

CLUSTER THEORY OF THE COHERENT SATAKE CATEGORY

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CONTENTS

1. Introduction	709
2. The coherent Satake category	718
3. Rigidity	727
4. Renormalized r -matrices	737
5. Chiral categories and r -matrices	746
6. Monoidal cluster categorification	754
References	774

1. INTRODUCTION

Let G be a complex reductive group. The Satake category $\mathcal{P}^{G(\mathcal{O})}(\mathrm{Gr}_G)$ of $G(\mathcal{O})$ -equivariant perverse sheaves on the affine Grassmannian Gr_G plays a fundamental role in geometric representation theory and, in particular, the geometric Langlands program. Its structure is well understood via the geometric Satake equivalence of [Lus83, Gin95, MV00], which states that $\mathcal{P}^{G(\mathcal{O})}(\mathrm{Gr}_G)$ is monoidally equivalent to the representation category of the Langlands dual group G^\vee .

Like its constructible counterpart, the derived category of $G(\mathcal{O})$ -equivariant coherent sheaves on Gr_G has a perverse t -structure [AB10, BFM05] which is stable under convolution. This gives us the *coherent* Satake category $\mathcal{P}_{coh}^{G(\mathcal{O})}(\mathrm{Gr}_G)$. In contrast with $\mathcal{P}^{G(\mathcal{O})}(\mathrm{Gr}_G)$ this monoidal category is not semisimple and is poorly understood.

The purpose of this paper is to establish the existence of duals and renormalized r -matrices in $\mathcal{P}_{coh}^{G(\mathcal{O})}(\mathrm{Gr}_G)$, and to explain how these lead to its structure being governed by the theory of cluster algebras. This in turn makes visible the physical meaning of $\mathcal{P}_{coh}^{G(\mathcal{O})}(\mathrm{Gr}_G)$: its simple objects correspond to Wilson-'t Hooft line operators in 4d $\mathcal{N} = 2$ gauge theory, just as those of $\mathcal{P}^{G(\mathcal{O})}(\mathrm{Gr}_G)$ correspond to 't Hooft operators in 4d $\mathcal{N} = 4$ gauge theory [KW07].

The coherent Satake category. Our starting point is the fact that $\mathcal{P}_{coh}^{G(\mathcal{O})}(\mathrm{Gr}_G)$ is finite length, as is the \mathbb{G}_m -equivariant coherent Satake category $\mathcal{P}_{coh}^{G(\mathcal{O}) \times \mathbb{G}_m}(\mathrm{Gr}_G)$ in which we further consider the action of loop rotation on Gr_G [AB10]. In either setting the simple objects can be explicitly enumerated: they are in bijection with

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isomorphism classes of simple equivariant vector bundles on $G(\mathcal{O})$ -orbits. Recall that $G(\mathcal{O})$ -orbits $\text{Gr}_G^{\lambda^\vee}$ in Gr_G are indexed by coweights λ^\vee of G . The most basic simple objects are the restrictions of line bundles on Gr_G to some $\overline{\text{Gr}}_G^{\lambda^\vee}$, shifted to lie in cohomological degree $-\frac{1}{2} \dim \text{Gr}_G^{\lambda^\vee}$ [Mir] (as in [BFM05] we allow formal half-integer degrees).

In general, a simple object with support $\overline{\text{Gr}}_G^{\lambda^\vee}$ restricts to a shifted equivariant vector bundle on $\text{Gr}_G^{\lambda^\vee}$ (one direction of the above bijection) but a priori lacks an elementary description itself. However, we will be able to deduce many nontrivial properties of convolution products of simple objects using the following structure.

Theorem 1.1 (Cf. Theorem 5.14). *Given nonzero objects $\mathcal{F}, \mathcal{G} \in D_{\text{coh}}^{G(\mathcal{O})}(\text{Gr}_G)$ there is a canonical nonzero map $\mathbf{r}_{\mathcal{F}, \mathcal{G}} : \mathcal{F} * \mathcal{G} \rightarrow \mathcal{G} * \mathcal{F}$, which we call a renormalized r -matrix. These maps are not isomorphisms in general and are not natural in the categorical sense, but satisfy a collection of axioms which control their failure to constitute a braiding.*

The construction and characteristic properties of the maps $\mathbf{r}_{\mathcal{F}, \mathcal{G}}$ are similar to those of the renormalized r -matrices which appear in the representation theory of quantum loop algebras [Kas02] and KLR (or quiver Hecke) algebras [KKK18], hence our terminology. Their existence is closely related to the commutativity constraint which appears in the proof of the geometric Satake equivalence.

Adding in loop rotations, there exist $\Lambda(\mathcal{F}, \mathcal{G}) \in \mathbb{Z}$ so that these r -matrices lift to maps

$$\mathbf{r}_{\mathcal{F}, \mathcal{G}} : \mathcal{F} * \mathcal{G} \rightarrow \mathcal{G} * \mathcal{F}\{\Lambda(\mathcal{F}, \mathcal{G})\}$$

in $D_{\text{coh}}^{G(\mathcal{O}) \rtimes \mathbb{G}_m}(\text{Gr}_G)$, where $\{\cdot\}$ denotes a \mathbb{G}_m -equivariance shift. The axioms alluded to above can be summarized as saying that renormalized r -matrices deviate from satisfying the axioms of a braiding in a precise way controlled by certain inequalities among these degrees (see Definition 4.1). Note that it would be impossible to define a true braiding on $D_{\text{coh}}^{G(\mathcal{O})}(\text{Gr}_G)$ since $\mathcal{F} * \mathcal{G}$ and $\mathcal{G} * \mathcal{F}$ are usually not isomorphic.

Theorem 1.2 (Cf. Theorem 3.1). *The monoidal category $D_{\text{coh}}^{G(\mathcal{O}) \rtimes \mathbb{G}_m}(\text{Gr}_G)$ is rigid. The left and right duals of $\mathcal{F} \in D_{\text{coh}}^{G(\mathcal{O}) \rtimes \mathbb{G}_m}(\text{Gr}_G)$ are $\mathcal{F}^L = \mathbb{D}(\mathcal{F}^*)$ and $\mathcal{F}^R = (\mathbb{D}\mathcal{F})^*$. Here $\mathcal{F} \mapsto \mathcal{F}^*$ is an involution and $\mathbb{D}\mathcal{F} = \mathcal{H}om(\mathcal{F}, \omega)$, where ω denotes the dualizing sheaf. The duals of a perverse coherent sheaf are again perverse, so $\mathcal{P}_{\text{coh}}^{G(\mathcal{O}) \rtimes \mathbb{G}_m}(\text{Gr}_G)$ is also rigid.*

This result turns out to be more complicated than its counterpart in the constructible case, where the existence of a fiber functor and commutativity constraint reduce rigidity to an elementary check [DM82, BR17].

As in the setting of quantum loop algebras and KLR algebras [KKKO15], renormalized r -matrices together with certain properties implied by rigidity strongly constrain the behavior of real simple objects—that is, objects whose convolution square is again simple (see Theorem 5.15). Moreover, the coherent Satake category has an abundance of real objects, namely the shifted restrictions of equivariant line bundles discussed above.

Since $\mathcal{P}_{\text{coh}}^{G(\mathcal{O}) \rtimes \mathbb{G}_m}(\text{Gr}_G)$ is finite length, the classes of simple objects form a canonical basis in the Grothendieck ring $K^{G(\mathcal{O}) \rtimes \mathbb{G}_m}(\text{Gr}_G)$. Following the framework developed in [KKKO18] to study the upper global or dual canonical bases of quantum

groups [Lus90, Kas91], Theorems 1.1 and 1.2 provide a mechanism by which this basis may come to include the structure of a quantum cluster algebra.

Cluster algebras. A cluster algebra $\mathcal{A}_{\tilde{B}}$ is a ring with a partial basis of a certain form: it contains special elements (cluster variables) grouped into overlapping subsets (clusters) such that the monomials in any cluster are again basis elements [FZ02]. Any two clusters are connected by a sequence of mutations, an operation which creates a new cluster by exchanging a single cluster variable for a new one. The form of these exchanges, hence the entire structure, is encoded by a skew-symmetrizable matrix \tilde{B} . In [FKRD18] it was shown that $\mathbb{C} \otimes K^{GL_n(\mathcal{O})}(\mathrm{Gr}_{GL_n})$ is isomorphic with the coordinate ring of a unipotent cell in LSL_2 , hence possesses a cluster structure following [GLS11].

Monoidal categorifications of cluster algebras were considered in [HL10] and its successors [HL13, Nak11, KQ14, Qin17] in the context of quantum loop algebras. The terminology is meant to distinguish from descriptions of cluster algebras via homological (not monoidal) properties of quiver representations [CC06, BMR+06, GLS06, CK06, Pla11, Ami09].

The algebra $\mathcal{A}_{\tilde{B}}$ has a family of q -deformations $\mathcal{A}_{(\tilde{B}, L)}$ (quantum cluster algebras) depending on an auxiliary matrix L , and in which the elements of any cluster have distinguished pairwise q -commuting lifts called quantum cluster variables [BZ05]. The proof in [KKKO18] of the existence of a quantum cluster structure on the upper global basis of a quantum unipotent coordinate algebra proceeds by establishing that one obtains monoidal quantum cluster categorifications from module categories of KLR algebras [KL09, Rou08].

Coherent Satake as a monoidal cluster categorification. For any particular G , to deduce a categorified cluster structure on $\mathcal{P}_{coh}^{G(\mathcal{O}) \rtimes \mathbb{G}_m}(\mathrm{Gr}_G)$ from Theorems 1.1 and 1.2 one must further show, by hand, that a finite subset of would-be cluster variables and mutations in $K^{G(\mathcal{O}) \rtimes \mathbb{G}_m}(\mathrm{Gr}_G)$ lift to suitable real simple objects and exact sequences. In this paper we carry this out fully in the case $G = GL_n$, discussing the expected extension to other G later in the introduction.

Theorem 1.3 (Cf. Theorem 6.18). *The coherent Satake category*

$$\mathcal{P}_{coh}^{GL_n(\mathcal{O}) \rtimes \mathbb{G}_m}(\mathrm{Gr}_{GL_n})$$

is a monoidal categorification of a quantum cluster algebra $\mathcal{A}_{(L_n, \tilde{B}_n)}^{loc}$ with localized frozen variables. That is, there is a $\mathbb{Z}[q^{\pm 1/2}]$ -algebra isomorphism of $\mathcal{A}_{(L_n, \tilde{B}_n)}^{loc}$ with $K^{GL_n(\mathcal{O}) \rtimes \mathbb{G}_m}(\mathrm{Gr}_{GL_n})$ such that

- (1) *quantum cluster variables are classes of certain real simple perverse coherent sheaves,*
- (2) *the convolution of two such sheaves in the same cluster is again simple and its isomorphism class is independent of the order up to a grading shift, and*
- (3) *the convolution in either order of such a sheaf and its mutation is the middle term of an exact sequence categorifying the relevant exchange relation.*

Here \tilde{B}_n is the signed adjacency matrix of the quiver in Figure 1.1. We may take as the initial cluster the classes of certain perverse coherent sheaves $\mathcal{P}_{k, \ell}$ for $1 \leq k \leq n$ and $0 \leq \ell \leq 1$. These sheaves are the restrictions of the line

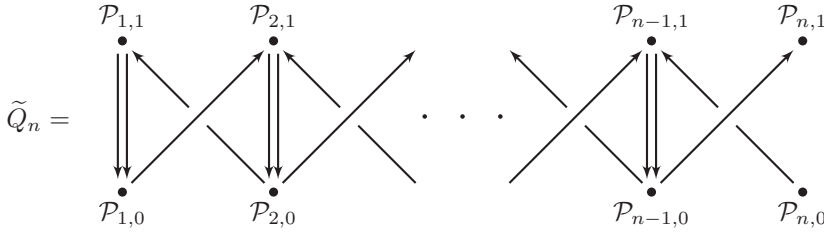


FIGURE 1.1

bundles $\mathcal{O}(\ell)$ to the closed orbits $\text{Gr}_{GL_n}^{\omega_k^\vee}$ with a cohomological/equivariant shift of $[\frac{1}{2} \dim \text{Gr}_{GL_n}^{\omega_k^\vee}] \{-\frac{1}{2} \dim \text{Gr}_{GL_n}^{\omega_k^\vee} - k\ell\}$.

The GL_n case is of particular interest because $K^{GL_n(\mathcal{O}) \rtimes \mathbb{G}_m}(\text{Gr}_{GL_n})$ is also the simplest (A_1) example of a quantized Coulomb branch of a 4d $\mathcal{N} = 2$ quiver gauge theory [Nak16, BFN16]. The complexified classical limits of such spaces are Poisson isomorphic to coordinate rings of open Richardson varieties in the affine flag variety [FKRD18], or equivalently unipotent cells of LSL_2 in the A_1 case. Given results of [GLS13, KO17], Theorem 1.3 implies that moreover $K^{GL_n(\mathcal{O}) \rtimes \mathbb{G}_m}(\text{Gr}_{GL_n})$ and the quantized unipotent cell $A_q(\mathcal{N}^{(s_1 s_0)^n})$ are isomorphic after tensoring both with $\mathbb{Q}(q^{1/2})$.

It follows that a subalgebra of $\mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm 1/2}]} K^{GL_n(\mathcal{O}) \rtimes \mathbb{G}_m}(\text{Gr}_{GL_n})$ also admits a categorification via KLR algebras of type $A_1^{(1)}$. At present we do not know a precise relationship between this categorification and the category of perverse coherent sheaves.

From another point of view, Theorem 1.3 is significant in that it asserts a precise way in which the geometry of the affine Grassmannian, viewed through the lens of the coherent Satake category, reflects enumerative aspects of an a priori unrelated 3-Calabi-Yau category. Indeed, the cluster algebra of any skew-symmetric matrix encodes (and is entirely determined by) invariants of moduli spaces related to the representation theory of the associated quiver, equipped with a suitable potential [CC06, CK06, Ami09, Pla11].

In our case the relevant quiver with potential (without frozen variables) describes coherent sheaves on a particular toric Calabi-Yau 3-fold [CDM+14], or in principle the Fukaya category of its mirror, a 3-fold related to the periodic Toda system [KS14, Smi15]. In this way Theorem 1.3 provides a concrete mathematical manifestation of geometric engineering [KKV97]: the affine Grassmannian encodes various structures in pure 4d $\mathcal{N} = 2$ gauge theory [Tel14, Cos14a, BFN16], hence should in turn encode properties of the Calabi-Yau that engineers the theory.

Chiral categories and renormalized r -matrices. The origin of renormalized r -matrices in $\mathcal{P}_{coh}^{G(\mathcal{O}) \rtimes \mathbb{G}_m}(\text{Gr}_G)$ is the Beilinson-Drinfeld (BD) Grassmannian [BD96]. The BD Grassmannians of \mathbb{A}^1 and its powers are ind-schemes $\text{Gr}_{G, \mathbb{A}^n}$ over \mathbb{A}^n for $n > 0$. Collectively they form the prototypical example of a factorization space, meaning they satisfy certain compatibility conditions on their restrictions to diagonals and disjoint loci.

To construct r -matrices we consider $\text{Gr}_{G, \mathbb{A}^2}$. The factorization property in this case is that the fibers of $\text{Gr}_{G, \mathbb{A}^2} \rightarrow \mathbb{A}^2$ are isomorphic to Gr_G over the diagonal

and to $\text{Gr}_G \times \text{Gr}_G$ away from it. Given $\mathcal{F}, \mathcal{G} \in D_{coh}^{G(\mathcal{O}) \rtimes G_m}(\text{Gr}_G)$, the factorization structure allows one to produce a pair of sheaves on $\text{Gr}_{G, \mathbb{A}^2}$ which

- (1) over the diagonal of \mathbb{A}^2 have fibers isomorphic to $\mathcal{F} * \mathcal{G}$ and $\mathcal{G} * \mathcal{F}$, respectively, and
- (2) over the complement of the diagonal each has fibers isomorphic to $\mathcal{F} \boxtimes \mathcal{G}$.

Restricting to $\text{Spec } \mathbb{C}[t] \cong \mathbb{A}^1 \times \{0\} \subset \mathbb{A}^2$ one can consider the identity endomorphism of $\mathcal{F} \boxtimes \mathcal{G}$ over $\mathbb{A}^1 \setminus \{0\}$. This defines a rational section of Hom between these sheaves. After multiplying by a suitable power of t and restricting to the fiber over $0 \in \mathbb{A}^1$ we obtain a canonical homomorphism $\mathbf{r}_{\mathcal{F}, \mathcal{G}} : \mathcal{F} * \mathcal{G} \rightarrow \mathcal{G} * \mathcal{F}$. This construction can be thought of as a meromorphic analogue of the Eckmann-Hilton argument.

We emphasize that it is only the factorization structure of the BD Grassmannian which is essential for defining these r -matrices. Abstractly, we only use the fact that we have a monoidal category \mathcal{C} which can be recovered from a suitably compatible chiral category on \mathbb{A}^1 as a subcategory of its fiber at 0. A chiral category consists of a collection of sheaves of categories over the powers of \mathbb{A}^1 which satisfy analogous compatibilities to those of factorization spaces or chiral/factorization algebras [Ras14].

In Section 5 we describe the construction of renormalized r -matrices in this level of generality (cf. Theorem 5.10). In Section 4 we distill from [KKKO15, KKKO18] a list of axioms (Definition 4.1) required of such a collection of morphisms for the proofs of the main results of loc. cit. to carry over—we call such a collection a system of renormalized r -matrices. We then recall following [KKKO18] how renormalized r -matrices reduce the existence of iterated mutations to the existence of one-step mutations given the presence of a quantum cluster structure on the Grothendieck ring. After showing that the chiral r -matrices of Section 5 satisfy the conditions in Definition 4.1 we thus obtain the following result.

Theorem 1.4 (Cf. Theorem 4.25, Theorem 5.10). *Let $\widehat{\mathcal{C}}$ be a graded monoidal abelian category contained in the subcategory of compact objects in the fiber at 0 of a compatible G_m -equivariant chiral category on \mathbb{A}^1 . Let $(\{\mathcal{F}_i\}_{i \in I}, \widetilde{B})$ be a quantum monoidal seed in $\widehat{\mathcal{C}}$ with coefficient matrix L . Suppose that*

- (1) $\widehat{\mathcal{C}}$ is rigid,
- (2) $(\{\mathcal{F}_i\}_{i \in I}, \widetilde{B})$ admits a mutation in every unfrozen direction, and
- (3) there is an isomorphism $K_0(\widehat{\mathcal{C}}) \cong \mathcal{A}_{(L, \widetilde{B})}$ taking the classes $\{[\mathcal{F}_i]\}_{i \in I}$ to the initial quantum cluster variables.

Then $\widehat{\mathcal{C}}$ is a monoidal categorification of the quantum cluster algebra $\mathcal{A}_{(L, \widetilde{B})}$.

Here a quantum monoidal seed is a collection of q -commuting real simple objects (that is, the products of any two in either order are isomorphic up to a grading shift) together with an exchange matrix (see Section 4.4). Compatibility means in particular that the sections over \mathbb{A}^1 and \mathbb{A}^2 of the relevant chiral category possess monoidal structures compatible with the factorization structure and the monoidal structure on $\widehat{\mathcal{C}}$ (in natural examples one will in fact have a chiral \mathbb{E}_1 -category, hence this condition will hold over \mathbb{A}^n for all n).

While we believe formulating our construction at this level of abstraction (rather than in the specific context of the BD Grassmannian) provides more conceptual

clarity, we also do so with an eye towards laying a foundation for future applications. In particular, one expects analogues of Theorem 1.3 involving the K -theoretic Coulomb branches of [BFN16] and suitable abelian categories when the associated 4d $\mathcal{N} = 2$ theory possesses a BPS quiver [ACC+14]. While it is not clear how to define the correct t -structures, it is clear that a factorization structure is present and will be crucial for generalizing Theorem 1.3.

Computations for GL_n . Given Theorems 1.1, 1.2, and 1.4, the proof of Theorem 1.3 reduces to two remaining steps. First, one needs to verify that the stated initial cluster consists of q -commuting real objects and to produce its mutations in all unfrozen directions. Given $k \in [1, n], \ell \in \mathbb{Z}$ we let $\mathcal{P}_{k,\ell} \in \mathcal{P}_{coh}^{GL_n(\mathcal{O}) \rtimes \mathbb{G}_m}(\mathrm{Gr}_{GL_n})$ denote the restriction of $\mathcal{O}(\ell)$ to $\mathrm{Gr}_{GL_n}^{\omega_k^\vee}$, shifted by $\langle \frac{1}{2} \dim \mathrm{Gr}_{GL_n}^{\omega_k^\vee} \rangle \{-k\ell\}$ (where $\langle 1 \rangle$ is short for $[1]\{-1\}$).

Theorem 1.5 (Cf. Propositions 2.9 and 2.16). *The perverse coherent sheaves $\mathcal{P}_{k,\ell}$ are simple and real for all $k \in [1, n], \ell \in \mathbb{Z}$. If $|\ell_1 - \ell_2| \leq 1$, then \mathcal{P}_{k_1,ℓ_1} and \mathcal{P}_{k_2,ℓ_2} q -commute. Moreover, there exist exact sequences*

$$\begin{aligned} 0 \rightarrow \mathcal{P}_{k-1,\ell} * \mathcal{P}_{k+1,\ell} \{-1\} &\rightarrow \mathcal{P}_{k,\ell+1} * \mathcal{P}_{k,\ell-1} \{-2k\} \rightarrow \mathcal{P}_{k,\ell} * \mathcal{P}_{k,\ell} \rightarrow 0, \\ 0 \rightarrow \mathcal{P}_{k,\ell} * \mathcal{P}_{k,\ell} &\rightarrow \mathcal{P}_{k,\ell-1} * \mathcal{P}_{k,\ell+1} \{2k\} \rightarrow \mathcal{P}_{k+1,\ell} * \mathcal{P}_{k-1,\ell} \{1\} \rightarrow 0. \end{aligned}$$

In particular, letting \tilde{B}_n denote the signed adjacency matrix of the quiver in Figure 1.1, the pair $(\{\mathcal{P}_{k,\ell}\}_{k \in [1,n], \ell \in \{0,1\}}, \tilde{B}_n)$ is a quantum monoidal seed admitting a mutation in all unfrozen directions.

Next, we must establish the third property listed in Theorem 1.4. We do this in Section 6.1 as part of the following result.

