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ERRATUM TO “BERTINI IRREDUCIBILITY THEOREMS OVER FINITE FIELDS”

FRANÇOIS CHARLES AND BJORN POONEN

Jiayu Zhao pointed out that in the proof of our Lemma 5.1 we were implicitly using what is called Lemma 3.6 below without first reducing to the case of a normal variety. To fix this, insert Lemma 3.6, replace Lemma 5.1 with the version below, and change the hypothesis on $\psi$ in Lemma 5.2 to require $\psi$ to be an immersion. There is also a notational error in the proof of Lemma 3.4: $X'_f$ should be $C_f$. The rest of the article is unchanged.

Lemma 3.6. Let $L \supseteq k$ be a Galois extension of fields. Let $\phi: V \to W$ be a morphism of irreducible $k$-varieties. If $W$ is normal, then $\# \text{Irr}_L W$ divides $\# \text{Irr}_V L$.

Proof. Let $G = \text{Gal}(L/k)$. Let $V_0 \in \text{Irr}_V L$. Since $W$ is normal, $W_L$ is normal by [Ray70, VII, Proposition 2.4], so the irreducible components of $W_L$ are disjoint. Thus $\phi(V_0) \subseteq W_0$ for a unique $W_0 \in \text{Irr}_W L$. For the action of $G$ on $\text{Irr}_V L$ and $\text{Irr}_W L$, the stabilizers satisfy $\text{Stab} V_0 \subseteq \text{Stab} W_0$. Thus $(G : \text{Stab} W_0) \text{ divides } (G : \text{Stab} V_0)$. Since $W$ is irreducible, $G$ acts transitively on $\text{Irr}_W L$, so $(G : \text{Stab} W_0) = \# \text{Irr}_W L$, and likewise $(G : \text{Stab} V_0) = \# \text{Irr}_V L$. $\square$

Lemma 5.1. Let $X$ and $Y$ be irreducible finite-type $\mathbb{F}$-schemes, with morphisms $X \xrightarrow{\phi} Y \xrightarrow{\psi} \mathbb{P}^n_\mathbb{F}$ such that $\pi$ is finite étale, $\psi: Y \to \overline{\psi(Y)}$ is smooth of relative dimension $s$, and $\dim \overline{\psi(Y)} \geq 2$.

For $f$ in a set of density 1, the implication

$$Y_f \text{ irreducible } \implies X_f \text{ irreducible}$$

holds.

Proof. Since irreducibility is a purely topological property, we may replace $Y$ by $Y_{\text{red}}$ and $\pi: X \to Y$ by its pullback to $Y_{\text{red}}$; then $X$ is reduced too. Irreducibility of $X_f$ only becomes harder to achieve if $X$ is replaced by a higher finite étale cover of $Y$. In particular, we may replace $X$ by a cover corresponding to a Galois closure of $\kappa(X)/\kappa(Y)$. So assume from now on that $X \to Y$ is Galois étale, say with Galois group $G$.

Choose a finite extension $\mathbb{F}_r$ of $\mathbb{F}_q$ with a morphism $\psi': Y' \to \mathbb{P}^n_{\mathbb{F}_r}$ and a Galois étale cover $\pi': X' \to Y'$ whose base extensions to $\mathbb{F}$ yield $\psi$ and $\pi$. Let $Z := \overline{\psi'(Y')}$. Let $m := \dim Z' = \dim \overline{\psi(Y)} \geq 2$. Then $\dim Y' = \dim Y = s + m$. The morphism $\psi': Y' \to Z'$ is smooth, so it maps $(Y')_{\text{smooth}}$ into $(Z')_{\text{smooth}}$.

Given a closed point $y \in Y'$, let $\text{Frob}_y$ be the associated Frobenius conjugacy class in $G$. We will prove that the following claims hold for $f$ in a set of density 1:

Claim 1. The $\text{Frob}_y$ for $y \in (Y'_f)_{\text{smooth}}$ cover all conjugacy classes of $G$.

Claim 2. The $\mathbb{F}_r$-scheme $(X'_f)_{\text{smooth}}$ contains two closed points whose degrees over $\mathbb{F}_r$ are coprime.

