookrightarrow W$ of T in the smooth \mathbb{K} -scheme W , where the two expressions on the right-hand side of (6.6) are equal by (6.5), since

$$I_{T,W} = ((f \circ a|_Z)^{-1})^{-1}((a^*(d_{dR}\Phi))) = ((g \circ b|_Z)^{-1})^{-1}((b^*(d_{dR}\check{\Phi}))).$$

We now have

$$\begin{aligned} \iota_{T,W}(s_R|_T) &= ((f \circ a|_Z)^{-1})^{-1}(a_{\#}) \circ \iota_{R,U}|_T(s_R|_T) \\ &= ((f \circ a|_Z)^{-1})^{-1}(a_{\#}) \circ ((f^{-1})^{-1}(\Phi) + I_{R,U}^2)|_T \\ &= ((f \circ a|_Z)^{-1})^{-1}(a^*(\Phi)) + I_{T,W}^2 = \iota_{T,W}(s_T), \end{aligned}$$

using (6.1) with $T, W, (f \circ a|_Z)^{-1}, R, U, f^{-1}, a$ in place of R, U, i, S, V, j, Ψ in the first step, (6.4) in the second, and (6.6) in the fourth. Hence, $s_R|_T = s_T$, as $\iota_{T,W}$ is injective in Theorem 6.1(i). Similarly $s_S|_T = s_T$, so $s_R|_T = s_S|_T$. As we can cover $R \cap S$ by such open $x \in T \subseteq R \cap S$, this implies that $s_R|_{R \cap S} = s_S|_{R \cap S}$, and the first part of Theorem 6.6 follows.

For the second part of the theorem, let $A, \omega, \Phi, \mathbf{f}, \mathbf{R}, U, R, f$ be as in Theorem 6.6, so that (R, U, Φ, f^{-1}) is a critical chart on (X, s) , and write $Y = \text{Spec } H^0(A) \subseteq U$, so that $f : Y \rightarrow R$ is an isomorphism. Then Theorem 6.4(i) gives a natural isomorphism

$$(6.7) \quad \iota_{R,U,\Phi,f^{-1}} : K_{X,s}|_{R^{\text{red}}} \longrightarrow (f^{-1})^*(K_U^{\otimes 2})|_{R^{\text{red}}}.$$

Also $\mathbb{L}_{\mathbf{f}} : \mathbf{f}^*(\mathbb{L}_{\mathbf{X}}) \rightarrow \mathbb{L}_A \simeq \Omega_A^1$ is a quasi-isomorphism as \mathbf{f} is a Zariski open inclusion. Hence $\det(\mathbb{L}_{\mathbf{f}})|_{Y^{\text{red}}} : \mathbf{f}^*(\det(\mathbb{L}_{\mathbf{X}})|_{R^{\text{red}}}) \rightarrow \det(\Omega_A^1)|_{Y^{\text{red}}}$ is an isomorphism, so pulling back by $f^{-1}|_{R^{\text{red}}}$ gives an isomorphism

$$(6.8) \quad (f^{-1}|_{R^{\text{red}}})^*(\det(\mathbb{L}_{\mathbf{f}})|_{Y^{\text{red}}}) : \det(\mathbb{L}_{\mathbf{X}})|_{R^{\text{red}}} \longrightarrow (f^{-1}|_{R^{\text{red}}})^*(\det(\Omega_A^1)|_{Y^{\text{red}}}).$$

Now by the material of §2.3 and §3.3, we have a natural isomorphism

$$\Omega_A^1|_{Y^{\text{red}}} \cong [TU|_{Y^{\text{red}}} \xrightarrow{\partial^2 \Phi|_{Y^{\text{red}}}} T^*U|_{Y^{\text{red}}}],$$

with $TU|_{Y^{\text{red}}}$ in degree -1 and $T^*U|_{Y^{\text{red}}}$ in degree 0 . Thus we have a natural isomorphism

$$(6.9) \quad \det(\Omega_A^1)|_{Y^{\text{red}}} \cong K_U^{\otimes 2}|_{Y^{\text{red}}}.$$