Theorem 1.6 (Cf. Proposition 6.10, Lemma 6.17). *There is a $\mathbb{Z}[q^{\pm 1/2}]$ -algebra isomorphism between $K^{GL_n(\mathcal{O}) \rtimes \mathbb{G}_m}(\mathrm{Gr}_{GL_n})$ and $\mathcal{A}_{(L_n, \tilde{B}_n)}^{loc}$. The initial quantum cluster variables are the images of the classes $\{\mathcal{P}_{k,\ell}\}_{k \in [1,n], \ell \in \{0,1\}}$. Moreover, $[\mathcal{P}_{k,\ell}]$ is a quantum cluster variable for all $k \in [1, n], \ell \in \mathbb{Z}$ (up to multiplication by powers of q and frozen variables).*

Here we use the aforementioned construction in [GLS13] of quantum cluster structures on quantum unipotent coordinate rings. Since $\mathcal{A}_{(L_n, \tilde{B}_n)}$ also appears in this context, we obtain a finite set of generators for $\mathcal{A}_{(L_n, \tilde{B}_n)}^{loc}$ from known generating sets of quantum coordinate rings. Using these generators we establish the isomorphism in Theorem 1.6 by showing in Section 2.5 that a certain subset of the $[\mathcal{P}_{k,\ell}]$ generates $K^{GL_n(\mathcal{O}) \rtimes \mathbb{G}_m}(\mathrm{Gr}_{GL_n})$ (a similar result was recently obtained in [FT17]).

Categorification of the quantum twist. Many cluster varieties appearing in Lie theory have distinguished automorphisms called twists [BZ97, FZ99, GY16, MS16, MS17]. These were introduced for applications in total positivity [BFZ96], but have since been understood as significant from various categorical viewpoints [GLS12, KO17]. In particular, after erasing frozen variables they become special cases [GLS12, Wen16a, Wen16b] of the operators introduced in [KS08] to encode the wall-crossing of generalized Donaldson-Thomas invariants—these operators have been referred to as spectrum generators [GMN13c, GMN13b], (half) monodromy operators [ACC+14], DT transformations [GS18], or total DT invariants [Kel11].

In our case we obtain the following interpretation in terms of convolution.

Theorem 1.7 (Cf. Theorem 6.28, Corollary 6.30). *The isomorphism*

$$\mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm 1/2}]} K^{GL_n(\mathcal{O}) \rtimes \mathbb{G}_m}(\mathrm{Gr}_{GL_n}) \cong \mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm 1}]} A_q(N^{(s_1 s_0)^n})$$

identifies the antiautomorphism $[\mathcal{F}] \mapsto [\mathcal{F}^L]$ sending a sheaf to its left dual with the composition $X \mapsto \overline{\eta_{(s_1 s_0)^n}(X)}$ of the quantum twist and the bar involution of $A_{(L_n, \tilde{B}_n)}^{loc}$. Specializing frozen variables (hence necessarily q) to 1 identifies $[\mathcal{F}] \mapsto [\mathcal{F}^L]$ with the total DT invariant of \mathcal{A}_{B_n} .

Here B_n is the matrix obtained by erasing the frozen rows of \tilde{B}_n . The bar involution of a quantum cluster algebra is an antiautomorphism which inverts q and fixes the quantum cluster variables [BZ05]. In Section 6.2 we lift the bar involution of $A_{(L_n, \tilde{B}_n)}^{loc}$ to an autoequivalence of $D_{coh}^{GL_n(\mathcal{O}) \rtimes \mathbb{G}_m}(\mathrm{Gr}_{GL_n})$, the result being similar in flavor to its analogues in the setting of the Steinberg variety and nilpotent cone [Lus98, Ost00].

Theorem 1.7 implies the squared twist is integrable: the double dual $[\mathcal{F}] \mapsto [\mathcal{F}^{LL}]$ preserves the symmetric monoidal subcategory $\mathrm{Rep}(GL_n(\mathcal{O}) \rtimes \mathbb{G}_m)$, embedded as sheaves supported on the identity. In fact, the double dual is very close to being a power of the autoequivalence given by tensoring with $\mathcal{O}(1)$, which itself manifestly leaves invariant objects supported on the identity. This provides a categorical interpretation of the discrete integrability of the $A_{n-1}^{(1)}$ Q -system [Ked08] (see also [Wil16, CDZ14]), which we now recall.

Relation to Q -systems. Q -systems are recursion relations that describe the characters of Kirillov-Reshetikhin (KR) modules, special Yangian modules labeled by multiples of fundamental weights [KR87, HKO+99]. In [Ked08] it was shown that the relations of the Q -system can be normalized to coincide with exchange relations in the cluster algebra \mathcal{A}_{B_n} (or another suitable cluster algebra in other affine types [DFK09, Wil15]).

This connection emerges as follows in our setup. Recall that $\mathrm{Gr}_{GL_n}^{\omega_k^\vee}$ is isomorphic to the finite Grassmannian $\mathrm{Gr}_{k,n}$ of subspaces in \mathbb{C}^n of codimension k , and that $\mathcal{P}_{k,\ell}$ is an equivariant shift of $\mathcal{O}_{\mathrm{Gr}_{k,n}}(\ell)[\frac{1}{2} \dim \mathrm{Gr}_{k,n}]$. Thus, for $\ell \geq 0$, the global sections functor recovers (up to the cohomological shift) the irreducible GL_n -representation of highest weight $\ell\omega_k$. These are exactly the classical limits of the KR modules in type $A_{n-1}^{(1)}$.

On the other hand, the mutations described in Section 2.4 can be used for any $\ell \in \mathbb{Z}$ to produce the sheaf $\mathcal{P}_{k,\ell}$ from the initial cluster $\{\mathcal{P}_{k,\ell}\}_{k \in [1,n], \ell \in \{0,1\}}$. Given a suitable compatibility between convolution of sheaves and tensor products of global sections (see Proposition 6.31), it follows that the associated exchange relations describe characters of KR modules normalized by shifts dictated by the perverse t -structure. These exchange relations are exactly those that appear in [Ked08]. Since we work with sheaves in formal half-integer degrees, the degree normalizations will in general be 4th roots of unity, as found in [DFK09].

We also note that through the coherent Satake category we find a conceptual explanation for the appearance of identical quivers in the context of Q -systems and separately in the context of BPS quivers [ACC+14, CDZ12]. We turn to the latter topic next.

Perspective from gauge theory. Coulomb branches of gauge theories with eight supercharges in three and four dimensions have recently been the subject of intense

study in both physics and mathematics [CHZ14, Nak16, BFN16, BDG17, BDGH16, FT17]. In this language, the spectrum of $K^{G(\mathcal{O})}(\mathrm{Gr}_G)$ is the Coulomb branch of pure 4d $\mathcal{N} = 2$ gauge theory on $\mathbb{R}^3 \times S^1$ with compact gauge group G_c .

Of relevance to us is the monoidal category of line operators in the holomorphic-topological twist of [Kap] on $\mathbb{C} \times \mathbb{R}^2$, with monoidal structure given by collision of lines in \mathbb{R}^2 [KS09]—expectation values of such operators wrapped on S^1 give functions on the Coulomb branch. It was proposed by Costello that this category should be $D_{coh}^{G(\mathcal{O})}(\mathrm{Gr}_G)$ —this is the higher geometric quantization of the shifted cotangent bundle of $G(\mathcal{O}) \backslash \mathrm{Gr}_G$, which appears as a derived space of solutions to the equations of motion of the twisted theory [Cos14a, EY18].

It is expected that the Grothendieck ring of the category of line operators has a basis consisting of Wilson-’t Hooft loop operators [KS09]. Mathematically, the most natural origin of a basis in a Grothendieck ring would be a finite-length t -structure. It was suggested in [KS09] that Wilson-’t Hooft line operators in 4d $\mathcal{N} = 4$ gauge theory correspond to suitable analogues of perverse coherent sheaves on the affine Grassmannian Steinberg variety.

Theorem 1.3 implicitly claims that Wilson-’t Hooft line operators in pure 4d $\mathcal{N} = 2$ gauge theory are simple perverse coherent sheaves on the affine Grassmannian itself. Indeed, irreducible line operators in 4d $\mathcal{N} = 2$ theories satisfying certain finiteness conditions are partially described by the cluster algebras of their BPS quivers [GMN13a, CN14, CDZ17]. The quiver \tilde{Q}_n from Figure 1.1 is exactly the BPS quiver of pure $SU(n)$ gauge theory [ACC+14] (up to frozen vertices present because we focus on GL_n not SL_n). Thus Theorem 1.3 asserts that simple perverse coherent sheaves have the cluster-theoretic properties expected of Wilson-’t Hooft operators.

Theorem 1.4, on the other hand, provides an illuminating conceptual perspective on why cluster algebras should appear in the context of line operators of $\mathcal{N} = 2$ theories at all, complementary to any considerations involving (framed) BPS states or specific properties of class S theories. Indeed, the language of chiral categories succinctly captures the holomorphic dependence of categories of line operators in 4d holomorphic-topological field theories. In this sense Theorem 1.4 says that the phenomenon of iterated mutation of clusters is simply a structural consequence of the basic formal properties of holomorphic-topological field theory. Furthermore, we expect the least satisfying aspect of the theorem (the need to know in advance a cluster structure on the algebra of loop operators) can ultimately be removed; see Conjecture 1.9.

We note that relations between r -matrices and holomorphic-topological field theory are also explored in [Cos13, Cos14b, CWY18a, CWY18b], which establish a direct connection between affine quantum groups and deformed $\mathcal{N} = 1$ gauge theory. Of course, it is quantum loop algebras which provided the examples of monoidal cluster categorifications originally considered in [HL10].

Beyond GL_n . We expect the following extension of Theorem 1.3 to other types.

Conjecture 1.8. *Let C be a finite-type Cartan matrix, let G be the associated simply-connected simple algebraic group, and let G_{Ad} be its adjoint form. Then $\mathcal{P}_{coh}^{G(\mathcal{O}) \rtimes G_m}(\mathrm{Gr}_{G_{Ad}})$ is a monoidal categorification of the quantum cluster algebra*

$\mathcal{A}_{(L_C, \tilde{B}_C)}^{loc}$, where

$$\tilde{B}_C = \begin{pmatrix} C^T - C & -C^T \\ C & 0 \end{pmatrix}$$

and L_C is a suitable coefficient matrix. The elements of the initial monoidal cluster are the sheaves $\mathcal{P}_{\omega_i^\vee, 0}$ and $\mathcal{P}_{\omega_i^\vee, \omega_i}$ for ω_i^\vee a fundamental coweight.

Here $\mathcal{P}_{\lambda^\vee, \mu}$ denotes the simple perverse coherent sheaf supported on $\overline{\text{Gr}}_{G_{Ad}}^{\lambda^\vee}$ and associated to a weight μ (see Section 2.1). On one hand, the matrix \tilde{B}_C is the adjacency matrix of the BPS quiver of 4d $\mathcal{N} = 2$ gauge theory with compact gauge group G_c [ACC+14, CDZ12], so the conjecture is consistent with a correspondence between simple perverse coherent sheaves and simple line operators.

On the other hand, the normalized Q -system associated to G is described by relations among specific cluster variables in $\mathcal{A}_{\tilde{B}_C}$ attached to the weights $\ell\omega_i$ [DFK09]. Under the conjectural isomorphism $K^{G(\mathcal{O})}(\text{Gr}_{G_{Ad}}) \cong \mathcal{A}_{\tilde{B}_C}$ these cluster variables should coincide with the classes of the simple objects $\mathcal{P}_{\omega_i^\vee, \ell\omega_i}$. Indeed, when $\mathcal{P}_{\omega_i^\vee, \ell\omega_i}$ is the restriction of a line bundle on $\text{Gr}_{G_{Ad}}$ (in the appropriate cohomological degree) its global sections are known to be isomorphic as a $G(\mathcal{O})$ -module to the classical limit of the KR module of weight $\ell\omega_i$ [FL07]. This is the case when ℓ satisfies a simple divisibility condition (which is trivial if G is simply laced). We also anticipate an analogue of Conjecture 1.8 for twisted types involving the skew-symmetrizable matrices of [Wil15].

Potentials and monoidal categorification. Recall that a potential on a quiver is a formal sum of oriented cycles [Gin06, DWZ08]. Mutation of quivers with potential is similar to ordinary quiver mutation, except that arbitrary sequences of mutations are not always allowed. Instead, a possible obstruction arises when one mutates at a vertex of an oriented 3-cycle.

Precisely this situation arises in the proof of Theorem 1.4: one can show that a monoidal seed admits mutation sequences that avoid vertices of oriented 3-cycles without assuming in advance the existence of a cluster structure on $K_0(\hat{\mathcal{C}})$ (and without needing to work in a graded setting).

Conjecture 1.9. *Let \mathcal{C} be a rigid monoidal abelian category with a system of renormalized r -matrices, and let $(\{\mathcal{F}_i\}_{i \in I}, \tilde{B})$ be a monoidal seed where \tilde{B} is the signed adjacency matrix of a quiver Q . To this data one can associate a potential W on Q such that a mutation sequence can be performed on $(\{\mathcal{F}_i\}_{i \in I}, \tilde{B})$ in \mathcal{C} if and only if it can be performed on (Q, W) . This potential should encode monoidal factorization properties of objects constructed from the \mathcal{F}_i .*

From the point of view of the conjecture, the hypothesis $K_0(\hat{\mathcal{C}}) \cong \mathcal{A}_{(L, \tilde{B})}$ in Theorem 1.4 should be regarded as a stand-in for knowing the potential associated to $(\{\mathcal{F}_i\}_{i \in I}, \tilde{B})$ is nondegenerate (i.e., allows arbitrary mutation sequences).

Conjecture 1.9 is also consistent with considerations from geometric engineering. Line operators in the 4d $\mathcal{N} = 2$ theory arising by compactifying type IIA/IIB string theory on a noncompact Calabi-Yau 3-fold X may be produced from noncompact B/A -branes in X . When the relevant category of branes is described by a quiver with potential (Q, W) the mutation of suitable collections of branes in X will be controlled by W , hence so should mutation of monoidal seeds in the resulting monoidal category of line operators.

2. THE COHERENT SATAKE CATEGORY

We begin by discussing the category of perverse coherent sheaves on the affine Grassmannian following [AB10, BFM05]. We then describe some structural results in the case $G = GL_n$.

2.1. Definitions and basic properties. Let G be a reductive complex algebraic group and

$$\mathrm{Gr}_G := G(\mathcal{K})/G(\mathcal{O})$$

its affine Grassmannian, where $\mathcal{K} := \mathbb{C}((t))$ and $\mathcal{O} := \mathbb{C}[[t]]$. We fix Cartan and Borel subgroups $H \subset B \subset G$, and let P_+^\vee denote the associated set of dominant elements of the coweight lattice P^\vee of H . The assignment $\lambda^\vee \mapsto G(\mathcal{O}) \cdot [t^{\lambda^\vee}]$ yields a bijection between P_+^\vee and the set of left $G(\mathcal{O})$ -orbits in Gr_G .

The (spherical) Schubert variety $\overline{\mathrm{Gr}}_G^{\lambda^\vee}$ is the closure of this orbit endowed with its reduced scheme structure. We have $\overline{\mathrm{Gr}}_G^{\mu^\vee} \subset \overline{\mathrm{Gr}}_G^{\lambda^\vee}$ exactly when $\mu^\vee \leq \lambda^\vee$. The complement of $\bigcup_{\mu^\vee < \lambda^\vee} \overline{\mathrm{Gr}}_G^{\mu^\vee}$ inside $\overline{\mathrm{Gr}}_G^{\lambda^\vee}$ is the Schubert cell $\mathrm{Gr}_G^{\lambda^\vee}$, which set-theoretically is the $G(\mathcal{O})$ -orbit of t^{λ^\vee} and is open inside $\overline{\mathrm{Gr}}_G^{\lambda^\vee}$. In general Gr_G is not reduced, and as an ind-scheme its reduced locus $(\mathrm{Gr}_G)_{red}$ is the colimit $\varinjlim \overline{\mathrm{Gr}}_G^{\leq \lambda^\vee}$.

The left $G(\mathcal{O})$ -action extends to an action of $G(\mathcal{O}) \rtimes \mathbb{G}_m$, where by convention our \mathbb{G}_m will denote the 4-fold cover of the standard loop rotation. We will write $D_{coh}^{G(\mathcal{O}) \rtimes \mathbb{G}_m}(\mathrm{Gr}_G)$ for the bounded derived category of $G(\mathcal{O}) \rtimes \mathbb{G}_m$ -equivariant coherent sheaves on $(\mathrm{Gr}_G)_{red}$. The reader may refer to [GR14] for generalities on coherent sheaves on ind-schemes. We note in particular that any $\mathcal{F} \in D_{coh}^{G(\mathcal{O}) \rtimes \mathbb{G}_m}(\mathrm{Gr}_G)$ is a pushforward from $D_{coh}^{G(\mathcal{O}) \rtimes \mathbb{G}_m}(\overline{\mathrm{Gr}}_G^{\lambda^\vee})$ for a sufficiently large $\overline{\mathrm{Gr}}_G^{\lambda^\vee}$.

Except in Section 5, we will not really need to distinguish between the classical derived category and its DG/ ∞ -enhancement—the reader may interpret D_{coh} at whichever level they prefer. In the same spirit we write $D_{qcoh}^{G(\mathcal{O}) \rtimes \mathbb{G}_m}(\overline{\mathrm{Gr}}_G^{\lambda^\vee})$ for the (possibly enhanced) unbounded derived category of $G(\mathcal{O}) \rtimes \mathbb{G}_m$ -equivariant quasi-coherent sheaves on $\overline{\mathrm{Gr}}_G^{\lambda^\vee}$.

Note that $D_{coh}^{G(\mathcal{O}) \rtimes \mathbb{G}_m}(\mathrm{Gr}_G)$ has a block decomposition indexed by $\pi_0(\mathrm{Gr}_G) \cong \pi_1(G)$. Crucially for what follows, the dimensions of the $G(\mathcal{O})$ -orbits within a fixed component are of equal parity. With this in mind, we will always consider the formal variant of $D_{coh}^{G(\mathcal{O}) \rtimes \mathbb{G}_m}(\mathrm{Gr}_G)$ where the block attached to a component with odd-dimensional orbits consists of complexes supported in formal degrees $\mathbb{Z} + \frac{1}{2}$.

Definition 2.1 ([AB10]). A $G(\mathcal{O}) \rtimes \mathbb{G}_m$ -equivariant *perverse coherent sheaf* on Gr_G is an object $\mathcal{F} \in D_{coh}^{G(\mathcal{O}) \rtimes \mathbb{G}_m}(\mathrm{Gr}_G)$ such that for every orbit $i_{\lambda^\vee} : \mathrm{Gr}_G^{\lambda^\vee} \hookrightarrow \mathrm{Gr}_G$

- (1) $i_{\lambda^\vee}^*(\mathcal{F}) \in D_{qcoh}^{G(\mathcal{O}) \rtimes \mathbb{G}_m}(\mathrm{Gr}_G^{\lambda^\vee})$ is supported in degrees $\leq -\frac{1}{2} \dim \mathrm{Gr}_G^{\lambda^\vee}$,
- (2) $i_{\lambda^\vee}^!(\mathcal{F}) \in D_{qcoh}^{G(\mathcal{O}) \rtimes \mathbb{G}_m}(\mathrm{Gr}_G^{\lambda^\vee})$ is supported in degrees $\geq -\frac{1}{2} \dim \mathrm{Gr}_G^{\lambda^\vee}$.

We write $\mathcal{P}_{coh}^{G(\mathcal{O}) \rtimes \mathbb{G}_m}(\mathrm{Gr}_G) \subset D_{coh}^{G(\mathcal{O}) \rtimes \mathbb{G}_m}(\mathrm{Gr}_G)$ for the full subcategory of perverse coherent sheaves.

Inherent in the definition is the choice of middle perversity function. By virtue of the above parity condition on orbits, this notion of perverse coherent sheaves enjoys comparable formal properties to its constructible counterpart.

Theorem 2.2 ([AB10]). *The category $\mathcal{P}_{coh}^{G(\mathcal{O}) \times \mathbb{G}_m}(\mathrm{Gr}_G)$ is the heart of a finite-length t -structure on $D_{coh}^{G(\mathcal{O}) \times \mathbb{G}_m}(\mathrm{Gr}_G)$ which is preserved by the duality functor \mathbb{D} . The restriction to $\mathrm{Gr}_G^{\lambda^\vee}$ of a simple perverse coherent sheaf whose support is $\overline{\mathrm{Gr}}_G^{\lambda^\vee}$ is a simple equivariant vector bundle concentrated in cohomological degree $-\frac{1}{2} \dim \mathrm{Gr}_G^{\lambda^\vee}$. This determines a bijection between isomorphism classes of simple objects in*

$$\mathcal{P}_{coh}^{G(\mathcal{O}) \times \mathbb{G}_m}(\mathrm{Gr}_G)$$

and pairs of an orbit together with a simple equivariant vector bundle on it.

Recall the convolution diagram

$$\overline{m} : (G(\mathcal{K}) \times \mathbb{G}_m) \times_{G(\mathcal{O}) \times \mathbb{G}_m} \mathrm{Gr}_G \rightarrow \mathrm{Gr}_G$$

induced from multiplication in $G(\mathcal{K}) \times \mathbb{G}_m$. Given $\mathcal{F}, \mathcal{G} \in D_{coh}^{G(\mathcal{O}) \times \mathbb{G}_m}(\mathrm{Gr}_G)$, we write $\mathcal{F} \widetilde{\boxtimes} \mathcal{G} \in D_{coh}^{G(\mathcal{O}) \times \mathbb{G}_m}((G(\mathcal{K}) \times \mathbb{G}_m) \times_{G(\mathcal{O}) \times \mathbb{G}_m} \mathrm{Gr}_G)$ for their twisted product. This sheaf is characterized by having pullback to $(G(\mathcal{K}) \times \mathbb{G}_m) \times \mathrm{Gr}_G$ isomorphic to the pullback of $\mathcal{F} \boxtimes \mathcal{G} \in D_{coh}^{G(\mathcal{O}) \times \mathbb{G}_m}(\mathrm{Gr}_G \times \mathrm{Gr}_G)$ along projection in the first factor. If \mathcal{F} and \mathcal{G} are pushforwards from $\overline{\mathrm{Gr}}_G^{\lambda_1^\vee}$ and $\overline{\mathrm{Gr}}_G^{\lambda_2^\vee}$, respectively, then $\mathcal{F} \widetilde{\boxtimes} \mathcal{G}$ is a pushforward from the finite-dimensional convolution variety

$$\overline{\mathrm{Gr}}_G^{\lambda_1^\vee} \widetilde{\times} \overline{\mathrm{Gr}}_G^{\lambda_2^\vee} \subset (G(\mathcal{K}) \times \mathbb{G}_m) \times_{G(\mathcal{O}) \times \mathbb{G}_m} \mathrm{Gr}_G.$$

The convolution map determines a monoidal structure on $D_{coh}^{G(\mathcal{O}) \times \mathbb{G}_m}(\mathrm{Gr}_G)$ via

$$\mathcal{F} * \mathcal{G} := \overline{m}_*(\mathcal{F} \widetilde{\boxtimes} \mathcal{G}).$$

Theorem 2.3 ([BFM05]). *If $\mathcal{F}, \mathcal{G} \in D_{coh}^{G(\mathcal{O}) \times \mathbb{G}_m}(\mathrm{Gr}_G)$ are perverse, then so is $\mathcal{F} * \mathcal{G}$.*

If X is a variety equipped with an action of \mathbb{G}_m we denote by $\mathcal{O}_X\{k\}$ the structure sheaf with the \mathbb{G}_m action of weight $2k$. In other words, given a local function $f \in \mathcal{O}_X(U)$, the corresponding section $f' \in \mathcal{O}_X\{k\}(U)$ transforms as $a \cdot f' = a^{-2k}(a \cdot f)$, where $a \cdot f$ denotes the pullback of f along the action of a on U . For example, under the standard \mathbb{G}_m -action on $X = \mathrm{Spec} \mathbb{C}[t]$ multiplication by t induces a morphism $\mathcal{O}_X \rightarrow \mathcal{O}_X\{\frac{1}{2}\}$.