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Let $C$ be a conjugacy class in $G$. Let $c := \#C/\#G$. In the arguments below, for fixed $X'$, $Y'$, $\psi'$, $\pi'$, $G$, and $C$, the expression $o(1)$ denotes a function of $e$ that tends to 0 as $e \to \infty$. By a function field analogue of the Chebotarev density theorem [Lan56] last display on p. 393 (which, in this setting, follows from applying the Lang–Weil estimates to all twists of the cover $Y' \to X'$), the number of closed points $y \in (Y')^{\text{smooth}}$ with residue field $\mathbb{F}_r$ satisfying $\text{Frob}_y = C$ is $(c + o(1)) r^{(x+y)e}/e$. Since each nonempty fiber of $\psi'$ has dimension $s$, there exists $c' > 0$ such that the images of these points in $\mathbb{F}_q^m$ are at least $(c' + o(1)) r^{me}/e$ closed points $z \in (Z')^{\text{smooth}}$ with residue field of size at most $r^e$. For any such $z$, say with residue field of size $r^e$, the density of $\{ f : z \notin H_f \}$ is $1 - r^{-e}$, and the density of $\{ f : z \in H_f \}$ is $r^{-e}$, so the union of these two disjoint sets has density $1 - r^{-e} + r^{-e(1+m)} \leq 1 - r^{-e}/2 \leq 1 - r^{-e}/2$. These conditions at the finitely many $z$ are independent, so the density of the set $\mathcal{Q}_{C,e}$ of $f$ such that they hold at all $z$ is at most $(1 - r^{-e}/2)(c' + o(1)) r^{me}/e$, which tends to 0 as $e \to \infty$ since $m \geq 2$. If the condition at some $z$ fails, then $z \in (Z')^{\text{smooth}}$, and any $y \in (Y')^{\text{smooth}}$ with residue field $\mathbb{F}_r$ with $\psi'(y) = z$ lies in $(Y')^{\text{smooth}}$, since $\psi' : Y' \to Z'$ is smooth. Thus the complement $\mathcal{P}_{C,e}$ of $\mathcal{Q}_{C,e}$ equals the set of $f$ for which there exists $y \in (Y')^{\text{smooth}}$ such that $\kappa(y) = \mathbb{F}_r$, and $\text{Frob}_y = C$. The lower density of $\mathcal{P}_{C,e}$ tends to 1 as $e \to \infty$.

Proof of Claim 1: There are only finitely many $C$, so the previous sentence shows that the lower density of $\cap_C \mathcal{P}_{C,e}$ tends to 1 as $e \to \infty$.

Proof of Claim 2: If $f \in \mathcal{P}_{1,e}$, then there exists $y \in (Y')^{\text{smooth}}$ with $\kappa(y) = \mathbb{F}_r$ and $\text{Frob}_y = 1$, and any preimage $x \in X'_f$ is a point of $(X'_f)^{\text{smooth}}$ satisfying $\kappa(x) = \mathbb{F}_r$, since $X'_f \to Y'_f$ is finite étale and $\text{Frob}_y = 1$. The lower density of $\mathcal{P}_{1,e} \cap \mathcal{P}_{1,e'}$ tends to 1 as $(e,e') \to \infty$.

To complete the proof of the lemma, we show that if $Y_f$ is irreducible and Claims 1 and 2 hold, then $X_f$ is irreducible. Assume that $Y_f$ is irreducible, so $Y'_f$ is geometrically irreducible. The only subgroup of $G$ that meets all conjugacy classes is $G$ itself, so Claim 1 implies that $(X'_f)^{\text{smooth}} \to (Y'_f)^{\text{smooth}}$ is a finite Galois irreducible cover (with Galois group $G$).

If $x'$ is a closed point of $(X'_f)^{\text{smooth}}$ of degree $e$ over $\mathbb{F}_r$, then applying Lemma 3.6 to $\mathbb{F} \supset \mathbb{F}_r$ and $\{ x' \} \leftrightarrow (X'_f)^{\text{smooth}}$ shows that $\# \text{Irr } X'_f^{\text{smooth}}$ divides $e$. Applying this to both points in Claim 2 shows that $\# \text{Irr } X'_f^{\text{smooth}} = 1$, so $X'_f^{\text{smooth}}$ is irreducible. On the other hand, $Y_f$ is irreducible and $Y'_f^{\text{smooth}}$ is nonempty, so $Y'_f^{\text{smooth}}$ is dense in $Y'_f$; since $X_f \to Y_f$ is finite étale, $X_f^{\text{smooth}}$ is dense in $X_f$ too. Combining the previous two sentences shows that $X_f$ is irreducible. □

References

