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ERRATUM TO "BERTINI IRREDUCIBILITY THEOREMS OVER FINITE FIELDS"

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Jiayu Zhao pointed out that in the proof of our Lemma 5.1 we were implicitly using what is called Lemma 3.6 below without first reducing to the case of a normal variety. To fix this, insert Lemma 3.6, replace Lemma 5.1 with the version below, and change the hypothesis on ψ in Lemma 5.2 to require ψ to be an immersion. There is also a notational error in the proof of Lemma 3.4: X'_f should be C_f. The rest of the article is unchanged.

Lemma 3.6. *Let $L \supseteq k$ be a Galois extension of fields. Let $\phi\colon V \to W$ be a morphism of irreducible k-varieties. If W is normal, then $\#\operatorname{Irr} W_L$ divides $\#\operatorname{Irr} V_L$.*

Proof. Let $G = \operatorname{Gal}(L/k)$. Let $V_0 \in \operatorname{Irr} V_L$. Since W is normal, W_L is normal by [Ray70, VII, Proposition 2], so the irreducible components of W_L are disjoint. Thus $\phi(V_0) \subseteq W_0$ for a unique $W_0 \in \operatorname{Irr} W_L$. For the action of G on $\operatorname{Irr} V_L$ and $\operatorname{Irr} W_L$, the stabilizers satisfy $\operatorname{Stab} V_0 \subseteq \operatorname{Stab} W_0$. Thus $(G : \operatorname{Stab} W_0)$ divides $(G : \operatorname{Stab} V_0)$. Since W is irreducible, G acts transitively on $\operatorname{Irr} W_L$, so $(G : \operatorname{Stab} W_0) = \#\operatorname{Irr} W_L$, and likewise $(G : \operatorname{Stab} V_0) = \#\operatorname{Irr} V_L$. □

Lemma 5.1. *Let X and Y be irreducible finite-type $\mathbb{F}$-schemes, with morphisms $X \xrightarrow{\pi} Y \xrightarrow{\psi} \mathbb{P}^n_{\mathbb{F}}$ such that π is finite étale, $\psi\colon Y \to \overline{\psi(Y)}$ is smooth of relative dimension s, and $\dim \overline{\psi(Y)} \geq 2$. For f in a set of density 1, the implication*

$$Y_f \text{ irreducible} \implies X_f \text{ irreducible}$$

holds.

Proof. Since irreducibility is a purely topological property, we may replace Y by Y_{red} and $\pi\colon X \to Y$ by its pullback to Y_{red}; then X is reduced, too. Irreducibility of X_f only becomes harder to achieve if X is replaced by a higher finite étale cover of Y. In particular, we may replace X by a cover corresponding to a Galois closure of $\kappa(X)/\kappa(Y)$. So assume from now on that $X \to Y$ is Galois étale, say with Galois group G.

Choose a finite extension $\mathbb{F}_r$ of $\mathbb{F}_q$ with a morphism $\psi'\colon Y' \to \mathbb{P}^n_{\mathbb{F}_r}$ and a Galois étale cover $\pi'\colon X' \to Y'$ whose base extensions to $\mathbb{F}$ yield ψ and π. Let $Z' := \overline{\psi'(Y')}$. Let $m := \dim Z' = \dim \overline{\psi(Y)} \geq 2$. Then $\dim Y' = \dim Y = s + m$. The morphism $\psi'\colon Y' \to Z'$ is smooth, so it maps $(Y')^{\mathrm{smooth}}$ into $(Z')^{\mathrm{smooth}}$.

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Given a closed point $y \in Y'$, let Frob_y be the associated Frobenius conjugacy class in G. We will prove that the following claims hold for f in a set of density 1.

Claim 1. The Frob_y for $y \in (Y'_f)^{\mathrm{smooth}}$ cover all conjugacy classes of G.

Claim 2. The $\mathbb{F}_r$-scheme $(X'_f)^{\mathrm{smooth}}$ contains two closed points whose degrees over $\mathbb{F}_r$ are coprime.