Tensoring with $\mathcal{O}_X\{k\}$ gives us an autoequivalence $\mathcal{F} \mapsto \mathcal{F}\{k\}$ of $D_{coh}^{\mathbb{G}_m}(X)$. At the level of K-theory, the equivariant Grothendieck ring $K^{\mathbb{G}_m}(X)$ is a module over $K^{\mathbb{G}_m}(pt) = \mathbb{Z}[q^{\pm 1/2}]$. Here q acts as shift by $\{-1\}$ (tensoring with $\mathcal{O}_X\{-1\}$).

We write $K^{G(\mathcal{O}) \times \mathbb{G}_m}(\mathrm{Gr}_G)$ for the Grothendieck ring of $D_{coh}^{G(\mathcal{O}) \times \mathbb{G}_m}(\mathrm{Gr}_G)$ (and hence of $\mathcal{P}_{coh}^{G(\mathcal{O}) \times \mathbb{G}_m}(\mathrm{Gr}_G)$). In this case we can also think of multiplication by q as convolution with $[\mathcal{O}_e\{-1\}]$, where e denotes the point Gr_G^0 . It will turn out to be useful to define $\langle 1 \rangle := [1]\{-1\}$.

Since $\mathcal{P}_{coh}^{G(\mathcal{O}) \times \mathbb{G}_m}(\mathrm{Gr}_G)$ has finite length the classes of simple objects provide a basis of $K^{G(\mathcal{O}) \times \mathbb{G}_m}(\mathrm{Gr}_G)$. To a pair $(\lambda^\vee, \mu) \in P^\vee \times P$ we can attach a simple perverse coherent sheaf $\mathcal{P}_{\lambda^\vee, \mu}$ as follows. The orbit $\mathrm{Gr}_G^{\lambda^\vee}$ is an affine space bundle over its \mathbb{G}_m -fixed locus, which is isomorphic to $G/P_{w\lambda^\vee}$ for $P_{w\lambda^\vee}$ the parabolic associated to the unique dominant conjugate $w\lambda^\vee$ of λ^\vee . Acting by the stabilizer of λ^\vee in W we can conjugate μ to become dominant for the Levi factor of $P_{w\lambda^\vee}$ in a unique way. This defines a simple G -equivariant vector bundle on $G/P_{w\lambda^\vee}$.

We extend this to a $G(\mathcal{O}) \rtimes \mathbb{G}_m$ -equivariant bundle by letting t act by zero and letting \mathbb{G}_m act on fibers by the character

$$a \mapsto a^{2\langle \lambda^\vee, \mu \rangle + \dim \text{Gr}_G^{\lambda^\vee}}.$$

The pullback of this bundle to $\text{Gr}_G^{\lambda^\vee}$, shifted to lie in cohomological degree $-\frac{1}{2} \dim \text{Gr}_G^{\lambda^\vee}$, is the restriction of a unique simple perverse coherent sheaf $\mathcal{P}_{\lambda^\vee, \mu}$ whose support is $\overline{\text{Gr}}_G^{\lambda^\vee}$.

Remark 2.4. The choice of \mathbb{G}_m -equivariant structure of $\mathcal{P}_{\lambda^\vee, \mu}$ will be explained by the discussion of the bar involution in Section 6.2. It also results in the mutation exact sequences of Section 2.4 aligning correctly with the conventions of quantum cluster theory.

We refer to pairs $(\lambda^\vee, \mu) \in P^\vee \times P$ for which λ^\vee is dominant and μ is dominant for the Levi factor of P_{λ^\vee} as **dominant pairs**. From the above construction it follows that simple objects (up to \mathbb{G}_m -equivariant shifts) are labeled by $(P^\vee \times P)/W$ (equivalently, by dominant pairs).

Proposition 2.5. *Given dominant pairs (λ_1^\vee, μ_1) and (λ_2^\vee, μ_2) the product $\mathcal{P}_{\lambda_1^\vee, \mu_1} * \mathcal{P}_{\lambda_2^\vee, \mu_2}$ is supported on $\overline{\text{Gr}}_G^{\lambda_1^\vee + \lambda_2^\vee}$. Moreover, in $K^{G(\mathcal{O}) \rtimes \mathbb{G}_m}(\text{Gr}_G)$ we have*

$$[\mathcal{P}_{\lambda_1^\vee, \mu_1} * \mathcal{P}_{\lambda_2^\vee, \mu_2}] = q^{\langle \lambda_1^\vee, \mu_2 \rangle - \langle \lambda_2^\vee, \mu_1 \rangle} [\mathcal{P}_{\lambda_1^\vee + \lambda_2^\vee, \mu_1 + \mu_2}] + \sum_{(\lambda^\vee, \mu) \in S} p_{\lambda^\vee, \mu} [\mathcal{P}_{\lambda^\vee, \mu}],$$

where $p_{\lambda^\vee, \mu} \in \mathbb{Z}[q^{\pm 1/2}]$ and S is a finite collection of dominant pairs (λ^\vee, μ) with either $\lambda^\vee < \lambda_1^\vee + \lambda_2^\vee$ or $\lambda^\vee = \lambda_1^\vee + \lambda_2^\vee$ and $\|\mu\|^2 \leq \|\mu_1\|^2 + \|\mu_2\|^2$ for any W -invariant quadratic form $\|\cdot\|^2$.

Proof. The map $\overline{m} : \overline{\text{Gr}}_G^{\lambda_1^\vee} \times \overline{\text{Gr}}_G^{\lambda_2^\vee} \rightarrow \overline{\text{Gr}}_G^{\lambda_1^\vee + \lambda_2^\vee}$ restricts to an isomorphism over $\text{Gr}_G^{\lambda_1^\vee + \lambda_2^\vee}$. In particular, the restriction of $\mathcal{P}_{\lambda_1^\vee, \mu_1} * \mathcal{P}_{\lambda_2^\vee, \mu_2}$ to $\text{Gr}_G^{\lambda_1^\vee + \lambda_2^\vee}$ is a vector bundle whose fiber over the $P_{\lambda_1^\vee + \lambda_2^\vee}$ -fixed point $[t^{\lambda_1^\vee + \lambda_2^\vee}]$ is $V_{\mu_1} \otimes V_{\mu_2}$ placed in cohomological degree $-\frac{1}{2} \dim \text{Gr}_G^{\lambda_1^\vee + \lambda_2^\vee}$. Here we write V_{μ_i} for the irreducible $P_{\lambda_i^\vee}$ -representation of highest weight μ_i . This bundle need not be equivariantly simple as $V_{\mu_1} \otimes V_{\mu_2}$ need not be simple as a $P_{\lambda_1^\vee + \lambda_2^\vee}$ representation, but any weight μ of $V_{\mu_1} \otimes V_{\mu_2}$ lies in the convex hull of the sums of the W -orbits of μ_1 and μ_2 . Thus it satisfies $\|\mu\|^2 \leq \|\mu_1\|^2 + \|\mu_2\|^2$ for any W -invariant form.

The coefficient of $[\mathcal{P}_{\lambda_1^\vee + \lambda_2^\vee, \mu_1 + \mu_2}]$ is determined by our normalizations of the $\mathcal{P}_{\lambda^\vee, \mu}$ and the fact that $(1, a) \in G(\mathcal{O}) \rtimes \mathbb{G}_m$ acts on $(G(\mathcal{K}) \rtimes \mathbb{G}_m) \times_{G(\mathcal{O}) \rtimes \mathbb{G}_m} \text{Gr}_G$ by

$$(1, a)[t^{\lambda_1^\vee}, 1, [t^{\lambda_2^\vee}]] = [t^{\lambda_1^\vee} a^{4\lambda_1^\vee}, a, [t^{\lambda_2^\vee}]] = [t^{\lambda_1^\vee}, 1, (a^{4\lambda_1^\vee}, a)[t^{\lambda_2^\vee}]].$$

This contributes a factor of $q^{2\langle \lambda_1^\vee, \mu_2 \rangle}$ to the final expression, which is partially canceled by the normalization of $\mathcal{P}_{\lambda_1^\vee + \lambda_2^\vee, \mu_1 + \mu_2}$ to yield the leading coefficient $q^{\langle \lambda_1^\vee, \mu_2 \rangle - \langle \lambda_2^\vee, \mu_1 \rangle}$. □

Recall that a simple object in a monoidal abelian category is **real** if its product with itself is again simple [HL10].

Lemma 2.6 ([Mir]). *Let \mathcal{L} be a $G(\mathcal{O}) \rtimes \mathbb{G}_m$ -equivariant line bundle on Gr_G . Then the restriction of $\mathcal{L}[\frac{1}{2} \dim \text{Gr}_G^{\lambda^\vee}]$ to $\overline{\text{Gr}}_G^{\lambda^\vee}$ is a real simple perverse coherent sheaf for any $\lambda^\vee \in P^\vee$.*

Proof. Perversity and simplicity follow from the trivial case of $\mathcal{L} = \mathcal{O}_{\text{Gr}_G}$ (cf. [AB10, Section 4.1]) since tensoring with \mathcal{L} is an autoequivalence which preserves perversity. Letting \mathcal{P} denote the indicated restriction, we further have

$$\begin{aligned} \mathcal{P} * \mathcal{P} &\cong \overline{m}_* \left(\mathcal{O}_{\overline{\text{Gr}}_G^{\lambda^\vee} \times \overline{\text{Gr}}_G^{\lambda^\vee}} \otimes \overline{m}^* \mathcal{L} \right) [\dim \text{Gr}_G^{\lambda^\vee}] \\ &\cong \mathcal{O}_{\overline{\text{Gr}}_G^{2\lambda^\vee}} \otimes \mathcal{L} \left[\frac{1}{2} \dim \text{Gr}_G^{2\lambda^\vee} \right], \end{aligned}$$

where the second isomorphism follows from the projection formula and the fact that (spherical) Schubert varieties have rational singularities. Reality of \mathcal{P} then follows since as we have just seen the restriction of $\mathcal{L} \left[\frac{1}{2} \dim \text{Gr}_G^{2\lambda^\vee} \right]$ to $\overline{\text{Gr}}_G^{2\lambda^\vee}$ is simple. \square

2.2. Geometry of Gr_{GL_n} . In the remainder of Section 2 we study $\mathcal{P}_{\text{coh}}^{G(\mathcal{O}) \times \mathbb{G}_m}(\text{Gr}_G)$ in greater detail for $G = GL_n$. In this case the geometry of Gr_G can be described via the identification

$$\begin{aligned} \text{Gr}_{GL_n} &\xrightarrow{\sim} \{ \mathcal{O}\text{-lattices inside } \mathcal{K} \otimes_{\mathbb{C}} V \} \\ [g] &\mapsto gL_0, \end{aligned}$$

where $V := \mathbb{C}^n$ and L_0 denotes the standard lattice $\mathcal{O} \otimes_{\mathbb{C}} V$.

When $\lambda^\vee = \omega_k^\vee$ this induces an identification

$$\text{Gr}_{GL_n}^{\omega_k^\vee} = \{ L \overset{k}{\subset} L_0 : tL_0 \subset L \},$$

where $L \overset{k}{\subset} L_0$ indicates that $\dim(L_0/L) = k$. This space is isomorphic to a (finite) Grassmannian of quotient spaces of $V = L_0/tL_0$ and we denote it by $\text{Gr}_{GL_n}^k$ for short.

The convolution spaces of such varieties can similarly be described as

$$\begin{aligned} \text{Gr}_{GL_n}^{k_1} \widetilde{\times} \cdots \widetilde{\times} \text{Gr}_{GL_n}^{k_m} &\xrightarrow{\sim} \{ L_m \overset{k_m}{\subset} \cdots \overset{k_2}{\subset} L_1 \overset{k_1}{\subset} L_0 : tL_{i-1} \subset L_i \} \\ [g_1, \dots, g_m] &\mapsto g_1 \cdots g_m L_0 \subset \cdots \subset g_1 L_0 \subset L_0. \end{aligned}$$

This is an iterated bundle of (finite) Grassmannians which, for convenience, we denote $\text{Gr}_{GL_n}^{(k_1, \dots, k_m)}$. The multiplication map \overline{m} to Gr_{GL_n} is then given by the forgetful map

$$\begin{aligned} \overline{m} : \text{Gr}_{GL_n}^{(k_1, \dots, k_m)} &\rightarrow \text{Gr}_{GL_n} \\ (L_m \subset \cdots \subset L_1 \subset L_0) &\mapsto (L_m \subset L_0). \end{aligned}$$

These spaces carry some obvious tautological vector bundles. For example, one has the vector bundle of rank k_i whose fiber over a point $(L_m \subset \cdots \subset L_0)$ is (L_{i-1}/L_i) . We abuse notation and denote this bundle as L_{i-1}/L_i .

If $i_k : \text{Gr}_{GL_n}^k \hookrightarrow \text{Gr}_{GL_n}$ is the natural inclusion we will write

$$(2.7) \quad \mathcal{P}_{k,\ell} := i_{k*} \left(\mathcal{O}_{\text{Gr}_{GL_n}^k} \otimes \det(L_0/L)^\ell \right) \left\langle \frac{1}{2} \dim \text{Gr}_{GL_n}^k \right\rangle \{-k\ell\},$$

where $\langle 1 \rangle := [1]\{-1\}$. In the notation of Section 2.1 we have $\mathcal{P}_{k,\ell} = \mathcal{P}_{\omega_k^\vee, \ell \omega_k}$. We sometimes simply write \mathcal{P}_k for $\mathcal{P}_{k,0}$. Note that $\mathcal{P}_{0,\ell} \cong \mathcal{O}_e$ for all $\ell \in \mathbb{Z}$, where $e = \text{Gr}_{GL_n}^0$.

We also have the analogous varieties

$$\text{Gr}_{GL_n}^{-k} := \text{Gr}_{GL_n}^{\omega_k^{\vee*}} = \{ L_0 \overset{k}{\subset} L : tL \subset L_0 \}.$$

Here $\omega_k^{\vee*} = -w_0 \omega_k^{\vee}$, where w_0 is the longest element of W . We then define

$$(2.8) \quad \mathcal{P}_{-k,\ell} := i_{-k*} \left(\mathcal{O}_{\text{Gr}_{GL_n}^{-k}} \otimes \det(L/L_0)^\ell \right) \langle \frac{1}{2} \dim \text{Gr}_{GL_n}^{-k} \rangle \{-k\ell\},$$

where $i_{-k} : \text{Gr}_{GL_n}^{-k} \hookrightarrow \text{Gr}_{GL_n}$ is the inclusion. By Lemma 2.6 it follows that $\mathcal{P}_{k,\ell}$ and $\mathcal{P}_{-k,\ell}$ are real. Moreover, it is not difficult to check that the $\mathcal{P}_{n,\ell}$ are invertible and that $\mathcal{P}_{n,\ell}^{-1} \cong \mathcal{P}_{-n,-\ell}$.

2.3. Commutativity relations. To prove Theorem 1.5 we require the following q -commutativity result. It implies in particular that the collection $\{\mathcal{P}_{k,\ell}\}_{k \in [1,n], \ell \in \{0,1\}}$ is a quantum monoidal cluster in $\mathcal{P}_{coh}^{GL_n(\mathcal{O}) \times \mathbb{G}_m}(\text{Gr}_{GL_n})$.

Proposition 2.9. *We have*

$$\mathcal{P}_{k_1,\ell_1} * \mathcal{P}_{k_2,\ell_2} \cong \mathcal{P}_{k_2,\ell_2} * \mathcal{P}_{k_1,\ell_1} \{2(\ell_1 - \ell_2) \min\{k_1, k_2\}\}$$

whenever $|\ell_1 - \ell_2| \leq 1$.

Proof. Consider the convolution diagram

$$\{L_2 \supseteq^{k_2} L_1 \supseteq^{k_1} L_0 : tL_0 \subset L_1, tL_1 \subset L_2\} = \text{Gr}_{GL_n}^{(k_1,k_2)} \xrightarrow{\overline{m}} \overline{\text{Gr}}_{GL_n}^{\omega_{k_1}^{\vee} + \omega_{k_2}^{\vee}}.$$

Then

$$\begin{aligned} & \mathcal{P}_{k_1,\ell_1} * \mathcal{P}_{k_2,\ell_2} \{k_1\ell_1 + k_2\ell_2\} \\ & \cong \overline{m}_* (\mathcal{O}_{\text{Gr}_{GL_n}^{(k_1,k_2)}} \otimes \det(L_0/L_1)^{\ell_1} \otimes \det(L_1/L_2)^{\ell_2}) \langle \frac{1}{2} \dim \text{Gr}_{GL_n}^{(k_1,k_2)} \rangle. \end{aligned}$$

Now $\overline{m}^*(\det(L_0/L_2)) \cong \det(L_0/L_1) \otimes \det(L_1/L_2)$. So, if $\ell_1 = \ell_2 = \ell$, then

$$\begin{aligned} \mathcal{P}_{k_1,\ell} * \mathcal{P}_{k_2,\ell} \{(k_1 + k_2)\ell\} & \cong \overline{m}_* (\mathcal{O}_{\text{Gr}_{GL_n}^{(k_1,k_2)}} \otimes \overline{m}^*(\det(L_0/L_2)^\ell)) \langle \frac{1}{2} \dim \text{Gr}_{GL_n}^{(k_1,k_2)} \rangle \\ & \cong \overline{m}_* (\mathcal{O}_{\text{Gr}_{GL_n}^{(k_1,k_2)}}) \otimes \det(L_0/L_2)^\ell \langle \frac{1}{2} \dim \text{Gr}_{GL_n}^{(k_1,k_2)} \rangle \\ & \cong \mathcal{O}_{\overline{\text{Gr}}_{GL_n}^{\omega_{k_1}^{\vee} + \omega_{k_2}^{\vee}}} \otimes \det(L_0/L_2)^\ell \langle \frac{1}{2} \dim \text{Gr}_{GL_n}^{(k_1,k_2)} \rangle, \end{aligned}$$

where $\overline{m}_*(\mathcal{O}_{\text{Gr}_{GL_n}^{(k_1,k_2)}}) \cong \mathcal{O}_{\overline{\text{Gr}}_{GL_n}^{\omega_{k_1}^{\vee} + \omega_{k_2}^{\vee}}}$ since any $\overline{\text{Gr}}_{GL_n}^{\lambda^\vee}$ has rational singularities. By symmetry this equals $\mathcal{P}_{k_2,\ell} * \mathcal{P}_{k_1,\ell} \{(k_1 + k_2)\ell\}$.