Combining (6.7)–(6.9) gives a natural isomorphism

$$(6.10) \quad K_{X,s}|_{R^{\text{red}}} \longrightarrow \det(\mathbb{L}_{\mathbf{X}})|_{R^{\text{red}}}$$

for each critical chart (R, U, Φ, f^{-1}) constructed from $A, \omega, \Phi, \mathbf{f}, \mathbf{R}$, as above. Combining Example 5.22 on comparing the charts (R, U, Φ, f^{-1}) with the material of [16, §2.4] defining the isomorphism J_{Ψ} in Theorem 6.4(ii), one can show that the canonical isomorphisms (6.10) on $R^{\text{red}}, S^{\text{red}}$ from two such charts (R, U, Φ, f^{-1}) and $(S, V, \check{\Phi}, g^{-1})$ are equal on the overlap $(R \cap S)^{\text{red}}$. Therefore, the isomorphisms

(6.10) glue to give a global canonical isomorphism $K_{X,s} \cong \det(\mathbb{L}_{\mathbf{X}})|_{X^{\text{red}}}$. This completes the proof of Theorem 6.6.

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REFERENCES

- [1] K. Behrend, *Donaldson–Thomas type invariants via microlocal geometry*, Ann. of Math. (2) **170** (2009), no. 3, 1307–1338, DOI 10.4007/annals.2009.170.1307. MR2600874
- [2] K. Behrend and B. Fantechi, *The intrinsic normal cone*, Invent. Math. **128** (1997), no. 1, 45–88, DOI 10.1007/s002220050136. MR1437495
- [3] O. Ben-Bassat, C. Brav, V. Bussi, and D. Joyce, *A ‘Darboux theorem’ for shifted symplectic structures on derived Artin stacks, with applications*, Geom. Topol. **19** (2015), no. 3, 1287–1359, DOI 10.2140/gt.2015.19.1287. MR3352237
- [4] D. Ben-Zvi and D. Nadler, *Loop spaces and connections*, J. Topol. **5** (2012), no. 2, 377–430, DOI 10.1112/jtopol/jts007. MR2928082
- [5] D. Borisov and D. Joyce, *Virtual fundamental classes for moduli spaces of sheaves on Calabi–Yau four-folds*, Geom. Topol. **21** (2017), no. 6, 3231–3311, DOI 10.2140/gt.2017.21.3231. MR3692967
- [6] E. Bouaziz and I. Grojnowski, *A d -shifted Darboux theorem*, arXiv:1309.2197 (2013).
- [7] C. Brav, V. Bussi, D. Dupont, D. Joyce, and B. Szendrői, *Symmetries and stabilization for sheaves of vanishing cycles*, J. Singul. **11** (2015), 85–151. With an appendix by Jörg Schürmann. MR3353002
- [8] V. Bussi, *Categorification of Lagrangian intersections on complex symplectic manifolds using perverse sheaves of vanishing cycles*, arXiv:1404.1329 (2014).
- [9] V. Bussi, D. Joyce and S. Meinhardt, *On motivic vanishing cycles of critical loci*, arXiv: 1305. 6428 (2013).
- [10] I. Emmanouil, *The cyclic homology of affine algebras*, Invent. Math. **121** (1995), no. 1, 1–19, DOI 10.1007/BF01884288. MR1345282
- [11] S. I. Gelfand and Y. I. Manin, *Methods of homological algebra*, 2nd ed., Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003. MR1950475
- [12] P. Goerss and K. Schemmerhorn, *Model categories and simplicial methods*, Interactions between homotopy theory and algebra, Contemp. Math., vol. 436, Amer. Math. Soc., Providence, RI, 2007, pp. 3–49, DOI 10.1090/conm/436/08403. MR2355769
- [13] T. G. Goodwillie, *Cyclic homology, derivations, and the free loop space*, Topology **24** (1985), no. 2, 187–215, DOI 10.1016/0040-9383(85)90055-2. MR793184
- [14] K. Hess, *Rational homotopy theory: a brief introduction*, Interactions between homotopy theory and algebra, Contemp. Math., vol. 436, Amer. Math. Soc., Providence, RI, 2007, pp. 175–202, DOI 10.1090/conm/436/08409. MR2355774
- [15] D. Huybrechts and R. P. Thomas, *Deformation-obstruction theory for complexes via Atiyah and Kodaira–Spencer classes*, Math. Ann. **346** (2010), no. 3, 545–569, DOI 10.1007/s00208-009-0397-6. MR2578562
- [16] D. Joyce, *A classical model for derived critical loci*, J. Differential Geom. **101** (2015), no. 2, 289–367. MR3399099
- [17] D. Joyce and P. Safronov, *A Lagrangian neighbourhood theorem for shifted symplectic derived schemes*, to appear in Annales de la Faculté des Sciences de Toulouse, 2018. arXiv:1506.04024.
- [18] D. Joyce and Y. Song, *A theory of generalized Donaldson–Thomas invariants*, Mem. Amer. Math. Soc. **217** (2012), no. 1020, iv+199, DOI 10.1090/S0065-9266-2011-00630-1. MR2951762
- [19] M. Kontsevich and Y. Soibelman, *Motivic Donaldson–Thomas invariants: summary of results*, Mirror symmetry and tropical geometry, Contemp. Math., vol. 527, Amer. Math. Soc., Providence, RI, 2010, pp. 55–89, DOI 10.1090/conm/527/10400. MR2681792