Let C be a conjugacy class in G. Let $c := \#C/\#G$. In the arguments below, for fixed X', Y', ψ', π', G, and C, the expression $o(1)$ denotes a function of e that tends to 0 as $e \to \infty$. By a function field analogue of the Chebotarev density theorem [Lan56, last display on p. 393] (which, in this setting, follows from applying the Lang–Weil estimates to all twists of the cover $Y' \to X'$), the number of closed points $y \in (Y')^{\mathrm{smooth}}$ with residue field $\mathbb{F}_{r^e}$ satisfying $\mathrm{Frob}_y = C$ is $(c + o(1))\, r^{(s+m)e}/e$. Since each nonempty fiber of ψ' has dimension s, there exists $c' > 0$ such that the images of these points in $\mathbb{P}^n_{\mathbb{F}_q}$ are at least $(c' + o(1))\, r^{me}/e$ closed points $z \in (Z')^{\mathrm{smooth}}$ with residue field of size at most r^e. For any such z, say with residue field of size r^ϵ, the density of $\{f : z \notin H_f\}$ is $1-r^{-\epsilon}$, and the density of $\{f : z \in H_f$ and H_f is not transverse to Z' at $z\}$ is $r^{-\epsilon} r^{-\epsilon m}$, so the union of these two disjoint sets has density $1 - r^{-\epsilon} + r^{-\epsilon(1+m)} \le 1 - r^{-\epsilon}/2 \le 1 - r^{-e}/2$. These conditions at the finitely many z are independent, so the density of the set $\mathcal{Q}_{C,e}$ of f such that they hold at all z is at most $(1 - r^{-e}/2)^{(c'+o(1))r^{me}/e}$, which tends to 0 as $e \to \infty$ since $m \ge 2$. If the condition at some z fails, then $z \in (Z'_f)^{\mathrm{smooth}}$, and any $y \in (Y')^{\mathrm{smooth}}$ with residue field $\mathbb{F}_{r^e}$ with $\psi'(y) = z$ lies in $(Y'_f)^{\mathrm{smooth}}$, since $\psi' \colon Y' \to Z'$ is smooth. Thus the complement $\mathcal{P}_{C,e}$ of $\mathcal{Q}_{C,e}$ equals the set of f for which there exists $y \in (Y'_f)^{\mathrm{smooth}}$ such that $\kappa(y) = \mathbb{F}_{r^e}$ and $\mathrm{Frob}_y = C$. The lower density of $\mathcal{P}_{C,e}$ tends to 1 as $e \to \infty$.

Proof of Claim 1. There are only finitely many C, so the previous sentence shows that the lower density of $\bigcap_C \mathcal{P}_{C,e}$ tends to 1 as $e \to \infty$.

Proof of Claim 2. If $f \in \mathcal{P}_{1,e}$, then there exists $y \in (Y'_f)^{\mathrm{smooth}}$ with $\kappa(y) = \mathbb{F}_{r^e}$ and $\mathrm{Frob}_y = 1$, and any preimage $x \in X'_f$ is a point of $(X'_f)^{\mathrm{smooth}}$ satisfying $\kappa(x) = \mathbb{F}_{r^e}$, since $X'_f \to Y'_f$ is finite étale and $\mathrm{Frob}_y = 1$. The lower density of $\mathcal{P}_{1,e} \cap \mathcal{P}_{1,e'}$ tends to 1 as (e, e') runs through pairs of coprime integers with $\min(e, e') \to \infty$.

To complete the proof of the lemma, we show that if Y_f is irreducible and Claims 1 and 2 hold, then X_f is irreducible. Assume that Y_f is irreducible, so Y'_f is geometrically irreducible. The only subgroup of G that meets all conjugacy classes is G itself, so Claim 1 implies that $(X'_f)^{\mathrm{smooth}} \to (Y'_f)^{\mathrm{smooth}}$ is a finite Galois *irreducible* cover (with Galois group G).

If x' is a closed point of $(X'_f)^{\mathrm{smooth}}$ of degree e over $\mathbb{F}_r$, then applying Lemma 3.6 to $\mathbb{F} \supseteq \mathbb{F}_r$ and $\{x'\} \hookrightarrow (X'_f)^{\mathrm{smooth}}$ shows that $\#\operatorname{Irr} X_f^{\mathrm{smooth}}$ divides e. Applying this to both points in Claim 2 shows that $\#\operatorname{Irr} X_f^{\mathrm{smooth}} = 1$, so X_f^{smooth} is irreducible. On the other hand, Y_f is irreducible and Y_f^{smooth} is nonempty, so Y_f^{smooth} is dense in Y_f; since $X_f \to Y_f$ is finite étale, X_f^{smooth} is dense in X_f, too. Combining the previous two sentences shows that X_f is irreducible. ∎

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