For the second, more complicated case, it suffices by symmetry to let $\ell := \ell_1$ and assume $\ell_2 = \ell + 1$. We will assume $k_1 \leq k_2$, the case $k_1 \geq k_2$ being similar. Then

$$\begin{aligned} & \mathcal{P}_{k_1,\ell} * \mathcal{P}_{k_2,\ell+1} \{k_1\ell + k_2(\ell + 1)\} \\ & \cong \overline{m}_{1*} (\mathcal{O}_{\text{Gr}_{GL_n}^{(k_1,k_2)}} \otimes \det(L_1/L_2)) \otimes \det(L_0/L_2)^\ell \langle \frac{1}{2} \dim \text{Gr}_{GL_n}^{(k_1,k_2)} \rangle, \\ & \mathcal{P}_{k_2,\ell+1} * \mathcal{P}_{k_1,\ell} \{k_1\ell + k_2(\ell + 1)\} \\ & \cong \overline{m}_{2*} (\mathcal{O}_{\text{Gr}_{GL_n}^{(k_2,k_1)}} \otimes \det(L_0/L_1)) \otimes \det(L_0/L_2)^\ell \langle \frac{1}{2} \dim \text{Gr}_{GL_n}^{(k_2,k_1)} \rangle, \end{aligned}$$

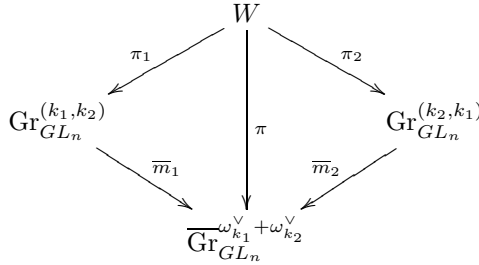
where \overline{m}_1 and \overline{m}_2 are the multiplication maps with domains $\text{Gr}_{GL_n}^{(k_1,k_2)}$ and $\text{Gr}_{GL_n}^{(k_2,k_1)}$, respectively. It remains to show that

$$(2.10) \quad \overline{m}_{1*} (\mathcal{O}_{\text{Gr}_{GL_n}^{(k_1,k_2)}} \otimes \det(L_1/L_2) \{2k_1\}) \cong \overline{m}_{2*} (\mathcal{O}_{\text{Gr}_{GL_n}^{(k_2,k_1)}} \otimes \det(L_0/L_1)).$$

To check this we consider the space

$$W := \{L_2 \overset{k_1}{\subset} L_1 \overset{k_2-k_1}{\subset} L'_1 \overset{k_1}{\subset} L_0 : tL_0 \subset L_1, tL'_1 \subset L_2\}.$$

It fits into a commutative diagram



where the maps π_1 and π_2 are given by forgetting L'_1 and L_1 , respectively. Since

$$\begin{aligned}
 \pi_{1*}(\mathcal{O}_W \otimes \det(L_1/L_2)) &\cong \mathcal{O}_{\mathrm{Gr}_{GL_n}^{(k_1, k_2)}} \otimes \det(L_1/L_2) \quad \text{and} \\
 \pi_{2*}(\mathcal{O}_W \otimes \det(L_0/L'_1)) &\cong \mathcal{O}_{\mathrm{Gr}_{GL_n}^{(k_2, k_1)}} \otimes \det(L_0/L_1)
 \end{aligned}$$

it follows that (2.10) is equivalent to

$$(2.11) \quad \pi_*(\mathcal{O}_W \otimes \det(L_0/L'_1)) \cong \pi_*(\mathcal{O}_W \otimes \det(L_1/L_2))\{2k_1\}.$$

Now consider the map of rank k_1 vector bundles on W given by

$$t : L_0/L'_1 \rightarrow L_1/L_2\{2\}.$$

This map is degenerate (i.e., not an isomorphism) along a divisor $D \subset W$ where

$$\mathcal{O}_W(D) \cong \mathcal{O}_W \otimes \det(L_0/L'_1)^{-1} \otimes \det(L_1/L_2\{2\}).$$

The standard triangle $\mathcal{O}_W(-D) \rightarrow \mathcal{O}_W \rightarrow \mathcal{O}_D$ then gives us the exact triangle

$$\mathcal{O}_W \otimes \det(L_0/L'_1) \otimes \det(L_1/L_2)^{-1}\{-2k_1\} \rightarrow \mathcal{O}_W \rightarrow \mathcal{O}_D.$$

After tensoring with $\det(L_1/L_2)\{2k_1\}$ we arrive at

$$\mathcal{O}_W \otimes \det(L_0/L'_1) \rightarrow \mathcal{O}_W \otimes \det(L_1/L_2)\{2k_1\} \rightarrow \mathcal{O}_D \otimes \det(L_1/L_2)\{2k_1\}.$$

Thus, to show (2.11), it remains to show that

$$(2.12) \quad \pi_*(\mathcal{O}_D \otimes \det(L_1/L_2)) = 0.$$

To show this, we note that D has a partial resolution $\tilde{D} \xrightarrow{f} D$, where

$$\tilde{D} := \{L_2 \overset{k_1-1}{\subset} L''_1 \overset{1}{\subset} L_1 \overset{k_2-k_1}{\subset} L'_1 \overset{k_1}{\subset} L_0 : tL_0 \subset L''_1, tL'_1 \subset L_2\}$$

and f forgets L''_1 . It suffices to show $\pi_* f_*(\mathcal{O}_{\tilde{D}} \otimes \det(L_1/L_2)) = 0$. But we can factor $\pi \circ f$ instead as $\tilde{D} \xrightarrow{f'} D' \rightarrow \overline{\mathrm{Gr}}_{GL_n}^{\omega_{k_1}^\vee + \omega_{k_2}^\vee}$, where

$$D' := \{L_2 \overset{k_1-1}{\subset} L''_1 \overset{k_2-k_1+1}{\subset} L'_1 \overset{k_1}{\subset} L_0 : tL_0 \subset L''_1, tL'_1 \subset L_2\}$$

and f' forgets L_1 . Since $tL_0 \subset L''_1$ the morphism f' is a $\mathbb{P}^{k_2-k_1}$ -bundle, because L_1 can be chosen arbitrarily inside L'_1/L''_1 . The restriction of $\det(L_1/L_2)$ to the fibers of f' is isomorphic to $\mathcal{O}_{\mathbb{P}^{k_2-k_1}}(-1)$. If $k_1 < k_2$, it follows that

$$f'_*(\mathcal{O}_{\tilde{D}} \otimes \det(L_1/L_2)) = 0.$$

If $k_1 = k_2$, then we can repeat the argument by also forgetting L'_1 to obtain this vanishing. This proves (2.12) and completes the proof. \square

Remark 2.13. A simple version of the argument used in the proof of the second case of Proposition 2.9 also shows that \mathcal{P}_{k,ℓ_1} and \mathcal{P}_{n,ℓ_2} q -commute for all ℓ_1, ℓ_2 with

$$\mathcal{P}_{k,\ell_1} * \mathcal{P}_{n,\ell_2} \cong \mathcal{P}_{n,\ell_2} * \mathcal{P}_{k,\ell_1} \{2k(\ell_1 - \ell_2)\}.$$

Lemma 2.14. *For any $k_1, k_2, k'_1, k'_2 \in \mathbb{Z}$ with $k_1 + k_2 = k'_1 + k'_2$ we have*

$$\mathcal{P}_{n,k_1} * \mathcal{P}_{n,k_2} \{2nk_2\} \cong \mathcal{P}_{n,k'_1} * \mathcal{P}_{n,k'_2} \{2nk'_2\}.$$

Proof. On $\text{Gr}_{GL_n}^{(n,n)}$ we have $t : L_0/L_1 \xrightarrow{\sim} L_1/L_2\{2\}$. This implies that

$$\begin{aligned} \mathcal{P}_{n,k_1} * \mathcal{P}_{n,k_2} \{(k_1 + k_2)n\} &\cong \overline{m}_*(\mathcal{O}_{\text{Gr}_{GL_n}^{(n,n)}} \otimes \det(L_0/L_1)^{k_1+k_2} \{-2nk_2\}) \\ &\cong \mathcal{P}_{n,k_1+k_2} * \mathcal{P}_{n,0} \{-2nk_2\}. \end{aligned}$$

Similarly we get $\mathcal{P}_{n,k'_1} * \mathcal{P}_{n,k'_2} \{(k'_1 + k'_2)n\} \cong \mathcal{P}_{n,k_1+k_2} * \mathcal{P}_{n,0} \{-2nk'_2\}$. \square

Remark 2.15. Lemma 2.14 applied repeatedly implies that for any $r \in \mathbb{N}$ we have

$$(\mathcal{P}_{n,1})^r \cong \mathcal{P}_{n,r} * (\mathcal{P}_{n,0})^{r-1} \{-nr(r-1)\}.$$

2.4. Mutation exact sequences. The following family of exact sequences will be used to show that the monoidal cluster $\{\mathcal{P}_{k,\ell}\}_{k \in [1,n], \ell \in \{0,1\}}$ admits mutations in every unfrozen direction.

Proposition 2.16. *For $k \in [0, n], \ell \in \mathbb{Z}$ we have exact sequences*

$$\begin{aligned} 0 \rightarrow \mathcal{P}_{k-1,\ell} * \mathcal{P}_{k+1,\ell} \{-1\} &\rightarrow \mathcal{P}_{k,\ell+1} * \mathcal{P}_{k,\ell-1} \{-2k\} \rightarrow \mathcal{P}_{k,\ell} * \mathcal{P}_{k,\ell} \rightarrow 0, \\ 0 \rightarrow \mathcal{P}_{k,\ell} * \mathcal{P}_{k,\ell} &\rightarrow \mathcal{P}_{k,\ell-1} * \mathcal{P}_{k,\ell+1} \{2k\} \rightarrow \mathcal{P}_{k+1,\ell} * \mathcal{P}_{k-1,\ell} \{1\} \rightarrow 0 \end{aligned}$$

in $\mathcal{P}_{coh}^{GL_n(\mathcal{O}) \rtimes \mathbb{G}_m}(\text{Gr}_{GL_n})$, where in these formulas $\mathcal{P}_{-1,\ell}$ and $\mathcal{P}_{n+1,\ell}$ are interpreted as the zero object.

Proof. We prove the first exact sequence. The second sequence can be proved similarly. Alternatively, the second can be derived from the first by applying the bar involution (see Section 6.2) or, in a more elementary fashion, by applying the functor \mathbb{D} .

We prove the case $\ell = 0$. The general case is proved similarly or by tensoring with the global line bundle $\mathcal{O}(\ell)$. Following the notation from the proof of Proposition 2.9 we have

$$\begin{aligned} \mathcal{P}_{k,1} * \mathcal{P}_{k,-1} &\cong \overline{m}_*(\mathcal{O}_{\text{Gr}_{GL_n}^{(k,k)}} \otimes \det(L_0/L_1) \otimes \det(L_1/L_2)^{-1}) \langle \frac{1}{2} \dim \text{Gr}_{GL_n}^{(k,k)} \rangle, \\ \mathcal{P}_k * \mathcal{P}_k &\cong \overline{m}_*(\mathcal{O}_{\text{Gr}_{GL_n}^{(k,k)}}) \langle \frac{1}{2} \dim \text{Gr}_{GL_n}^{(k,k)} \rangle. \end{aligned}$$

Now, on $\text{Gr}_{GL_n}^{(k,k)}$ we have the map of rank k vector bundles given by

$$t : L_0/L_1 \rightarrow L_1/L_2\{2\}.$$

As in the proof of Proposition 2.9, this map is degenerate along a divisor D where

$$\mathcal{O}_{\text{Gr}_{GL_n}^{(k,k)}}(D) \cong \mathcal{O}_{\text{Gr}_{GL_n}^{(k,k)}} \otimes \det(L_0/L_1)^{-1} \otimes \det(L_1/L_2\{2\}).$$

Substituting into the standard triangle $\mathcal{O}_{\text{Gr}_{GL_n}^{(k,k)}}(-D) \rightarrow \mathcal{O}_{\text{Gr}_{GL_n}^{(k,k)}} \rightarrow \mathcal{O}_D$ we obtain

$$\mathcal{O}_{\text{Gr}_{GL_n}^{(k,k)}} \otimes \det(L_0/L_1) \otimes \det(L_1/L_2)^{-1} \{-2k\} \rightarrow \mathcal{O}_{\text{Gr}_{GL_n}^{(k,k)}} \rightarrow \mathcal{O}_D.$$

Shifting by $\langle \frac{1}{2} \dim \mathrm{Gr}_{GL_n}^{(k,k)} \rangle$ and applying \overline{m}_* we arrive at the exact triangle

$$\overline{m}_*(\mathcal{O}_D \langle \frac{1}{2} \dim \mathrm{Gr}_{GL_n}^{(k,k)} \rangle [-1]) \rightarrow \mathcal{P}_{k,1} * \mathcal{P}_{k,-1} \{-2k\} \rightarrow \mathcal{P}_k * \mathcal{P}_k.$$

Since $\frac{1}{2} \dim \mathrm{Gr}_{GL_n}^{(k,k)} = k(n - k) = \frac{1}{2} \dim \mathrm{Gr}_{GL_n}^{(k-1,k+1)} + 1$ it remains to show that

$$\overline{m}_*(\mathcal{O}_D) \cong \overline{m}_*(\mathcal{O}_{\mathrm{Gr}_{GL_n}^{(k-1,k+1)}}).$$

This follows since D has a resolution

$$\tilde{D} = \{L_2 \overset{k-1}{\subset} L'_1 \overset{1}{\subset} L_1 \overset{k}{\subset} L_0 : tL_0 \subset L'_1, tL_1 \subset L_2\}$$

and we can then factor $\overline{m}_*(\mathcal{O}_D)$ as $\tilde{D} \rightarrow \mathrm{Gr}_{GL_n}^{(k-1,k+1)} \xrightarrow{\overline{m}} \mathrm{Gr}_{GL_n}^{\omega_{k-1}^\vee + \omega_{k+1}^\vee}$, where the first map forgets L_1 and the second forgets L'_1 . \square

2.5. Generating the Grothendieck ring. In addition to the $\mathcal{P}_{k,\ell}$ one can also consider the sheaves

$$\begin{aligned} \mathcal{P}_{k,\ell}^j &:= i_{k*} \left(\mathcal{O}_{\mathrm{Gr}_{GL_n}^k} \otimes \bigwedge^{j_1} (L_0/L) \otimes \cdots \otimes \bigwedge^{j_r} (L_0/L) \otimes \det(L_0/L)^\ell \right) \\ &\cdot \langle \frac{1}{2} k(n - k) \rangle \{-|j| - k\ell\}, \end{aligned}$$

where $\underline{j} = (j_1, \dots, j_r)$ is a sequence with $j_i \in [0, k]$, $|j| = \sum_i j_i$, and $\ell \in \mathbb{Z}$.

Lemma 2.17. *Let $\underline{j} = (j_1, \dots, j_r)$ be a sequence with $j_i \in [1, k]$. Then we have*

$$[\mathcal{P}_{k,\ell}^j] = \sum_{s=0}^{k-1} \sum_{\underline{e} \in \{0,1\}^r} (-1)^s (-q)^{-k+2s+1} [\mathcal{P}_{k-1,\ell}^{\underline{j}-\underline{e},s}] \cdot [\mathcal{P}_{1,-s+|\underline{e}|+\ell}]$$

in $K^{GL_n(\mathcal{O}) \rtimes \mathbb{G}_m}(\mathrm{Gr}_{GL_n})$.

Proof. Consider the varieties

$$\begin{aligned} \{L_2 \overset{1}{\subset} L_1 \overset{k-1}{\subset} L_0 : tL_0 \subset L_2\} &=: W \subset \mathrm{Gr}_{GL_n}^{(k-1,1)} \\ &= \{L_2 \overset{1}{\subset} L_1 \overset{k-1}{\subset} L_0 : tL_0 \subset L_1, tL_1 \subset L_2\}. \end{aligned}$$

Then

$$(2.18) \quad \mathcal{P}_{k,\ell}^j \{-|j| + k\ell\} \cong \overline{m}_* \left(\mathcal{O}_{\mathrm{Gr}_{GL_n}^{(k-1,1)}} \otimes \bigwedge^j (L_0/L_2) \otimes \det(L_0/L_2)^\ell \otimes \mathcal{O}_W \right) \langle \frac{1}{2} k(n - k) \rangle,$$

where $\overline{m} : \mathrm{Gr}_{GL_n}^{(k-1,1)} \rightarrow \mathrm{Gr}_{GL_n}$ is the usual multiplication map given by forgetting L_1 and $\bigwedge^j (L_0/L_2)$ is short for $\bigwedge^{j_1} (L_0/L_2) \otimes \cdots \otimes \bigwedge^{j_r} (L_0/L_2)$.

To deal with \mathcal{O}_W note that $W \subset \mathrm{Gr}_{GL_n}^{(k-1,1)}$ is carved out by the zero section of the map $t : L_0/L_1 \rightarrow L_1/L_2\{2\}$. Thus, if we denote by \mathcal{V} the vector bundle $(L_0/L_1) \otimes (L_1/L_2)^\vee \{-2\}$, we have a Koszul resolution

$$(2.19) \quad \bigwedge^{k-1} \mathcal{V} \rightarrow \cdots \rightarrow \bigwedge^2 \mathcal{V} \rightarrow \mathcal{V} \rightarrow \mathcal{O}_{\mathrm{Gr}_{GL_n}^{(k-1,1)}} \rightarrow \mathcal{O}_W$$

of \mathcal{O}_W on $\mathrm{Gr}_{GL_n}^{(k-1,1)}$. Moreover, we have

$$\bigwedge^s \mathcal{V} \cong \bigwedge^s (L_0/L_1) \otimes \det(L_1/L_2)^{-s} \{-2s\}$$

while, at the level of Grothendieck rings, $[(L_0/L_2)] = [(L_0/L_1)] + [(L_1/L_2)]$ which implies that

$$[\bigwedge^{j_i}(L_0/L_2)] = [\bigwedge^{j_i}(L_0/L_1)] + [\bigwedge^{j_i-1}(L_0/L_1)] \cdot [(L_1/L_2)].$$

Putting this together gives us

$$\begin{aligned} & \left[\overline{m}_* \left(\mathcal{O}_{\mathrm{Gr}_{GL_n}^{(k-1,1)}} \otimes \bigwedge^j(L_0/L_2) \otimes \det(L_0/L_2)^\ell \otimes \bigwedge^s \mathcal{V} \right) \right] \\ &= \sum_{\underline{e} \in \{0,1\}^r} q^{2s} \left[\overline{m}_* \left(\mathcal{O}_{\mathrm{Gr}_{GL_n}^{(k-1,1)}} \otimes \bigwedge^{j-\underline{e}}(L_0/L_1) \otimes \bigwedge^s(L_0/L_1) \right. \right. \\ & \quad \left. \left. \otimes \det(L_0/L_1)^\ell \otimes \det(L_1/L_2)^{-s+|\underline{e}|+\ell} \right) \right] \\ &= \sum_{\underline{e} \in \{0,1\}^r} q^{2s} q^{-|\underline{j}|-k\ell} [\mathcal{P}_{k-1,\ell}^{(j-\underline{e},s)} \langle -\frac{1}{2}(k-1)(n-k+1) \rangle] \\ & \quad \cdot [\mathcal{P}_{1,-s+|\underline{e}|+\ell} \langle -\frac{1}{2}(n-1) \rangle], \end{aligned}$$

where $|\underline{e}| := \sum_i e_i$. The result follows by combining this identity with (2.18) and the Koszul resolution (2.19) (while keeping in mind that the class of $\langle 1 \rangle$ in K-theory is $-q$). □

Corollary 2.20. *The Grothendieck ring of $\mathcal{P}_{coh}^{GL_n(\mathcal{O}) \times \mathbb{G}_m}(\mathrm{Gr}_{GL_n})$ is generated as a $\mathbb{Z}[q^{\pm 1/2}]$ -algebra by $[\mathcal{P}_{1,\ell}]$ for $\ell \in \mathbb{Z}$ and $[\mathcal{P}_n]^{-1}$.*

Proof. For any dominant $\lambda = \sum_{i=1}^r \omega_{k_i}$ we have the resolution

$$\overline{m} : \mathrm{Gr}_{GL_n}^{k_1} \widetilde{\times} \cdots \widetilde{\times} \mathrm{Gr}_{GL_n}^{k_r} \rightarrow \overline{\mathrm{Gr}}_{GL_n}^\lambda.$$

By the projection formula and the fact $\overline{\mathrm{Gr}}_{GL_n}^\lambda$ has rational singularities the map \overline{m}_* is surjective on K-theory. Moreover, $K^{GL_n(\mathcal{O}) \times \mathbb{G}_m}(\mathrm{Gr}_{GL_n}^{k_1} \widetilde{\times} \cdots \widetilde{\times} \mathrm{Gr}_{GL_n}^{k_r})$ is isomorphic to

$$K^{GL_n(\mathcal{O}) \times \mathbb{G}_m}(\mathrm{Gr}_{GL_n}^{k_1}) \otimes_{K^{GL_n(\mathcal{O}) \times \mathbb{G}_m}(pt)} \cdots \otimes_{K^{GL_n(\mathcal{O}) \times \mathbb{G}_m}(pt)} K^{GL_n(\mathcal{O}) \times \mathbb{G}_m}(\mathrm{Gr}_{GL_n}^{k_r}).$$

Thus it suffices to show generation of the (equivariant) K-theory of the fixed point $e \in \mathrm{Gr}_{GL_n}$ and of $\mathrm{Gr}_{GL_n}^k$ for $k = 1, \dots, n$.

The former is generated (as an algebra) by

$$\mathcal{O}_e \otimes [\bigwedge^i(L_0/tL_0)] \cong [\mathcal{P}_n^i] \cdot [\mathcal{P}_n^{-1}],$$

where, following the notation introduced above and used in Lemma 2.17, $\mathcal{P}_n^i = \mathcal{O}_{\mathrm{Gr}_{GL_n}^n} \otimes \bigwedge^i(L_0/L)$. On the other hand, applying Lemma 2.17 repeatedly shows that $[\mathcal{P}_n^i]$ is generated by the classes $\{[\mathcal{P}_{1,\ell}]\}_{\ell \in \mathbb{Z}}$.

The latter is generated, as an algebra, by the equivariant K-theory of $e \in \mathrm{Gr}_{GL_n}$ and by $[\mathcal{P}_k^i]$ for $i = 0, \dots, k$. Once again, Lemma 2.17 shows that the $[\mathcal{P}_k^i]$ are generated by the classes $\{[\mathcal{P}_{1,\ell}]\}_{\ell \in \mathbb{Z}}$. □

Remark 2.21. A similar argument as in the proof of Corollary 2.20 can be used to show that $\mathcal{P}_{coh}^{GL_n(\mathcal{O}) \times \mathbb{G}_m}(\mathrm{Gr}_{GL_n})$ is generated by $[\mathcal{P}_{1,\ell}]$ and $[\mathcal{P}_{-1,\ell}]$ (without the need for $[\mathcal{P}_n]^{-1}$). This gives another (more geometric) proof of [FT17, Theorem 9.3].

3. RIGIDITY

In this section we construct the left and right duals (of convolution) in the coherent Satake category, and more generally in $D_{coh}^{G(\mathcal{O}) \rtimes \mathbb{G}_m}(\mathrm{Gr}_G)$. We illustrate the construction with some explicit examples in the case $G = GL_n$.

Let us briefly contrast our setting with that of the constructible Satake category $\mathcal{P}^{G(\mathcal{O})}(\mathrm{Gr}_G)$. This category is symmetric monoidal and its global sections functor is a fiber functor. Thus following [DM82, Proposition 1.20] to establish its rigidity it suffices to show that objects with one-dimensional image under the fiber functor are invertible. This condition is moreover sufficient to apply Tannakian reconstruction, leading in the case of $\mathcal{P}^{G(\mathcal{O})}(\mathrm{Gr}_G)$ to the geometric Satake equivalence (cf. [BR17, Theorem 2.7(3)]).

However, the global sections functor on the coherent Satake category is not a fiber functor (it does not respect the monoidal structure), and its convolution product is not symmetric. This forces us to prove rigidity by directly checking the usual properties of duals. An adaptation of our proof should also apply to the constructible Satake category to recover rigidity in that case.

3.1. Left and right duals. We begin with the diagram

$$\mathrm{Gr}_G \xleftarrow{\pi} (G(\mathcal{K}) \rtimes \mathbb{G}_m) \times_{G(\mathcal{O}) \rtimes \mathbb{G}_m} \mathrm{Gr}_G \xrightarrow{\overline{m}} \mathrm{Gr}_G,$$

where $\pi(g, x) = [g]$ and $\overline{m}(g, x) = [gx]$. The projection π has a section

$$s : \mathrm{Gr}_G \rightarrow (G(\mathcal{K}) \rtimes \mathbb{G}_m) \times_{G(\mathcal{O}) \rtimes \mathbb{G}_m} \mathrm{Gr}_G$$

given by $s([g]) = (g, [g^{-1}])$.