- [20] M. Kontsevich and Y. Soibelman, *Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson-Thomas invariants*, Commun. Number Theory Phys. **5** (2011), no. 2, 231–352, DOI 10.4310/CNTP.2011.v5.n2.a1. MR2851153
- [21] J.-L. Loday, *Cyclic homology*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 301, Springer-Verlag, Berlin, 1992. Appendix E by María O. Ronco. MR1217970
- [22] J. Lurie, *Derived Algebraic Geometry*, PhD thesis, M.I.T., 2004. Available at <http://www.math.harvard.edu/~lurie/papers/DAG.pdf>.
- [23] J. Lurie, *Higher algebra*, prepublication book draft (2011); available at <http://www.math.harvard.edu/~lurie/papers/HA.pdf>.
- [24] T. Pantev, B. Toën, M. Vaquié, and G. Vezzosi, *Shifted symplectic structures*, Publ. Math. Inst. Hautes Études Sci. **117** (2013), 271–328, DOI 10.1007/s10240-013-0054-1. MR3090262
- [25] T. Schürg, B. Toën, and G. Vezzosi, *Derived algebraic geometry, determinants of perfect complexes, and applications to obstruction theories for maps and complexes*, J. Reine Angew. Math. **702** (2015), 1–40, DOI 10.1515/crelle-2013-0037. MR3341464
- [26] R. P. Thomas, *A holomorphic Casson invariant for Calabi-Yau 3-folds, and bundles on K3 fibrations*, J. Differential Geom. **54** (2000), no. 2, 367–438. MR1818182
- [27] B. Toën, *Higher and derived stacks: a global overview*, Algebraic geometry—Seattle 2005. Part 1, Proc. Sympos. Pure Math., vol. 80, Amer. Math. Soc., Providence, RI, 2009, pp. 435–487, DOI 10.1090/pspum/080.1/2483943. MR2483943
- [28] B. Toën and G. Vezzosi, *Homotopical algebraic geometry. II. Geometric stacks and applications*, Mem. Amer. Math. Soc. **193** (2008), no. 902, x+224, DOI 10.1090/memo/0902. MR2394633
- [29] B. Toën and G. Vezzosi, *From HAG to DAG: derived moduli stacks*, Axiomatic, enriched and motivic homotopy theory, NATO Sci. Ser. II Math. Phys. Chem., vol. 131, Kluwer Acad. Publ., Dordrecht, 2004, pp. 173–216, DOI 10.1007/978-94-007-0948-5_6. MR2061855
- [30] B. Toën and G. Vezzosi, *Algèbres simpliciales S^1 -équivariantes, théorie de de Rham et théorèmes HKR multiplicatifs* (French, with English and French summaries), Compos. Math. **147** (2011), no. 6, 1979–2000, DOI 10.1112/S0010437X11005501. MR2862069
- [31] G. Vezzosi, *A note on the cotangent complex in derived algebraic geometry*, arXiv:1008.0601 (2010).
- [32] G. Vezzosi, *Derived critical loci I: Basics*, arXiv:1109.5213 (2011).

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