For $\mathcal{F} \in D_{coh}^{G(\mathcal{O}) \rtimes \mathbb{G}_m}(\mathrm{Gr}_G)$ we define $\mathcal{F}^* := s^*(\mathcal{O} \boxtimes \mathcal{F})$. More concretely, suppose \mathcal{F} is supported on $\overline{\mathrm{Gr}}_G^{\lambda^\vee}$. Let $\lambda^{\vee*} := -w_0 \lambda^\vee$ denote the dominant conjugate of $-\lambda^\vee$. Then s restricts to a section

$$s : \overline{\mathrm{Gr}}_G^{\lambda^{\vee*}} \rightarrow \overline{\mathrm{Gr}}_G^{\lambda^{\vee*}} \widetilde{\times} \overline{\mathrm{Gr}}_G^{\lambda^\vee}$$

and $\mathcal{F}^* \cong s^*(\mathcal{O}_{\overline{\mathrm{Gr}}_G^{\lambda^{\vee*}}} \widetilde{\boxtimes} \mathcal{F})$. Since $\overline{\mathrm{Gr}}_G^{\lambda^{\vee*}} \widetilde{\times} \overline{\mathrm{Gr}}_G^{\lambda^\vee}$ is a locally trivial fibration it follows that \mathcal{F}^* is also coherent on $\overline{\mathrm{Gr}}_G^{\lambda^{\vee*}}$. It is easy to see that $\mathcal{F} \mapsto \mathcal{F}^*$ is an involution of $D_{coh}^{G(\mathcal{O}) \rtimes \mathbb{G}_m}(\mathrm{Gr}_G)$.

Theorem 3.1. *The monoidal category $D_{coh}^{G(\mathcal{O}) \rtimes \mathbb{G}_m}(\mathrm{Gr}_G)$ is rigid. The left and right duals of $\mathcal{F} \in D_{coh}^{G(\mathcal{O}) \rtimes \mathbb{G}_m}(\mathrm{Gr}_G)$ are $\mathcal{F}^L = \mathbb{D}(\mathcal{F}^*)$ and $\mathcal{F}^R = (\mathbb{D}\mathcal{F})^*$.*

We prove this result starting in Section 3.4. It yields the following corollary.

Corollary 3.2. *The left and right duals of a perverse sheaf are again perverse, hence $\mathcal{P}_{coh}^{G(\mathcal{O}) \rtimes \mathbb{G}_m}(\mathrm{Gr}_G)$ is rigid.*

Proof. Since \mathbb{D} preserves $\mathcal{P}_{coh}^{G(\mathcal{O}) \rtimes \mathbb{G}_m}(\mathrm{Gr}_G)$ it remains to show that $\mathcal{F} \mapsto \mathcal{F}^*$ also preserves it.

Consider the finite-dimensional convolution variety

$$i_{(\lambda^\vee, \lambda^{\vee*})} : \mathrm{Gr}_G^{\lambda^\vee} \widetilde{\times} \mathrm{Gr}_G^{\lambda^{\vee*}} \hookrightarrow (G(\mathcal{K}) \rtimes \mathbb{G}_m) \times_{G(\mathcal{O}) \rtimes \mathbb{G}_m} \mathrm{Gr}_G.$$

Since \mathcal{F} is perverse and $\mathrm{Gr}_G^{\lambda^\vee} \tilde{\times} \mathrm{Gr}_G^{\lambda^{\vee*}}$ is a Zariski-locally trivial fibration, $i_{(\lambda^\vee, \lambda^{\vee*})}^*(\mathcal{O} \tilde{\boxtimes} \mathcal{F})$ is supported in degrees $\leq \frac{1}{2} \dim \mathrm{Gr}_G^{\lambda^{\vee*}}$. From the commutativity of

$$\begin{array}{ccc} \mathrm{Gr}_G^{\lambda^\vee} & \xrightarrow{i_{\lambda^\vee}} & \mathrm{Gr}_G \\ \downarrow s & & \downarrow s \\ \mathrm{Gr}_G^{\lambda^\vee} \tilde{\times} \mathrm{Gr}_G^{\lambda^{\vee*}} & \xrightarrow{i_{(\lambda^\vee, \lambda^{\vee*})}} & (G(\mathcal{K}) \rtimes \mathbb{G}_m) \times_{G(\mathcal{O}) \rtimes \mathbb{G}_m} \mathrm{Gr}_G \end{array}$$

it follows that $i_{\lambda^\vee}^* \mathcal{F}^*$ is supported in degrees $\leq \frac{1}{2} \dim \mathrm{Gr}_G^{\lambda^\vee}$ (where we used that $\dim \mathrm{Gr}_G^{\lambda^\vee} = \dim \mathrm{Gr}_G^{\lambda^{\vee*}}$). The analogous statement for $i_{\lambda^\vee}^! \mathcal{F}^*$ follows similarly. \square

Remark 3.3. In general \mathcal{F}^L and \mathcal{F}^R are not isomorphic. This is in contrast to the constructible case where left and right duals in $\mathcal{P}^{G(\mathcal{O})}(\mathrm{Gr}_G) \cong \mathrm{Rep} G^\vee$ agree.

The difference between \mathcal{F}^L and \mathcal{F}^R in our case can be traced back to the fact that $(\omega_{\overline{\mathrm{Gr}}_G^{\lambda^\vee}})^*$ (defined using the pullback s^*) is *not* isomorphic to $\omega_{\overline{\mathrm{Gr}}_G^{\lambda^{\vee*}}}$. However, they are isomorphic in the constructible case (replacing the coherent dualizing sheaf with the constructible one).

3.2. The adjunction maps. We now describe the maps relating \mathcal{F} , \mathcal{F}^L , and \mathcal{F}^R . From here on fix $Y = \overline{\mathrm{Gr}}_G^{\lambda^\vee}$ and $Y^* = \overline{\mathrm{Gr}}_G^{\lambda^{\vee*}}$. We will write $D_{coh}(Y)$ for $D_{coh}^{G(\mathcal{O}) \rtimes \mathbb{G}_m}(\mathrm{Gr}_G^{\lambda^\vee})$ and likewise for Y^* . We have the map $s : Y^* \rightarrow Y^* \tilde{\times} Y$ and, for $\mathcal{F} \in D_{coh}(Y)$, we can consider the composition

$$(3.4) \quad \mathcal{F}^L \tilde{\boxtimes} \mathcal{F} = \mathbb{D}(\mathcal{F}^*) \tilde{\boxtimes} \mathcal{F} \xrightarrow{adj} s_* s^*(\mathcal{F}^L \tilde{\boxtimes} \mathcal{F}) \cong s_*(\mathbb{D}(\mathcal{F}^*) \otimes \mathcal{F}^*) \xrightarrow{m} s_* \omega_{Y^*},$$

where m is the natural evaluation map $\mathcal{H}om(\mathcal{F}^*, \omega_{Y^*}) \otimes \mathcal{F}^* \rightarrow \omega_{Y^*}$. Since $p := \overline{m} \circ s$ projects Y^* to the point $e \in \mathrm{Gr}_G$ we have an adjunction map

$$p_* \omega_{Y^*} = p_* p^! \mathcal{O}_e \rightarrow \mathcal{O}_e.$$

Composing with the pushforward of (3.4) to Gr_G we thus obtain

$$(3.5) \quad \mathcal{F}^L * \mathcal{F} \rightarrow \overline{m}_* s_* \omega_{Y^*} \xrightarrow{adj} \mathcal{O}_e.$$

To define the map $\mathcal{O}_e \rightarrow \mathcal{F} * \mathcal{F}^L$ we now let $s : Y \rightarrow Y \tilde{\times} Y^*$ and note that

$$(3.6) \quad s^!(\mathcal{F} \tilde{\boxtimes} \mathcal{F}^L) = s^!(\mathcal{F} \tilde{\boxtimes} \mathbb{D}(\mathcal{F}^*)) \cong \mathbb{D} s^* \mathbb{D}(\mathcal{F} \tilde{\boxtimes} \mathbb{D}(\mathcal{F}^*)) \cong \mathbb{D}(\mathbb{D}(\mathcal{F}) \otimes \mathcal{F}),$$

where in the last isomorphism we use the fact that $\mathcal{F}^{**} \cong \mathcal{F} \cong \mathbb{D} \mathbb{D}(\mathcal{F})$. Moreover, we have

$$\mathcal{O}_Y \xrightarrow{!} \mathcal{H}om(\mathbb{D}(\mathcal{F}), \mathbb{D}(\mathcal{F})) \cong \mathbb{D}(\mathbb{D}(\mathcal{F}) \otimes \mathcal{F})$$

which gives us maps

$$\mathcal{O}_Y \rightarrow s^!(\mathcal{F} \tilde{\boxtimes} \mathcal{F}^L) \Leftrightarrow s_* \mathcal{O}_Y \rightarrow \mathcal{F} \tilde{\boxtimes} \mathcal{F}^L.$$

We now push forward \overline{m} to Gr_G . As before, $p := \overline{m} \circ s$ projects Y to the point $e \in \mathrm{Gr}_G$ and we have a composition

$$(3.7) \quad \mathcal{O}_e \xrightarrow{adj} \overline{m}_* s_* \mathcal{O}_Y \rightarrow \mathcal{F} * \mathcal{F}^L,$$

where the first map is the natural adjunction $\mathcal{O}_e \rightarrow p_* p^* \mathcal{O}_e = p_* \mathcal{O}_Y$.

The maps involving \mathcal{F}^R are defined similarly. Equivalently they can be obtained from those above. For example, replacing \mathcal{F} with \mathcal{F}^R in (3.5) we get a map $(\mathcal{F}^R)^L * \mathcal{F}^R \rightarrow \mathcal{O}_e$. But $(\mathcal{F}^R)^L \cong \mathbb{D}(\mathbb{D}(\mathcal{F}^R)^{**}) \cong \mathcal{F}$ so this gives the map $\mathcal{F} * \mathcal{F}^R \rightarrow \mathcal{O}_e$.

3.3. Some examples. Before proceeding to the proof of Theorem 3.1 we illustrate the process of applying $(\cdot)^*$ and computing duals in the case $G = GL_n$.

Lemma 3.8. *We have $\mathcal{P}_{k,\ell}^* \cong \mathcal{P}_{-k,\ell} \cong \mathcal{P}_{n,-\ell}^{-1} * \mathcal{P}_{n-k,-\ell}$.*

Proof. Consider $\mathrm{Gr}_{GL_n}^{-k} \widetilde{\times} \mathrm{Gr}_{GL_n}^k = \{L_0 \overset{k}{\subset} L_1 \supset L_2 : tL_1 \subset L_0, tL_1 \subset L_2\}$. We have the section

$$s : \mathrm{Gr}_{GL_n}^{-k} \rightarrow \mathrm{Gr}_{GL_n}^{-k} \widetilde{\times} \mathrm{Gr}_{GL_n}^k, \quad (L_0 \overset{k}{\subset} L) \mapsto (L_0 \overset{k}{\subset} L \supset L_0),$$

from which we see that

$$\begin{aligned} \mathcal{P}_{k,\ell}^* &= s^*(\mathcal{O}_{\mathrm{Gr}_{GL_n}^{-k}} \widetilde{\boxtimes} \mathcal{P}_{k,\ell}) \\ &\cong s^*(\mathcal{O}_{\mathrm{Gr}_{GL_n}^{-k}} \widetilde{\boxtimes} \mathrm{Gr}_{GL_n}^k \otimes \det(L_1/L_2)^\ell \langle \frac{1}{2}k(n-k) \rangle \{-k\ell\}) \cong \mathcal{P}_{-k,\ell}. \end{aligned}$$

To rewrite $\mathcal{P}_{-k,\ell}$ consider the convolution

$$\mathcal{P}_{-k,\ell} * \mathcal{P}_n \cong \mathcal{O}_{\mathrm{Gr}_{GL_n}^{n-k}} \otimes \det(t^{-1}L/L_0)^\ell \langle \frac{1}{2}k(n-k) \rangle \{-k\ell\}.$$

Now on $\mathrm{Gr}_{GL_n}^{n-k}$ the isomorphism $t^{-1}L/L_0 \xrightarrow{\sim} L/tL_0\{2\}$ implies that

$$\det(t^{-1}L/L_0)^\ell \cong \det(L/tL_0)^\ell \{2k\ell\} \cong \det(L_0/tL_0)^\ell \otimes \det(L_0/L)^{-\ell} \{2k\ell\}.$$

It follows that

$$\mathcal{P}_{-k,\ell} * \mathcal{P}_n \cong \mathcal{P}_{n,\ell} * \mathcal{P}_n^{-1} * \mathcal{P}_{n-k,-\ell} \{2k\ell\},$$

where we use that tensoring with $\det(L_0/tL_0)^\ell$ is equivalent to convolving on the left with $\mathcal{P}_{n,\ell} * \mathcal{P}_{-n}$. The result now follows since by Remark 2.13 we have

$$\begin{aligned} \mathcal{P}_{n-k,-\ell} * \mathcal{P}_n^{-1} &\cong \mathcal{P}_n^{-1} * \mathcal{P}_{n-k,-\ell} \{2(n-k)\ell\}, \\ \mathcal{P}_{n,\ell} * \mathcal{P}_n^{-2} &\cong \mathcal{P}_{n,-\ell}^{-1} \{-2n\ell\}. \end{aligned} \quad \square$$

The usual calculation of the dualizing sheaf of the finite Grassmannian implies

$$(3.9) \quad \omega_{\mathrm{Gr}_{GL_n}^k} \cong \mathcal{O}_{\mathrm{Gr}_{GL_n}^k} \otimes \det(L_0/L)^{-n} \otimes \det(L_0/tL_0)^k [k(n-k)].$$

Using this it is not hard to show that

$$\mathbb{D}\mathcal{P}_{k,\ell} \cong \mathcal{P}_{n,k} * \mathcal{P}_n^{-1} * \mathcal{P}_{k,-\ell-n} \{k(n-k)\}.$$

Putting this together with Theorem 3.1 and Lemma 3.8 we obtain the following.

Proposition 3.10. *The left and right duals of $\mathcal{P}_{k,\ell}$ are given by*

$$\begin{aligned} \mathcal{P}_{k,\ell}^L &= \mathbb{D}(\mathcal{P}_{k,\ell}^*) \cong \mathcal{P}_{n,k+\ell-n}^{-1} * \mathcal{P}_{n-k,\ell-n} \{k(n-k)\}, \\ \mathcal{P}_{k,\ell}^R &= (\mathbb{D}\mathcal{P}_{k,\ell})^* \cong \mathcal{P}_{n,-k+\ell+n}^{-1} * \mathcal{P}_{n-k,\ell+n} \{-k(n-k)\}. \end{aligned}$$

Remark 3.11. In fact, if one does not insist on having everything written in terms of convolutions, the expressions above can be written more simply as

$$\begin{aligned} \mathcal{P}_{k,\ell}^L &\cong \mathcal{P}_{-k,-\ell} \otimes \omega_{\mathrm{Gr}_{GL_n}^{-k}} \langle -k(n-k) \rangle, \\ \mathcal{P}_{k,\ell}^R &\cong \mathcal{P}_{-k,-\ell} \otimes \omega_{\mathrm{Gr}_{GL_n}^{-k}}^{-1} \langle k(n-k) \rangle, \end{aligned}$$

where $\omega_{\mathrm{Gr}_{GL_n}^{-k}} \cong \mathcal{O}_{\mathrm{Gr}_{GL_n}^{-k}} \otimes \det(L/L_0)^n \otimes \det(t^{-1}L_0/L_0)^{-k} [k(n-k)]$ is the dualizing sheaf.

3.4. Outline of the proof. We begin the proof of Theorem 3.1 by outlining the argument.

It suffices to show that the composition

$$(3.12) \quad \mathcal{F} = \mathcal{O}_e * \mathcal{F} \rightarrow \mathcal{F} * \mathcal{F}^L * \mathcal{F} \rightarrow \mathcal{F} * \mathcal{O}_e = \mathcal{F}$$

is an isomorphism, as is the corresponding identity with \mathcal{F}^R in place of \mathcal{F}^L . As both identities may be shown via the same approach, we will explicitly focus on (3.12). To be more precise, showing composition (3.12) and its counterpart are the identity maps would establish Theorem 3.1 and that the unit and counit maps are exactly as indicated. However, suppose we only know that the composition in (3.12) is some isomorphism f . Then replacing the map $\mathcal{O}_e \rightarrow \mathcal{F} * \mathcal{F}^L$ with

$$\mathcal{O}_e \rightarrow \mathcal{F} * \mathcal{F}^L \xrightarrow{f^{-1} * id} \mathcal{F} * \mathcal{F}^L$$

makes (3.12) equal to the identity on the nose. This is sufficient to prove Theorem 3.1, and showing that (3.12) is an isomorphism (rather than the identity) will simplify our arguments.

To understand the composition in (3.12) the following diagram plays a large role:

$$(3.13) \quad \begin{array}{ccc} Y & \xrightarrow{j_1} & Y \widetilde{\times} Y^* \\ j_2 \downarrow & & \downarrow i_2 \\ Y \times Y & \xrightarrow{i_1} & Y \widetilde{\times} Y^* \widetilde{\times} Y \end{array}$$

Here j_1, i_1 , and i_2 are induced by s while j_2 is the diagonal. The map in (3.12) is obtained by pushing forward to Gr_G the composition

$$(3.14) \quad i_{1*}(\mathcal{O}_Y \boxtimes \mathcal{F}) \rightarrow \mathcal{F} \widetilde{\boxtimes} \mathbb{D}(\mathcal{F}^*) \widetilde{\boxtimes} \mathcal{F} \rightarrow i_{2*}(\mathcal{F} \widetilde{\boxtimes} \omega_{Y^*}).$$

This map is adjoint to the composition

$$(3.15) \quad \mathcal{O}_Y \boxtimes \mathcal{F} \rightarrow i_1^!(\mathcal{F} \widetilde{\boxtimes} \mathbb{D}(\mathcal{F}^*) \widetilde{\boxtimes} \mathcal{F}) \rightarrow i_1^! i_{2*}(\mathcal{F} \widetilde{\boxtimes} \omega_{Y^*}) \cong j_{2*} j_1^!(\mathcal{F} \widetilde{\boxtimes} \omega_{Y^*}) \cong j_{2*}(\mathcal{F}).$$

The first isomorphism is a consequence of the fact that the square in (3.13) is independent (see Lemma 3.16). The difficult part is to show that the composition in (3.15) is the obvious restriction morphism (see Section 3.6). The rest of the argument, which is easier, uses this fact to show that the composition in (3.12) is an isomorphism (see Section 3.7).

Of the maps in (3.15) it is the second one which is harder to understand. This is done in Section 3.5. We have $i_1^!(\mathcal{F} \widetilde{\boxtimes} \mathbb{D}(\mathcal{F}^*) \widetilde{\boxtimes} \mathcal{F}) \cong \mathcal{H}om(\mathbb{D}(\mathcal{F}), \mathbb{D}(\mathcal{F})) \widetilde{\boxtimes} \mathcal{F}$ and the goal is to show that the second map in (3.15) is equivalent to the composition

$$\begin{aligned} \mathcal{H}om(\mathbb{D}(\mathcal{F}), \mathbb{D}(\mathcal{F})) \widetilde{\boxtimes} \mathcal{F} &\rightarrow j_{2*} \mathcal{H}om(\mathbb{D}(\mathcal{F}), \mathbb{D}(\mathcal{F}) \otimes \mathcal{F}) \\ &\xrightarrow{m} j_{2*} \mathcal{H}om(\mathbb{D}(\mathcal{F}), \omega_Y) \cong j_{2*} \mathcal{F}, \end{aligned}$$

where the first map above is restriction composed with the natural map

$$\mathcal{H}om(\mathbb{D}(\mathcal{F}), \mathbb{D}(\mathcal{F})) \otimes \mathcal{F} \rightarrow \mathcal{H}om(\mathbb{D}(\mathcal{F}), \mathbb{D}(\mathcal{F}) \otimes \mathcal{F}).$$

This is the content of the diagram in Proposition 3.19 on taking $\mathcal{F} = \mathcal{G}$ so that $\mathcal{A} = \mathcal{F} \widetilde{\boxtimes} \mathbb{D}(\mathcal{F}^*) \widetilde{\boxtimes} \mathcal{F}$.

In carrying out this analysis we will be composing operations such as tensor product which need not preserve D_{coh} on the varieties involved, as they are generally singular. Thus our arguments necessarily take place in the unbounded derived

category D_{qcoh} of quasicohherent sheaves, even though the statement of Theorem 3.1 refers only to D_{coh} . We collect in an appendix various results about D_{qcoh} that we will need. Since on a technical level the rest of Section 3 is independent from those that follow, we emphasize that the reader who wishes to skip to Section 4 may safely do so.

We make frequent use of the natural maps $\iota : \mathcal{O}_X \rightarrow \mathcal{H}om(\mathcal{F}, \mathcal{F})$ and $m : \mathcal{H}om(\mathcal{F}, \mathcal{G}) \otimes \mathcal{F} \rightarrow \mathcal{G}$. A particular instance of m which will appear frequently is the map

$$\mathbb{D}(\mathcal{F}) \otimes \mathcal{F} = \mathcal{H}om(\mathcal{F}, \omega_X) \otimes \mathcal{F} \xrightarrow{m} \omega_X,$$

where $\mathbb{D}(\mathcal{F}) = \mathcal{H}om(\mathcal{F}, \omega_X)$ is the standard duality functor. For $\mathcal{F} \in D_{coh}(X)$ we have $\mathbb{D}\mathbb{D}(\mathcal{F}) \cong \mathcal{F}$, but for a general $\mathcal{F} \in D_{qcoh}(X)$ we just have a map $\mathcal{F} \rightarrow \mathbb{D}\mathbb{D}(\mathcal{F})$ induced by a natural transformation $\tau : id \rightarrow \mathbb{D}\mathbb{D}$.

Lemma 3.16. *The square in (3.13) is independent in the sense of Definition A.11.*

Proof. We need to show that $i_2^* i_{1*}(\mathcal{M}) \xrightarrow{\sim} j_{1*} j_2^*(\mathcal{M})$ for $\mathcal{M} \in D_{qcoh}(Y \times Y)$. Using the commutativity of [Lip09, Eq. 3.10.2.3] it suffices to consider the case $\mathcal{M} = \mathcal{O}_{Y \times Y}$.

Consider the following commutative diagram which extends (3.13):

$$(3.17) \quad \begin{array}{ccc} Y & \xrightarrow{j_1} & Y \widetilde{\times} Y^* \\ j_2 \downarrow & & \downarrow i_2 \\ Y \times Y & \xrightarrow{i_1} & Y \widetilde{\times} Y^* \widetilde{\times} Y \\ p_2 \downarrow & & \downarrow p_1 \\ Y & \xrightarrow{j_1} & Y \widetilde{\times} Y^* \end{array}$$

Here p_1 and p_2 forget the last factor Y and hence both vertical compositions are the identity. Now consider the composition

$$(3.18) \quad i_2^* p_1^* j_{1*}(\mathcal{O}_Y) \rightarrow i_2^* i_{1*} p_2^*(\mathcal{O}_Y) \rightarrow j_{1*} j_2^* p_2^*(\mathcal{O}_Y).$$

Since $p_2 \circ j_2$ and $p_1 \circ i_2$ are both the identity it follows from Lemma A.19 that the composition is an isomorphism. Moreover, the first map is an isomorphism since p_1 is a fiber bundle and hence flat. It follows that the second map in (3.18) is also an isomorphism, which is equivalent to what we needed to show. \square

3.5. Hard half of the composition. There are several natural transformations that we will use, including η, θ , and perhaps most interestingly ϕ . We refer to the appendix for their definition and properties.

Fix $\mathcal{F}, \mathcal{G} \in D_{coh}(Y)$ and write $\mathcal{A} := \mathcal{F} \boxtimes \mathbb{D}(\mathcal{G}^*) \boxtimes \mathcal{G}$. In this section we prove the following key result which will be used to understand the second map in (3.15).

Proposition 3.19. *There exists the following commutative diagram:*

$$\begin{array}{ccccc} i_1^!(\mathcal{A}) & \xrightarrow{adj} & i_1^! i_{2*} i_2^*(\mathcal{A}) & \xrightarrow{m} & i_1^! i_{2*}(\mathcal{F} \widetilde{\boxtimes} \omega_{Y^*}) \\ \downarrow \sim & & & & \downarrow \sim \\ \mathcal{H}om(\mathbb{D}(\mathcal{F}), \mathbb{D}(\mathcal{G})) \widetilde{\boxtimes} \mathcal{G} & \longrightarrow & j_{2*} \mathcal{H}om(\mathbb{D}(\mathcal{F}), \mathbb{D}(\mathcal{G}) \otimes \mathcal{G}) & \xrightarrow{m} & j_{2*} \mathcal{H}om(\mathbb{D}(\mathcal{F}), \omega_Y) \cong j_{2*} \mathcal{F} \end{array}$$

The rest of the section will contain the proof of Proposition 3.19. We will use the following commutative diagram:

$$(3.20) \quad \begin{array}{ccccc} i_1^!(\mathcal{A}) & \xrightarrow{adj} & i_1^!i_{2*}i_2^*(\mathcal{A}) & \longrightarrow & i_1^!i_{2*}(\mathcal{F}\tilde{\boxtimes}\omega_{Y^*}) \\ \downarrow adj & & \downarrow \sim & & \downarrow \sim \\ j_{2*}j_2^*i_1^!(\mathcal{A}) & \xrightarrow{\phi} & j_{2*}j_1^!i_2^*(\mathcal{A}) & \longrightarrow & j_{2*}j_1^!(\mathcal{F}\tilde{\boxtimes}\omega_{Y^*}) \\ & & \downarrow \theta & & \downarrow \theta \\ & & j_{2*}\mathbb{D}j_1^*\mathbb{D}i_2^*(\mathcal{A}) & \longrightarrow & j_{2*}\mathbb{D}j_1^*\mathbb{D}(\mathcal{F}\tilde{\boxtimes}\omega_{Y^*}) \end{array}$$

To check that the top left rectangle commutes one writes out $\phi \circ adj$ as

$$i_1^! \xrightarrow{adj} j_{2*}j_2^*i_1^! \xrightarrow{adj} j_{2*}j_2^*i_1^!i_{2*}i_2^* \xrightarrow{\sim} j_{2*}j_2^*j_{2*}j_1^!i_2^* \xrightarrow{adj} j_{2*}j_1^!i_2^*$$

which is easily rewritten as

$$i_1^! \xrightarrow{adj} i_1^!i_{2*}i_2^* \xrightarrow{\sim} j_{2*}j_1^!i_2^*.$$

The other rectangles commute for more obvious reasons. Note that the map we are interested in is the composition in the top row of (3.20).

We will first focus on the case $\mathcal{F} = \omega_Y$. Let us denote $\mathcal{B} = \omega_Y\tilde{\boxtimes}\mathbb{D}(\mathcal{G}^*)\tilde{\boxtimes}\mathcal{G}$.

Lemma 3.21. *There exists an isomorphism $\gamma : \mathbb{D}(\mathbb{D}(\mathcal{G}) \otimes \mathcal{G}) \xrightarrow{\sim} j_1^*\mathbb{D}i_2^*(\mathcal{B})$ which fits into the following commutative diagram:*

$$\begin{array}{ccccc} \mathbb{D}j_1^*\mathbb{D}i_2^*(\mathcal{B}) & \xrightarrow{\sim} & \mathbb{D}j_1^*\mathbb{D}(\omega_Y\tilde{\boxtimes}(\mathbb{D}(\mathcal{G}^*) \otimes \mathcal{G}^*)) & \xrightarrow{m} & \mathbb{D}j_1^*\mathbb{D}(\omega_Y\tilde{\boxtimes}\omega_{Y^*}) \\ \downarrow \mathbb{D}\gamma & & & & \downarrow \sim \\ \mathbb{D}\mathbb{D}(\mathbb{D}(\mathcal{G}) \otimes \mathcal{G}) & \xrightarrow{m} & \mathbb{D}\mathbb{D}(\omega_Y) & \xrightarrow{\sim} & \omega_Y \end{array}$$

Proof. We use the following two general facts. First, for any morphism $f : S \rightarrow T$ and $\mathcal{M} \in D_{qcoh}(T)$ the following diagram commutes:

$$\begin{array}{ccc} f^*\mathcal{O}_T & \xrightarrow{f^*\iota} & f^*\mathcal{H}om(\mathcal{M}, \mathcal{M}) \\ \downarrow \sim & & \downarrow \\ \mathcal{O}_S & \xrightarrow{\iota} & \mathcal{H}om(f^*\mathcal{M}, f^*\mathcal{M}) \end{array}$$

Second, identifying $\mathbb{D}(\mathbb{D}(\mathcal{M}) \otimes \mathcal{M}) \cong \mathcal{H}om(\mathcal{M}, \mathbb{D}\mathbb{D}(\mathcal{M}))$ the map

$$\mathbb{D}\mathcal{H}om(\mathcal{M}, \mathbb{D}\mathbb{D}(\mathcal{M})) \xrightarrow{\mathbb{D}\tau} \mathbb{D}\mathcal{H}om(\mathcal{M}, \mathcal{M}) \xrightarrow{\mathbb{D}\iota} \mathbb{D}\mathcal{O}_T \cong \omega_T$$

can be identified with the composition

$$\mathbb{D}\mathcal{H}om(\mathbb{D}(\mathcal{M}), \mathbb{D}(\mathcal{M})) \cong \mathbb{D}\mathbb{D}(\mathbb{D}(\mathcal{M}) \otimes \mathcal{M}) \xrightarrow{m} \mathbb{D}\mathbb{D}(\omega_T) \cong \omega_T.$$

Now take $f := j_1 : Y \rightarrow Y\tilde{\times}Y^*$ and $\mathcal{M} := \mathcal{O}_Y\tilde{\boxtimes}\mathcal{G}^*$ so that

$$i_2^*(\mathcal{B}) \cong \omega_Y\tilde{\boxtimes}(\mathbb{D}(\mathcal{G}^*) \otimes \mathcal{G}^*) \cong \mathbb{D}(\mathcal{M}) \otimes \mathcal{M}.$$

Consider now the map

$$\mathbb{D}j_1^*\mathbb{D}i_2^*(\mathcal{B}) \cong \mathbb{D}j_1^*\mathbb{D}(\mathbb{D}(\mathcal{M}) \otimes \mathcal{M}) \xrightarrow{\eta} j_1^!\mathbb{D}\mathbb{D}(\mathbb{D}(\mathcal{M}) \otimes \mathcal{M}) \xrightarrow{m} j_1^!\mathbb{D}\mathbb{D}(\omega_{Y\tilde{\times}Y^*}) \cong \omega_Y.$$

By the second fact above this map is equivalent to

$$\mathbb{D}j_1^* \mathbb{D}i_2^*(\mathcal{B}) \cong \mathbb{D}j_1^* \mathcal{H}om(\mathcal{M}, \mathbb{D}\mathbb{D}(\mathcal{M})) \xrightarrow{\eta} j_1^! \mathbb{D} \mathcal{H}om(\mathcal{M}, \mathbb{D}\mathbb{D}(\mathcal{M})) \xrightarrow{\mathbb{D}(\tau \circ \iota)} j_1^! \omega_{Y \times Y^*} \cong \omega_Y.$$

Since $\mathbb{D}\mathbb{D}\mathcal{M} \cong \mathcal{M}$ we can rewrite this as

$$\mathbb{D}j_1^* \mathbb{D}i_2^*(\mathcal{B}) \cong \mathbb{D}j_1^* \mathcal{H}om(\mathcal{M}, \mathcal{M}) \xrightarrow{\eta} j_1^! \mathbb{D} \mathcal{H}om(\mathcal{M}, \mathcal{M}) \xrightarrow{\mathbb{D}\iota} j_1^! \omega_{Y \times Y^*} \cong \omega_Y$$

and finally as

$$\mathbb{D}j_1^* \mathbb{D}i_2^*(\mathcal{B}) \cong \mathbb{D}j_1^* \mathcal{H}om(\mathcal{M}, \mathcal{M}) \xrightarrow{\mathbb{D}\iota} \mathbb{D}j_1^* \mathcal{O}_{Y \times Y^*} \cong \omega_Y.$$

Now the map $j_1^* \mathcal{H}om(\mathcal{M}, \mathcal{M}) \rightarrow \mathcal{H}om(j_1^* \mathcal{M}, j_1^* \mathcal{M}) \cong \mathcal{H}om(\mathcal{G}, \mathcal{G})$ is an isomorphism. So, using the first fact above, we can rewrite this last composition as

$$\mathbb{D}j_1^* \mathbb{D}i_2^*(\mathcal{B}) \cong \mathbb{D}j_1^* \mathcal{H}om(\mathcal{M}, \mathcal{M}) \xrightarrow{\sim} \mathbb{D} \mathcal{H}om(\mathcal{G}, \mathcal{G}) \xrightarrow{\mathbb{D}\iota} \omega_Y.$$

The result follows, where γ is the isomorphism $j_1^* \mathbb{D}i_2^*(\mathcal{B}) \cong \mathcal{H}om(\mathcal{G}, \mathcal{G}) \cong \mathbb{D}(\mathbb{D}(\mathcal{G}) \otimes \mathcal{G})$. \square

Lemma 3.22. *There exists an isomorphism $\gamma' : j_1^* \mathbb{D}i_2^*(\mathcal{B}) \xrightarrow{\sim} \mathbb{D}(\mathbb{D}(\mathcal{G}) \otimes \mathcal{G})$ which fits into the following commutative diagram:*

$$\begin{array}{ccccc} j_2^* i_1^! (\mathcal{B}) & \xrightarrow{\phi} & j_1^! i_2^* (\mathcal{B}) & \xrightarrow{\theta} & \mathbb{D}j_1^* \mathbb{D}i_2^* (\mathcal{B}) \\ \downarrow \sim & & & & \uparrow \mathbb{D}\gamma' \\ \mathbb{D}(\mathcal{G}) \otimes \mathcal{G} & \xrightarrow{\tau} & \mathbb{D}\mathbb{D}(\mathbb{D}(\mathcal{G}) \otimes \mathcal{G}). & & \end{array}$$

Proof. We do this in several steps. First consider the simpler case of $\mathcal{B}_1 := \mathcal{O}_Y \tilde{\boxtimes} \mathcal{G}^* \tilde{\boxtimes} \omega_Y$ and the map $j_1^* i_2^! (\mathcal{B}_1) \xrightarrow{\phi} j_2^! i_1^* (\mathcal{B}_1)$. We claim this map is an isomorphism.

To see this consider again the following commutative diagram from (3.17). Now $\mathcal{B}_1 = p_1^! (\mathcal{O}_Y \tilde{\boxtimes} \mathcal{G}^*)$ and we can look at the composition

$$j_1^* i_2^! p_1^! (\mathcal{O}_Y \tilde{\boxtimes} \mathcal{G}^*) \xrightarrow{\phi} j_2^! i_1^* p_1^! (\mathcal{O}_Y \tilde{\boxtimes} \mathcal{G}^*) \xrightarrow{\phi} j_2^! p_2^! j_1^* (\mathcal{O}_Y \tilde{\boxtimes} \mathcal{G}^*).$$

By Lemma A.19 this composition is just $j_1^* (p_1 \circ i_2)^! \xrightarrow{\phi} (p_2 \circ j_2)^! j_1^*$, which is an isomorphism since $p_1 \circ i_2$ and $p_2 \circ j_2$ are both the identity maps. Since

$$j_1^* i_2^! p_1^! (\mathcal{O}_Y \tilde{\boxtimes} \mathcal{G}^*) \cong j_2^! i_1^* p_1^! (\mathcal{O}_Y \tilde{\boxtimes} \mathcal{G}^*) \cong j_2^! p_2^! j_1^* (\mathcal{O}_Y \tilde{\boxtimes} \mathcal{G}^*) \cong \mathcal{G}$$

it follows that $j_1^* i_2^! (\mathcal{B}_1) \xrightarrow{\phi} j_2^! i_1^* (\mathcal{B}_1)$ is an isomorphism.

Next take $\mathcal{B}_2 := \mathcal{O}_Y \tilde{\boxtimes} \mathcal{G}^* \tilde{\boxtimes} \mathbb{D}(\mathcal{G}) \cong \mathcal{H}om(\mathcal{O}_Y \tilde{\boxtimes} \mathcal{O}_{Y^*} \tilde{\boxtimes} \mathcal{G}, \mathcal{B}_1)$. By Proposition A.23 we have the following commutative diagram:

$$\begin{array}{ccccc} j_1^* i_2^! \mathcal{H}om(\mathcal{O}_Y \tilde{\boxtimes} \mathcal{O}_{Y^*} \tilde{\boxtimes} \mathcal{G}, \mathcal{B}_1) & \xrightarrow{\sim} & j_1^* \mathcal{H}om(\mathcal{O}_Y \tilde{\boxtimes} \mathcal{G}^*, i_2^! \mathcal{B}_1) & \longrightarrow & \mathcal{H}om(\mathcal{G}, j_1^* i_2^! \mathcal{B}_1) \\ \downarrow \phi & & & & \downarrow \phi \\ j_2^! i_1^* \mathcal{H}om(\mathcal{O}_Y \tilde{\boxtimes} \mathcal{O}_{Y^*} \tilde{\boxtimes} \mathcal{G}, \mathcal{B}_1) & \longrightarrow & j_2^! \mathcal{H}om(\mathcal{O}_Y \tilde{\boxtimes} \mathcal{G}, i_1^* \mathcal{B}_1) & \xrightarrow{\sim} & \mathcal{H}om(\mathcal{G}, j_2^! i_1^* \mathcal{B}_1) \end{array}$$

It is not hard to check that the two unmarked horizontal morphisms are isomorphisms. The right-hand vertical map ϕ is an isomorphism by the argument above.

It follows that $j_1^* i_2^! (\mathcal{B}_2) \xrightarrow{\phi} j_2^! i_1^* (\mathcal{B}_2)$ is also an isomorphism.

Next consider the composition

$$j_1^* \mathbb{D} i_2^* \mathbb{D}(\mathcal{B}_2) \xrightarrow{\theta^{-1}} j_1^* i_2^! (\mathcal{B}_2) \xrightarrow{\phi} j_2^! i_1^* (\mathcal{B}_2) \xrightarrow{\theta} \mathbb{D} j_2^* \mathbb{D} i_1^* (\mathcal{B}_2).$$

In the notation of Proposition A.25 this composition is denoted ρ . Notice that, in this case, the map θ is an isomorphism and ϕ is an isomorphism by the argument above. This means ρ is an isomorphism.

Finally, note that $\mathbb{D}(\mathcal{B}) \cong \mathcal{B}_2$ and consider

$$(3.23) \quad j_2^* \mathbb{D} i_1^* \mathbb{D}(\mathcal{B}) \xrightarrow{\tau} \mathbb{D} \mathbb{D} j_2^* \mathbb{D} i_1^* \mathbb{D}(\mathcal{B}) \xrightarrow{\mathbb{D} \rho} \mathbb{D} j_1^* \mathbb{D} i_2^* \mathbb{D} \mathbb{D}(\mathcal{B}) \xrightarrow{\tau^{-1}} \mathbb{D} j_1^* \mathbb{D} i_2^* (\mathcal{B}).$$

Notice that the last two maps in (3.23) are obtained by applying \mathbb{D} to the composition

$$(3.24) \quad j_1^* \mathbb{D} i_2^* (\mathcal{B}) \xrightarrow{\mathbb{D} \tau^{-1}} j_1^* \mathbb{D} i_2^* \mathbb{D} \mathbb{D}(\mathcal{B}) \xrightarrow{\rho} \mathbb{D} j_2^* \mathbb{D} i_1^* \mathbb{D}(\mathcal{B}).$$

Moreover, if we identify $j_2^* \mathbb{D} i_1^* \mathbb{D}(\mathcal{B}) \cong j_2^* (\mathbb{D}(\mathcal{G}) \tilde{\boxtimes} \mathcal{G}) \cong \mathbb{D}(\mathcal{G}) \otimes \mathcal{G}$, then the composition in (3.23) is equivalent to

$$\mathbb{D}(\mathcal{G}) \otimes \mathcal{G} \xrightarrow{\tau} \mathbb{D} \mathbb{D}(\mathbb{D}(\mathcal{G}) \otimes \mathcal{G}) \xrightarrow{\mathbb{D} \gamma'} \mathbb{D} j_1^* \mathbb{D} i_2^* (\mathcal{B}),$$

where γ' is the composition from (3.24) (which is an isomorphism). To finish, by Proposition A.25, we can identify this last composition with

$$(3.25) \quad j_2^* \mathbb{D} i_1^* \mathbb{D}(\mathcal{B}) \xrightarrow{\theta^{-1}} j_2^* i_1^! (\mathcal{B}) \xrightarrow{\phi} j_1^! i_2^* (\mathcal{B}) \xrightarrow{\theta} \mathbb{D} j_1^* \mathbb{D} i_2^* (\mathcal{B}).$$

The result follows. □

Corollary 3.26. *There exists the following commutative diagram:*

$$\begin{array}{ccccccc} j_2^* i_1^! (\mathcal{B}) & \xrightarrow{\phi} & j_1^! i_2^* (\mathcal{B}) & \xrightarrow{\theta} & \mathbb{D} j_1^* \mathbb{D} i_2^* (\mathcal{B}) & \longrightarrow & \mathbb{D} j_1^* \mathbb{D}(\omega_Y \tilde{\boxtimes} \omega_{Y^*}) \\ \downarrow \sim & & & & & & \downarrow \sim \\ \mathbb{D}(\mathcal{G}) \otimes \mathcal{G} & \xrightarrow{\quad\quad\quad} & \xrightarrow{m} & \xrightarrow{\quad\quad\quad} & \xrightarrow{\quad\quad\quad} & \xrightarrow{\quad\quad\quad} & \omega_Y \end{array}$$

Proof. Combining Lemmas 3.21 and 3.22 we have a commutative diagram of the form

$$(3.27) \quad \begin{array}{ccccccc} j_2^* i_1^! (\mathcal{B}) & \xrightarrow{\phi} & j_1^! i_2^* (\mathcal{B}) & \xrightarrow{\theta} & \mathbb{D} j_1^* \mathbb{D} i_2^* (\mathcal{B}) & \longrightarrow & \mathbb{D} j_1^* \mathbb{D}(\omega_Y \tilde{\boxtimes} \omega_{Y^*}) \\ \downarrow \sim & & & & & & \downarrow \sim \\ \mathbb{D}(\mathcal{G}) \otimes \mathcal{G} & \xrightarrow{\tau} & \mathbb{D} \mathbb{D}(\mathbb{D}(\mathcal{G}) \otimes \mathcal{G}) & \xrightarrow{\mathbb{D} \gamma''} & \mathbb{D} \mathbb{D}(\mathbb{D}(\mathcal{G}) \otimes \mathcal{G}) & \xrightarrow{m} & \omega_Y \end{array}$$

for some isomorphism γ'' of $\mathbb{D}(\mathbb{D}(\mathcal{G}) \otimes \mathcal{G})$.

Now if we identify $\mathbb{D}(\mathbb{D}(\mathcal{G}) \otimes \mathcal{G}) \cong \mathcal{H}om(\mathcal{G}, \mathcal{G})$, then, as in the proof of Lemma 3.21, the composition $m \circ \mathbb{D} \gamma''$ in the diagram above is equivalent to \mathbb{D} applied to the following composition:

$$(3.28) \quad \mathcal{O}_Y \xrightarrow{\iota} \mathcal{H}om(\mathcal{G}, \mathcal{G}) \xrightarrow{\gamma''} \mathcal{H}om(\mathcal{G}, \mathcal{G}).$$

Now, by adjunction, $\text{Hom}(\mathcal{O}_Y, \mathcal{H}om(\mathcal{G}, \mathcal{G})) \cong \text{Hom}(\mathcal{G}, \mathcal{G})$. This identifies the composition in (3.28) with some $\delta \in \text{Hom}(\mathcal{G}, \mathcal{G})$. One can then write the composition

in (3.28) as $\mathcal{O}_Y \xrightarrow{L} \mathcal{H}om(\mathcal{G}, \mathcal{G}) \xrightarrow{\delta} \mathcal{H}om(\mathcal{G}, \mathcal{G})$, where δ acts on the second factor. This allows us to write the composition $m \circ \mathbb{D}\gamma''$ as

$$\mathbb{D}\mathbb{D}(\mathbb{D}(\mathcal{G}) \otimes \mathcal{G}) \xrightarrow{\mathbb{D}\delta} \mathbb{D}\mathbb{D}(\mathbb{D}(\mathcal{G}) \otimes \mathcal{G}) \xrightarrow{m} \omega_Y.$$

Finally this allows us to rewrite the bottom row in the commutative diagram (3.27) as

$$\mathbb{D}(\mathcal{G}) \otimes \mathcal{G} \xrightarrow{\mathbb{D}\delta} \mathbb{D}(\mathcal{G}) \otimes \mathcal{G} \xrightarrow{\tau} \mathbb{D}\mathbb{D}(\mathbb{D}(\mathcal{G}) \otimes \mathcal{G}) \xrightarrow{m} \omega_Y.$$

The result follows since the composition $\mathbb{D}(\mathcal{G}) \otimes \mathcal{G} \xrightarrow{\tau} \mathbb{D}\mathbb{D}(\mathbb{D}(\mathcal{G}) \otimes \mathcal{G}) \xrightarrow{m} \omega_Y$ is equivalent to m and we can include the isomorphism $\mathbb{D}\delta$ into the left vertical isomorphism in (3.27). □

Now we are ready to consider the general case of $\mathcal{A} \cong \mathcal{F}\tilde{\boxtimes}\mathbb{D}(\mathcal{G}^*)\tilde{\boxtimes}\mathcal{G}$.

Proposition 3.29. *There exists the following commutative diagram:*

$$\begin{array}{ccccc} j_2^* i_1^! (\mathcal{A}) & \xrightarrow{\phi} & j_1^! i_2^* (\mathcal{A}) & \longrightarrow & j_1^! (\mathcal{F}\tilde{\boxtimes}\omega_{Y^*}) \\ \downarrow \sim & & & & \downarrow \sim \\ \mathcal{H}om(\mathbb{D}(\mathcal{F}), \mathbb{D}(\mathcal{G})) \otimes \mathcal{G} & \longrightarrow & \mathcal{H}om(\mathbb{D}(\mathcal{F}), \mathbb{D}(\mathcal{G}) \otimes \mathcal{G}) & \xrightarrow{m} & \mathcal{H}om(\mathbb{D}(\mathcal{F}), \omega_Y) \cong \mathcal{F} \end{array}$$

Proof. We will use the fact that $\mathcal{A} \cong \mathcal{H}om(\mathbb{D}(\mathcal{F})\tilde{\boxtimes}\mathcal{O}_{Y^*}\tilde{\boxtimes}\mathcal{O}_Y, \mathcal{B})$. By Proposition A.23 we have the commutative diagram (3.30)

$$\begin{array}{ccccc} j_2^* i_1^! (\mathcal{A}) & \xrightarrow{\phi} & j_1^! i_2^* (\mathcal{A}) & \longrightarrow & j_1^! (\mathcal{F}\tilde{\boxtimes}\omega_{Y^*}) \\ \downarrow & & \downarrow & & \downarrow \sim \\ \mathcal{H}om(\mathbb{D}(\mathcal{F}), j_2^* i_1^! (\mathcal{B})) & \xrightarrow{\phi} & \mathcal{H}om(\mathbb{D}(\mathcal{F}), j_1^! i_2^* (\mathcal{B})) & \longrightarrow & \mathcal{H}om(\mathbb{D}(\mathcal{F}), j_1^! (\omega_Y \tilde{\boxtimes} \omega_{Y^*})) \end{array}$$

where we have used the fact that $j_2^* i_1^* (\mathbb{D}(\mathcal{F})\tilde{\boxtimes}\mathcal{O}_{Y^*}\tilde{\boxtimes}\mathcal{O}_Y) \cong \mathbb{D}(\mathcal{F})$. The left vertical map is the composition

$$(3.31) \quad \begin{aligned} j_2^* i_1^! \mathcal{H}om(\mathbb{D}(\mathcal{F})\tilde{\boxtimes}\mathcal{O}_{Y^*}\tilde{\boxtimes}\mathcal{O}_Y, \mathcal{B}) &\xrightarrow{\sim} j_2^* \mathcal{H}om(\mathbb{D}(\mathcal{F})\tilde{\boxtimes}\mathcal{O}_Y, i_1^! (\mathcal{B})) \\ &\rightarrow \mathcal{H}om(\mathbb{D}(\mathcal{F}), j_2^* i_1^! (\mathcal{B})). \end{aligned}$$

Now we have $i_1^! (\mathcal{B}) \cong \mathbb{D}(\mathcal{G})\tilde{\boxtimes}\mathcal{G}$, $j_2^* i_1^! (\mathcal{B}) \cong \mathbb{D}(\mathcal{G}) \otimes \mathcal{G}$ and

$$\mathcal{H}om(\mathbb{D}(\mathcal{F})\tilde{\boxtimes}\mathcal{O}_Y, i_1^! (\mathcal{B})) \cong \mathcal{H}om(\mathbb{D}(\mathcal{F}), \mathbb{D}(\mathcal{G}))\tilde{\boxtimes}\mathcal{G}.$$

Subsequently, the second map in (3.31) can be identified with the natural morphism

$$j_2^* (\mathcal{H}om(\mathbb{D}(\mathcal{F}), \mathbb{D}(\mathcal{G}))\tilde{\boxtimes}\mathcal{G}) \cong \mathcal{H}om(\mathbb{D}(\mathcal{F}), \mathbb{D}(\mathcal{G})) \otimes \mathcal{G} \rightarrow \mathcal{H}om(\mathbb{D}(\mathcal{F}), \mathbb{D}(\mathcal{G}) \otimes \mathcal{G}).$$

Combining this with Corollary 3.26 the commutative diagram (3.30) gives us the commutative diagram in Proposition 3.29. □

Finally, applying j_{2*} to the commutative diagram in Proposition 3.29 and comparing with the bottom two rows in (3.20) we arrive at the commutative diagram (3.32)

$$\begin{array}{ccccc}
 i_1^!(\mathcal{A}) & \xrightarrow{\text{adj}} & i_1^!i_{2*}i_2^*(\mathcal{A}) & \longrightarrow & i_1^!i_{2*}(\mathcal{F}\tilde{\boxtimes}\omega_{Y^*}) \\
 \downarrow & & & & \downarrow \sim \\
 j_{2*}\mathcal{H}om(\mathbb{D}(\mathcal{F}), \mathbb{D}(\mathcal{G}) \otimes \mathcal{G}) & \xrightarrow{m} & j_{2*}\mathcal{H}om(\mathbb{D}(\mathcal{F}), \omega_Y) & \cong & j_{2*}\mathcal{F}
 \end{array}$$

where the vertical map on the left is the composition

$$i_1^!(\mathcal{A}) \xrightarrow{\text{adj}} j_{2*}j_2^*i_1^!(\mathcal{A}) \xrightarrow{\sim} j_{2*}(\mathcal{H}om(\mathbb{D}(\mathcal{F}), \mathbb{D}(\mathcal{G})) \otimes \mathcal{G}) \rightarrow j_{2*}\mathcal{H}om(\mathbb{D}(\mathcal{F}), \mathbb{D}(\mathcal{G}) \otimes \mathcal{G}).$$

This map can be factored as

$$i_1^!(\mathcal{A}) \xrightarrow{\sim} \mathcal{H}om(\mathbb{D}(\mathcal{F}), \mathbb{D}(\mathcal{G}))\tilde{\boxtimes}\mathcal{G} \rightarrow j_{2*}\mathcal{H}om(\mathbb{D}(\mathcal{F}), \mathbb{D}(\mathcal{G}) \otimes \mathcal{G}).$$

Together with (3.32) this now recovers the diagram in Proposition 3.19.

3.6. Other half of the composition. We are now ready to identify more explicitly the composition in (3.15). We start with the following commutative diagram:

$$\begin{array}{ccccc}
 \mathcal{O}_Y\boxtimes\mathcal{F} & \longrightarrow & i_1^!(\mathcal{F}\tilde{\boxtimes}\mathbb{D}(\mathcal{F}^*)\tilde{\boxtimes}\mathcal{F}) & \longrightarrow & i_1^!i_{2*}(\mathcal{F}\tilde{\boxtimes}\omega_{Y^*}) \\
 \downarrow = & & \downarrow \sim & & \downarrow \sim \\
 \mathcal{O}_Y\boxtimes\mathcal{F} & \xrightarrow{\iota} & \mathcal{H}om(\mathbb{D}(\mathcal{F}), \mathbb{D}(\mathcal{F}))\tilde{\boxtimes}\mathcal{F} & \longrightarrow & j_{2*}\mathcal{F}
 \end{array}
 \tag{3.33}$$

The right rectangle is the large commutative rectangle from Proposition 3.19 (we take $\mathcal{F} = \mathcal{G}$ in the notation of Section 3.5). The composition in the bottom row is equivalent to the following composition:

$$\begin{aligned}
 \mathcal{O}_Y\boxtimes\mathcal{F} &\rightarrow j_{2*}\mathcal{F} \xrightarrow{\iota} j_{2*}(\mathcal{H}om(\mathbb{D}(\mathcal{F}), \mathbb{D}(\mathcal{F})) \otimes \mathcal{F}) \\
 &\rightarrow j_{2*}\mathcal{H}om(\mathbb{D}(\mathcal{F}), \mathbb{D}(\mathcal{F}) \otimes \mathcal{F}) \xrightarrow{m} j_{2*}\mathcal{H}om(\mathbb{D}(\mathcal{F}), \omega_Y) \cong j_{2*}\mathcal{F}.
 \end{aligned}$$

This is subsequently equivalent to the natural restriction map $\mathcal{O}_Y\boxtimes\mathcal{F} \rightarrow j_{2*}\mathcal{F}$ because the composition

$$\mathcal{F} \xrightarrow{\iota} \mathcal{H}om(\mathbb{D}(\mathcal{F}), \mathbb{D}(\mathcal{F})) \otimes \mathcal{F} \rightarrow \mathcal{H}om(\mathbb{D}(\mathcal{F}), \mathbb{D}(\mathcal{F}) \otimes \mathcal{F}) \xrightarrow{m} \mathcal{H}om(\mathbb{D}(\mathcal{F}), \omega_Y) \cong \mathcal{F}$$

is an isomorphism.

3.7. Final step in the proof of rigidity. Pushing forward the map in (3.14) to Gr_G we obtain the composition in (3.12). The last step is showing that this composition is an isomorphism.

The composition in the top row of (3.33) is by adjunction equivalent to the one in (3.14), so we get the following commutative diagram:

$$\begin{array}{ccc}
 i_{1*}(\mathcal{O}_Y\boxtimes\mathcal{F}) & \longrightarrow & i_{1*}i_1^!i_{2*}(\mathcal{F}\tilde{\boxtimes}\omega_{Y^*}) \xrightarrow{\text{adj}} i_{2*}(\mathcal{F}\tilde{\boxtimes}\omega_{Y^*}) \\
 \downarrow = & & \downarrow \sim \\
 i_{1*}(\mathcal{O}_Y\boxtimes\mathcal{F}) & \longrightarrow & i_{1*}j_{2*}\mathcal{F}
 \end{array}
 \tag{3.34}$$

Here the composition in the top row is the one from (3.14) and the map in the bottom row is restriction. Now by Lemma 3.36 below the composition

$$i_{1*}i_1^!i_{2*}(\mathcal{F}\widetilde{\boxtimes}\omega_{Y^*}) \xrightarrow{\sim} i_{1*}j_{2*}j_1^!(\mathcal{F}\widetilde{\boxtimes}\omega_{Y^*}) = i_{2*}j_{1*}j_1^!(\mathcal{F}\widetilde{\boxtimes}\omega_{Y^*}) \xrightarrow{adj} i_{2*}(\mathcal{F}\widetilde{\boxtimes}\omega_{Y^*})$$

is equal to the adjunction map $i_{1*}i_1^!i_{2*}(\mathcal{F}\widetilde{\boxtimes}\omega_{Y^*}) \rightarrow i_{2*}(\mathcal{F}\widetilde{\boxtimes}\omega_{Y^*})$. Thus we can complete the diagram in (3.34) to get (3.35)

$$\begin{array}{ccc} i_{1*}(\mathcal{O}_Y\boxtimes\mathcal{F}) & \xrightarrow{\hspace{15em}} & i_{2*}(\mathcal{F}\widetilde{\boxtimes}\omega_{Y^*}) \\ \downarrow = & & \downarrow = \\ i_{1*}(\mathcal{O}_Y\boxtimes\mathcal{F}) & \longrightarrow i_{1*}j_{2*}\mathcal{F} \xrightarrow{\sim} i_{2*}j_{1*}j_1^!(\mathcal{F}\widetilde{\boxtimes}\omega_{Y^*}) \xrightarrow{adj} & i_{2*}(\mathcal{F}\widetilde{\boxtimes}\omega_{Y^*}) \end{array}$$

where the top row is the map from (3.14), while the middle isomorphism is the fact that $j_1^!(\mathcal{F}\widetilde{\boxtimes}\omega_{Y^*}) \cong \mathcal{F}$ and $i_1 \circ j_2 = i_2 \circ j_1$.

Now we are ready to push down to Gr_G . Let us denote by $p : Y \times Y \rightarrow Y$ the projection onto the second factor and similarly by $q : Y \times Y^* \rightarrow Y$ the projection onto the first factor. Then the pushforward of $i_{1*}(\mathcal{O}_Y\boxtimes\mathcal{F})$ to Gr_G can be identified with $p_*(\mathcal{O}_Y\boxtimes\mathcal{F}) \cong p_*p^*(\mathcal{F})$. Similarly, the pushforward of $i_{2*}(\mathcal{F}\widetilde{\boxtimes}\omega_{Y^*})$ can be identified with $q_*(\mathcal{F}\widetilde{\boxtimes}\omega_{Y^*}) \cong q_*q^!(\mathcal{F})$. Subsequently, using the commutative diagram (3.35), the composition

$$\mathcal{F} = \mathcal{O}_e * \mathcal{F} \rightarrow \mathcal{F} * \mathcal{F}^L * \mathcal{F} \rightarrow \mathcal{F} * \mathcal{O}_e = \mathcal{F}$$

can be identified with a composition

$$\mathcal{F} \xrightarrow{adj} p_*p^*(\mathcal{F}) \xrightarrow{adj} p_*j_{2*}j_2^*p^*(\mathcal{F}) \xrightarrow{\sim} q_*j_{1*}j_1^!q^!(\mathcal{F}) \xrightarrow{adj} q_*q^!(\mathcal{F}) \xrightarrow{adj} \mathcal{F}.$$

Notice that the composition of the first two adjunction maps is the adjunction map for $p \circ j_2$. Since $p \circ j_2$ is the identity this composition is an isomorphism. Likewise, since $q \circ j_1$ is the identity map, the composition of the last two adjunctions is also an isomorphism. It follows that the entire composition is an isomorphism. This completes the proof of Theorem 3.1.

Lemma 3.36. *Using the notation from (A.12) the composition*

$$i_{1*} \xrightarrow{adj} i_{1*}j_{2*}j_2^* = i_{2*}j_{1*}j_1^* \xrightarrow{\sim} i_{2*}i_2^*i_{1*}$$

is equal to $i_{1*} \xrightarrow{adj} i_{2*}i_2^*i_{1*}$. Similarly, the composition

$$i_{1*}i_1^!i_{2*} \xrightarrow{\sim} i_{1*}j_{2*}j_1^! = i_{2*}j_{1*}j_1^! \xrightarrow{adj} i_{2*}$$

is equal to $i_{1*}i_1^!i_{2*} \xrightarrow{adj} i_{2*}$.

Proof. This is an elementary computation which we leave to the reader. □

4. RENORMALIZED r -MATRICES

In this section we develop the notion of a renormalized r -matrix in an arbitrary monoidal category \mathcal{C} , abstracting the essential features of the r -matrices introduced in the setting of KLR algebras in [KKK18]. Following [KKKO18], we recall how they provide a framework for the monoidal categorification of cluster algebras.

4.1. Definitions and examples.

Definition 4.1. Let $(\mathcal{C}, *)$ be a \mathbb{C} -linear monoidal category. A **system of renormalized r -matrices** in \mathcal{C} is an assignment to each pair of objects $M, N \in \mathcal{C}$ a map

$$\mathbf{r}_{M,N} : M * N \rightarrow N * M$$

and an element $\Lambda(M, N) \in \mathbb{Z} \cup \{-\infty\}$ satisfying the following properties:

- (i) For any $M \in \mathcal{C}$ the morphisms $\mathbf{r}_{M,1_{\mathcal{C}}}$ and $\mathbf{r}_{1_{\mathcal{C}},M}$ are given by composing the unit isomorphisms of $M * 1_{\mathcal{C}}$ and $1_{\mathcal{C}} * M$ with M .
- (ii) $\mathbf{r}_{M,N}$ is nonzero if and only if $\Lambda(M, N) \neq -\infty$.
- (iii) For any $M, N_1, N_2 \in \mathcal{C}$ we have

$$\Lambda(M, N_1 * N_2) \leq \Lambda(M, N_1) + \Lambda(M, N_2).$$

If equality holds, then

$$\mathbf{r}_{M, N_1 * N_2} = (id_{N_1} * \mathbf{r}_{M, N_2}) \circ (\mathbf{r}_{M, N_1} * id_{N_2})$$

while the right-hand composition is zero if the inequality is strict. The corresponding statement with M on the other side also holds.

- (iv) For $M, N \in \mathcal{C}$ we have $\Lambda(M, N) + \Lambda(N, M) \geq 0$. Moreover

$$\mathbf{r}_{N,M} \circ \mathbf{r}_{M,N} \neq 0 \iff \Lambda(M, N) + \Lambda(N, M) = 0.$$

- (v) For any $M, N_1, N_2 \in \mathcal{C}$ and morphism $f : N_1 \rightarrow N_2$ consider the following diagram:

$$\begin{array}{ccc} M * N_1 & \xrightarrow{\mathbf{r}_{M, N_1}} & N_1 * M \\ id_M * f \downarrow & & \downarrow f * id_M \\ M * N_2 & \xrightarrow{\mathbf{r}_{M, N_2}} & N_2 * M \end{array}$$

- If $\Lambda(M, N_1) = \Lambda(M, N_2)$, the diagram commutes.
- If $\Lambda(M, N_1) < \Lambda(M, N_2)$, the bottom left composition is zero.
- If $\Lambda(M, N_1) > \Lambda(M, N_2)$, the top right composition is zero.

The corresponding statements hold when the product with M is taken on the other side.

As the definition suggests, renormalized r -matrices are rarely invertible in motivating examples and need not be natural in the categorical sense. The last property above says that their failure to be natural is controlled by the quantity $\Lambda(M, N)$.

Example 4.2. Let \mathcal{C} be a braided monoidal category. Then letting $\mathbf{r}_{M,N}$ be the braiding isomorphism and setting $\Lambda(M, N) = 0$ for all M, N we obtain a system of renormalized r -matrices in \mathcal{C} .

Example 4.3. Let A be a symmetric generalized Cartan matrix and Q_+ the associated semigroup of positive roots. For each $\beta \in Q_+$, let $R(\beta)$ be the symmetric KLR (or quiver Hecke) algebra associated to some choice of auxiliary polynomials. Let $R(\beta)\text{-mod}$ be the category of gradable finite-dimensional $R(\beta)$ -modules. Then the direct sum $\bigoplus_{\beta \in Q_+} R(\beta)\text{-mod}$, equipped with its convolution product, admits a system of renormalized r -matrices constructed in [KKK18]. We note that this category is not rigid.

Example 4.4. Our main example in this paper comes by way of a monoidal category \mathcal{C} which extends in a suitable fashion to a chiral category over \mathbb{A}^1 . The construction of renormalized r -matrices in this case is explained in Section 5 (cf. Theorem 5.10). It is in light of this construction that we find it more natural not to assume \mathcal{C} is abelian in the definition.

Example 4.5. The constructions of renormalized r -matrices in Examples 4.3 and 4.4 parallel the construction of renormalized r -matrices for finite-dimensional representations of a quantum loop algebra [Kas02]. We expect the latter fit into the present axiomatic setup or a close relative thereof. Indeed, following [KKKO15] they share many of the key properties described in the following section. Moreover, the formalism of meromorphic tensor categories considered in [Soi97, Soi99], of which representations of quantum loop algebras provide central motivating examples, shares many of the key features of the chiral formalism considered in Section 5 and should allow a similar general construction.

Having a system of renormalized r -matrices becomes a stronger condition given additional hypotheses on the category \mathcal{C} .

Lemma 4.6. *If M or N has a left or right dual, then $\mathbf{r}_{M,N} \neq 0$ unless $M * N = 0$ or $N * M = 0$.*

Proof. Suppose N has a right dual N^R (the other cases are similar). Consider the following diagram:

$$\begin{array}{ccc}
 M * (N * N^R) & \xrightarrow{\mathbf{r}_{M,N * N^R}} & (N * N^R) * M \\
 \text{id}_M * \text{adj} \downarrow & & \downarrow \text{adj} * \text{id}_M \\
 M * 1_{\mathcal{C}} & \xrightarrow{\mathbf{r}_{M,1_{\mathcal{C}}}} & 1_{\mathcal{C}} * M
 \end{array}$$

The composition along the bottom left is just the adjunction map $M * N * N^R \rightarrow M$ which is nonzero (unless $M * N = 0$). By Definition 4.1(v) this implies that

$$\Lambda(M, N * N^R) \geq \Lambda(M, 1_{\mathcal{C}}).$$

It follows from (i) and (ii) that $\Lambda(M, 1_{\mathcal{C}}) \neq -\infty$ ((iii) then implying $\Lambda(M, 1_{\mathcal{C}}) = 0$), hence $\Lambda(M, N * N^R) \neq -\infty$ as well. On the other hand, by (iii) we have $\Lambda(M, N) + \Lambda(M, N^R) \geq \Lambda(M, N * N^R)$. Thus $\Lambda(M, N) \neq -\infty$ and the result follows by (ii). \square

Since in a rigid abelian monoidal category products of nonzero objects are nonzero, we obtain the following corollary.

Corollary 4.7. *If \mathcal{C} is rigid and abelian, then $\mathbf{r}_{M,N} \neq 0$ for all nonzero M, N .*

4.2. Consequences. When \mathcal{C} is abelian and satisfies some natural conditions, the existence of renormalized r -matrices strongly constrains the underlying monoidal structure. In this section we discuss some key consequences. The results here and in the remainder of Section 4 are straightforward generalizations from the KLR case treated in [KKKO15, KKKO18], hence we refer to their antecedents in loc. cit. for detailed proofs.

We assume now that \mathcal{C} is a locally finite abelian category whose monoidal product is biexact, and such that $\text{End}(1_{\mathcal{C}}) \cong \mathbb{C} \cdot \text{id}$. In other words, \mathcal{C} is a \mathbb{C} -linear ring

category in the terminology of [EGNO15]. We further assume that \mathcal{C} has a system of renormalized r -matrices for which $\mathbf{r}_{M,N}$ is nonzero for all nonzero M, N , and that \mathcal{C} has the following property considered in [KKKO15].

Definition 4.8. We say \mathcal{C} has *separable triple products* if the following condition holds. Suppose M_1, M_2 , and M_3 are objects with $X \subset M_1 * M_2$ and $Y \subset M_2 * M_3$ such that $X * M_3 \subset M_1 * Y$. Then there exists a subobject $N \subset M_2$ such that $X \subset M_1 * N$ and $N * M_3 \subset Y$. Likewise if $M_1 * Y \subset X * M_3$, then there is an $N \subset M_2$ such that $Y \subset N * M_3$ and $M_1 * N \subset X$.

By Corollary 4.7, the nonvanishing of $\mathbf{r}_{M,N}$ is guaranteed if \mathcal{C} is rigid. This is also the case for the condition of having separable triple products.

Proposition 4.9 (Cf. Lemma 3.10, [KKKO15]). *If \mathcal{C} is rigid, then it has separable triple products.*

Lemma 4.10 (Cf. Lemma 3.1.5, [KKKO18]). *If M is a nonzero simple object in \mathcal{C} , then for any nonzero N_1, N_2 we have*

$$\Lambda(M, N_1 * N_2) = \Lambda(M, N_1) + \Lambda(M, N_2), \quad \Lambda(N_1 * N_2, M) = \Lambda(N_1, M) + \Lambda(N_2, M).$$

Lemma 4.11. *Let L be a subquotient of N . Then for any M we have*

$$\Lambda(M, L) \leq \Lambda(M, N), \quad \Lambda(L, M) \leq \Lambda(N, M).$$

Proof. It suffices to assume that L is either a subobject or a quotient of N . In either case one of the compositions in condition (v) must be nonzero, which forces the stated inequality. □

Proposition 4.12 (Cf. Theorem 3.2, Corollary 3.3, [KKKO15]). *Let M and N be nonzero objects such that $\mathbf{r}_{M,M}$ is a multiple of the identity and N is simple. Then:*

- (i) $\text{im } \mathbf{r}_{M,N}$ is simple, is the socle of $N * M$, and is the head of $M * N$.
- (ii) $\text{im } \mathbf{r}_{N,M}$ is simple, is the socle of $M * N$, and is the head of $N * M$.
- (iii) M is simple.

In particular, an object M is real if and only if $\mathbf{r}_{M,M}$ is a multiple of the identity.

We say M and N *commute* if $M * N$ and $N * M$ are isomorphic. When either object is real, the following criteria follow immediately from the preceding results and Definition 4.1.

Proposition 4.13 (Cf. Lemma 3.2.3, Proposition 3.2.5, [KKKO18]). *Let M and N be simple objects and suppose either M or N is real. Then the following conditions are equivalent:*

- (i) M and N commute.
- (ii) $\mathbf{r}_{M,N} \circ \mathbf{r}_{N,M}$ is a multiple of the identity.
- (iii) $M * N$ or $N * M$ is simple.
- (iv) $\Lambda(M, N) + \Lambda(N, M) = 0$.

*Moreover, if M_1, \dots, M_n are pairwise commuting real simple objects, then $M_1 * \dots * M_n$ is also real and simple.*

A useful consequence of an object M being real is that we can often control the integers $\Lambda(M, L)$ through the following two results. Here we follow [KKOP18] in writing $M \nabla N$ and $M \Delta N$ for the head and socle of $M * N$, respectively.

Proposition 4.14 (Cf. Proposition 3.2.11, [KKKO18]). *Let $L, M,$ and N be simple objects, and suppose that L is real and commutes with M . Then*

$$\Lambda(L, M \nabla N) = \Lambda(L, M * N), \quad \Lambda(N \nabla M, L) = \Lambda(N * M, L),$$

and the following diagrams commute:

$$\begin{array}{ccccc} L * M * N & \xrightarrow{\mathbf{r}_{L, M * N}} & M * N * L & & N * M * L & \xrightarrow{\mathbf{r}_{N * M, L}} & L * N * M \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ L * (M \nabla N) & \xrightarrow{\mathbf{r}_{L, M \nabla N}} & (M \nabla N) * L & & (N \nabla M) * L & \xrightarrow{\mathbf{r}_{N \nabla M, L}} & L * (N \nabla M) \end{array}$$

Setting $L = M$ above, we obtain

$$\Lambda(M, M \nabla N) = \Lambda(M, M * N) = \Lambda(M, N).$$

In fact, this characterizes $M \nabla N$ among simple subquotients of $M * N$.

Proposition 4.15 (Cf. Theorem 4.1.1, [KKKO18]). *Let M, N be simple and M real. If L is a simple subquotient of $M * N$ we have $\Lambda(M, L) < \Lambda(M, N)$ unless L is its head and $\Lambda(L, M) < \Lambda(N, M)$ unless L is its socle. Likewise, if L is a simple subquotient of $N * M$ we have $\Lambda(M, L) < \Lambda(M, N)$ unless L is its socle and $\Lambda(L, M) < \Lambda(N, M)$ unless L is its head. Thus $M \nabla N$ and $M \Delta N$ each appear exactly once as a factor in any composition series of $M * N$ or $N * M$.*

4.3. Graded realizations. A category $\widehat{\mathcal{C}}$ is \mathbb{Z} -graded if it is endowed with an autoequivalence, called the grading shift, which we denote by $\{\frac{1}{2}\}$. If $\widehat{\mathcal{C}}$ is \mathbb{Z} -graded we can forget the grading to obtain a category \mathcal{C} whose objects are the same but where

$$\text{Hom}_{\mathcal{C}}(M, N) = \bigoplus_{k \in \mathbb{Z}} \text{Hom}_{\widehat{\mathcal{C}}}(\widehat{M}, \widehat{N}\{\frac{k}{2}\}).$$

Here we write \widehat{M} for an object in $\widehat{\mathcal{C}}$ and M for its image in \mathcal{C} . We say that $\widehat{\mathcal{C}}$ is a **graded realization** of \mathcal{C} .

If $\widehat{\mathcal{C}}$ is monoidal we require that $\{\frac{1}{2}\}$ is compatible with the monoidal structure in the sense that, for any $\widehat{M}, \widehat{N} \in \widehat{\mathcal{C}}$, we have isomorphisms

$$(\widehat{M}\{\frac{1}{2}\}) * \widehat{N} \cong (\widehat{M} * \widehat{N})\{\frac{1}{2}\} \cong \widehat{M} * (\widehat{N}\{\frac{1}{2}\}).$$

Note that if \mathcal{C} has a system of renormalized r -matrices, then

$$\mathbf{r}_{M, N} \in \text{Hom}_{\mathcal{C}}(M * N, N * M) = \bigoplus_{k \in \mathbb{Z}} \text{Hom}_{\widehat{\mathcal{C}}}(\widehat{M} * \widehat{N}, \widehat{N} * \widehat{M}\{\frac{k}{2}\}).$$

Definition 4.16. A system of renormalized r -matrices in \mathcal{C} is **compatible with the graded realization** $\widehat{\mathcal{C}}$ if $\mathbf{r}_{M, N}$ has pure degree $\Lambda(M, N)$ for all $M, N \in \mathcal{C}$. This means that for all $\widehat{M}, \widehat{N} \in \widehat{\mathcal{C}}$ we have a morphism

$$\mathbf{r}_{\widehat{M}, \widehat{N}} : \widehat{M} * \widehat{N} \rightarrow \widehat{N} * \widehat{M}\{\Lambda(M, N)\}.$$

Example 4.17. Recall from Example 4.3 the monoidal category $\bigoplus_{\beta \in Q_+} R(\beta)\text{-mod}$, where $R(\beta)\text{-mod}$ is the category of gradable modules over the symmetric KLR algebra $R(\beta)$ of the root β . By definition, this has a graded realization given by the direct sum $\bigoplus_{\beta \in Q_+} R(\beta)\text{-gmod}$ of the associated categories of graded modules.

This realization is compatible with the renormalized r -matrices of [KKK18] in the above sense.

Example 4.18. Suppose \mathcal{C} inherits a system of renormalized r -matrices from a compatible \mathbb{G}_m -equivariant chiral category $\widehat{\mathcal{C}}^{ch}$ on \mathbb{A}^1 via the construction of Section 5.2. Then these r -matrices are compatible with the natural graded realization $\widehat{\mathcal{C}}$ contained in the homotopy category of the fiber of $\widehat{\mathcal{C}}^{ch}$ at $\{0\}/\mathbb{G}_m$ (cf. Proposition 5.12).

The shift $\{\frac{1}{2}\}$ is equivalent to the functor $M \mapsto M * 1_{\widehat{\mathcal{C}}}\{\frac{1}{2}\}$. We denote by $q^{-1/2} \in K_0(\widehat{\mathcal{C}})$ the class of $1_{\widehat{\mathcal{C}}}\{\frac{1}{2}\}$ (so that multiplication by $q^{-1/2}$ corresponds to shifting by $\{\frac{1}{2}\}$). Thus $K_0(\widehat{\mathcal{C}})$ inherits the structure of a $\mathbb{Z}[q^{\pm 1/2}]$ -algebra. We say two objects $\widehat{M}, \widehat{N} \in \widehat{\mathcal{C}}$ q -commute if $\widehat{M} * \widehat{N}$ and $\widehat{N} * \widehat{M}$ are isomorphic up to a grading shift. In particular, their images in \mathcal{C} commute.

Clearly in any \mathbb{Z} -graded monoidal category if \widehat{M} and \widehat{N} q -commute, then their classes in the Grothendieck ring q -commute. By the following result the existence of a compatible system of renormalized r -matrices guarantees that the converse also holds when the objects are simple and one of them is real. Note that in this case for q -commuting simples \widehat{M}, \widehat{N} we must have $\widehat{M} * \widehat{N} \cong \widehat{N} * \widehat{M} \{\Lambda(M, N)\}$.

In the statement below we assume \mathcal{C} is as in the previous section and let $\{\widehat{S}_b\}_{b \in \mathcal{B}}$ be a set of representatives up to grading shifts of all isomorphism classes of simple objects in $\widehat{\mathcal{C}}$. As we have assumed \mathcal{C} , hence $\widehat{\mathcal{C}}$, to be of finite length, the classes $\{[\widehat{S}_b]\}_{b \in \mathcal{B}}$ form a $\mathbb{Z}[q^{\pm 1/2}]$ -basis of $K_0(\widehat{\mathcal{C}})$.

Theorem 4.19 (Cf. Theorem 4.1.3, [KKKO18]). *Consider an element*

$$\Phi = \sum_{b \in \mathcal{B}} a_b [\widehat{S}_b] \in \mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm 1/2}]} K_0(\widehat{\mathcal{C}}),$$

and suppose that \widehat{M} is a real simple object of $\widehat{\mathcal{C}}$ such that $\Phi[\widehat{M}] = q^n[\widehat{M}]\Phi$ for some $n \in \frac{1}{2}\mathbb{Z}$. Then for any $b \in \mathcal{B}$ with $a_b \neq 0$ the objects \widehat{M} and \widehat{S}_b q -commute with $\Lambda(M, S_b) = n$.

4.4. Mutation via r -matrices. We are now ready to discuss the relationship between renormalized r -matrices and monoidal categorifications of cluster algebras, following [KKKO18]. We first recall some basic notions from [FZ02] and [BZ05].

Let $I = I_{ex} \sqcup I_{fr}$ be a finite index set divided into two subsets called exchangeable and frozen. Given an $I \times I$ integer matrix \widetilde{B} we refer to its $I_{ex} \times I_{ex}$ submatrix B as its **principal part**. We say $\widetilde{B} = (b_{ij})$ is an **exchange matrix** if B is skew-symmetrizable (that is, DB is skew-symmetric for some diagonal matrix D with positive diagonal entries). When B is skew-symmetric, it is convenient to encode \widetilde{B} as a quiver with $Q_0 = I$, no self-loops or oriented 2-cycles, no arrows between frozen vertices, and

$$b_{ij} = \#\{\text{arrows } i \rightarrow j\} - \#\{\text{arrows } j \rightarrow i\}.$$

A skew-symmetric $I \times I$ matrix $L = (\lambda_{ij})$ is **compatible** with \widetilde{B} if $L\widetilde{B}$ is zero except along the diagonal of its principal part, where it has positive integer entries. We refer to (L, \widetilde{B}) as a **compatible pair**.

The **quantum torus** $\mathcal{T}_q(L)$ attached to L is the $\mathbb{Z}[q^{\pm 1/2}]$ -algebra generated by elements $\{X^v\}_{v \in \mathbb{Z}^I}$ with multiplication

$$X^u X^v = q^{\frac{1}{2}u^T L v} X^{u+v}.$$

Thus, for example, $X^u X^v = q^{u^T L v} X^v X^u$. We write $\mathcal{F}_q(L)$ for the division ring of fractions of $\mathcal{T}_q(L)$, and $\mathcal{R}_q(L) \subset \mathcal{T}_q(L)$ for the quantum affine space spanned as a $\mathbb{Z}[q^{\pm 1/2}]$ -module by $\{X^v\}_{v \in \mathbb{N}^I}$.

A **quantum seed** $S = (\{x_i\}_{i \in I}, L, \tilde{B})$ in a division ring \mathcal{F} is the data of

- (1) a compatible pair (L, \tilde{B}) ,
- (2) nonzero elements $x_i \in \mathcal{F}$ such that $x_i x_j = q^{\lambda_{ij}} x_j x_i$ and such that the induced map $M_S : \mathcal{F}_q(L) \rightarrow \mathcal{F}$ is injective.

The set $\{x_i\}_{i \in I}$ is a **quantum cluster** and its elements **quantum cluster variables**. We refer to an element of \mathcal{F} of the form $M_S(X^u)$ for $u \in \mathbb{N}^I \subset \mathbb{Z}^I$ as a **quantum cluster monomial**.

Given $k \in I_{ex}$, the **mutation of S in direction k** is the quantum seed

$$\mu_k(S) = (\{\mu_k(x_i)\}_{i \in I}, \mu_k(L), \mu_k(\tilde{B}))$$

defined as follows. Consider auxiliary matrices $E = (e_{ij})$, $F = (f_{ij})$ given by

$$e_{ij} = \begin{cases} \delta_{ij}, & j \neq k, \\ -1, & i = k = j, \\ \max(0, -b_{ik}), & i \neq k = j, \end{cases} \quad f_{ij} = \begin{cases} \delta_{ij}, & i \neq k, \\ -1, & i = k = j, \\ \max(0, b_{kj}), & i = k \neq j. \end{cases}$$

With these in hand we set

$$\mu_k(L) := E^T L E, \quad \mu_k(\tilde{B}) := E \tilde{B} F.$$

These have the property that $\mu_k(L)\mu_k(\tilde{B}) = L\tilde{B}$. We then set

$$\mu_k(x_i) = \begin{cases} M_S(X^{v_+}) + M_S(X^{v_-}), & i = k, \\ x_i, & i \neq k, \end{cases}$$

where

$$v_+ = -e_k + \sum_{b_{ik} > 0} b_{ik} e_i, \quad v_- = -e_k - \sum_{b_{ik} < 0} b_{ik} e_i.$$

The pair (L, \tilde{B}) defines a canonical quantum seed in $\mathcal{F}_q(L)$ with $x_i = X^{e_i}$. The **quantum cluster algebra** $\mathcal{A}_{(L, \tilde{B})}$ is the $\mathbb{Z}[q^{\pm 1/2}]$ -subalgebra of $\mathcal{F}_q(L) \subset \mathcal{F}$ generated by all quantum cluster variables of all seeds obtained by iterated mutation from this initial one. By the quantum cluster variables of $\mathcal{A}_{(L, \tilde{B})}$ we mean the set of quantum cluster variables from all such seeds, likewise for quantum cluster monomials. We also write $\mathcal{A}_{(L, \tilde{B})}^{loc}$ for the localization of $\mathcal{A}_{(L, \tilde{B})}$ at its frozen variables. The **cluster algebra** $\mathcal{A}_{\tilde{B}}$ is the commutative ring and is defined analogously but with all notions replaced by their specializations at $q = 1$.

Definition 4.20. A \mathbb{Z} -graded, finite length, monoidal abelian category $\hat{\mathcal{C}}$ is a **monoidal categorification** of $\mathcal{A}_{(L, \tilde{B})}$ if there is a $\mathbb{Z}[q^{\pm 1/2}]$ -algebra isomorphism $K_0(\hat{\mathcal{C}}) \cong \mathcal{A}_{(L, \tilde{B})}$ which identifies all quantum cluster monomials with classes of simple objects.

Similarly, if \mathcal{C} does not have a \mathbb{Z} -grading we say it is a monoidal categorification of $\mathcal{A}_{\tilde{B}}$ if there is an isomorphism $K_0(\mathcal{C}) \cong \mathcal{A}_{\tilde{B}}$ which identifies all cluster monomials with classes of simple objects. Monoidal categorifications of $\mathcal{A}_{(L, \tilde{B})}^{loc}$ and $\mathcal{A}_{\tilde{B}}^{loc}$ are defined the same way. Note that the simple objects whose classes are (quantum) cluster variables are necessarily real and that two such objects in a common cluster necessarily (q -)commute.

Remark 4.21. The notion of monoidal cluster categorification was introduced in [HL10] and the quantum version studied in [KKKO18]. We note that [HL10] asks for the stronger condition that all cluster variables are classes of prime objects, but as in [KKKO18] we do not (although see Remark 6.19). Conversely, as in [HL10] but not [KKKO18] we do not insist on any particular exact sequences in our definition. In the quantum case we also deviate from [KKKO18] in omitting any discussion of self-dual or bar-invariant objects (although see Section 6.2).

We have the following categorical counterparts of the algebraic notions reviewed above, again fixing a ring category \mathcal{C} with a system of renormalized r -matrices and separable triple products. We say that a collection $\{M_i\}_{i \in I}$ of pairwise commuting real simples in \mathcal{C} is a **monoidal cluster** if the objects $\{\ast_{i \in I} M_i^{a_i}\}_{(a_i) \in \mathbb{N}^I}$ are pairwise nonisomorphic. This data together with an $I \times I_{ex}$ exchange matrix constitutes a **monoidal seed**. The **coefficient matrix** $L = (\lambda_{ij})$ of a monoidal cluster is defined by $\lambda_{ij} := -\Lambda(M_i, M_j)$.

We say $(\{M_i\}_{i \neq k} \cup \{M'_k\}, \mu_k(\tilde{B}))$ is a **mutation of $(\{M_i\}_{i \in I}, \tilde{B})$ in direction k** if $M_k \ast M'_k$ fits into an exact sequence

$$(4.22) \quad 0 \rightarrow \ast_{b_{ik} > 0} M_i^{b_{ik}} \rightarrow M_k \ast M'_k \rightarrow \ast_{b_{ik} < 0} M_i^{(-b_{ik})} \rightarrow 0.$$

When clearer, we also write $\mu_k(M_i) := M_i$ for $i \neq k$ and $\mu_k(M_k) := M'_k$.

Proposition 4.23 (Cf. Proposition 7.1.2, [KKKO18]). *Let $(\{M_i\}_{i \in I}, \tilde{B})$ be a monoidal seed such that the k th column of \tilde{B} is nonzero, and let M'_k be a simple object such that $M_k \ast M'_k$ fits into the exact sequence (4.22). Then*

- (i) $(\{M_i\}_{i \neq k} \cup \{M'_k\}, \mu_k(\tilde{B}))$ is a mutation of $(\{M_i\}_{i \in I}, \tilde{B})$ in direction k if M'_k commutes with M_i for $i \neq k$ and $K_0(\mathcal{C})$ is an integral domain, and
- (ii) M'_k commutes with M_i for $i \neq k$ if the coefficient matrix L of $\{M_i\}_{i \in I}$ is compatible with \tilde{B} .

Suppose $\hat{\mathcal{C}}$ is a graded realization of \mathcal{C} compatible with its r -matrices, and let $\{\widehat{M}_i\}_{i \in I}$ be a collection of pairwise q -commuting real simple objects in $\hat{\mathcal{C}}$. Given a sequence of these objects (possibly with repetitions) we define the normalized product

$$\widehat{M}_{i_1} \odot \cdots \odot \widehat{M}_{i_n} := (\widehat{M}_{i_1} \ast \cdots \ast \widehat{M}_{i_n}) \left\{ -\frac{1}{2} \sum_{j < k} \Lambda(M_{i_j}, M_{i_k}) \right\}.$$

This is independent of the ordering of the \widehat{M}_{i_j} , hence given $(a_i) \in \mathbb{N}^I$ we have a well-defined object $\odot_{i \in I} \widehat{M}_i^{a_i}$. We say $\{\widehat{M}_i\}_{i \in I}$ is a **quantum monoidal cluster** if the objects $\{\odot_{i \in I} \widehat{M}_i^{a_i}\}_{(a_i) \in \mathbb{N}^I}$ are pairwise nonisomorphic, even up to grading shifts (equivalently the map $\mathcal{R}_q(L) \rightarrow K_0(\hat{\mathcal{C}})$ generated by the classes $\{[\widehat{M}_i]\}_{i \in I}$ is injective).

The additional data of an exchange matrix compatible with L defines a **quantum monoidal seed** $(\{\widehat{M}_i\}_{i \in I}, \widetilde{B})$. We say $(\{\widehat{M}_i\}_{i \neq k} \cup \{\widehat{M}'_k\}, \mu_k(\widetilde{B}))$ is a mutation of $(\{\widehat{M}_i\}_{i \in I}, \widetilde{B})$ in direction k if $\widehat{M}_k * \widehat{M}'_k$ fits into an exact sequence

$$(4.24) \quad 0 \rightarrow \left(\bigoplus_{b_{ik} > 0} \widehat{M}_i^{b_{ik}} \right) \{-1\} \rightarrow (\widehat{M}_k * \widehat{M}'_k) \left\{ -\frac{1}{2} \Lambda(M_k, M'_k) \right\} \rightarrow \left(\bigoplus_{b_{ik} < 0} \widehat{M}_i^{(-b_{ik})} \right) \rightarrow 0.$$

If \mathcal{C} is rigid, then mutations of monoidal seeds are unique: it follows from (4.22) that M'_k is determined by the two identities

$$M'_k \cong \left(\bigstar_{b_{ik} < 0} M_i^{(-b_{ik})} \right) \nabla M_k^R \quad \text{and} \quad M'_k \cong M_k^L \nabla \left(\bigstar_{b_{ik} > 0} M_i^{b_{ik}} \right).$$

However, for an arbitrary monoidal seed one cannot generally expect that the right-hand sides of these formulas are isomorphic and fit into the needed sequence to constitute a mutation. In particular, a monoidal seed need not admit arbitrary sequences of mutations.

Nonetheless, the following result gives two criteria for when the existence of a single initial mutation in every direction guarantees the existence of further sequences of mutations.

Theorem 4.25 (Cf. Theorem 7.1.3, [KKKO18]). *Let $(\{M_i\}_{i \in I}, \widetilde{B})$ be a monoidal seed that admits a mutation in direction k for all $k \in I_{ex}$, and suppose $K_0(\mathcal{C})$ is an integral domain. Then $(\{M_i\}_{i \in I}, \widetilde{B})$ admits a sequence $\mu_{i_n} \circ \dots \circ \mu_{i_1}$ of mutations if either of the following holds:*

- (i) *For all $1 \leq j \leq n - 1$, i_j is not a vertex of an oriented 3-cycle of the quiver associated to $\mu_{i_{j-1}} \circ \dots \circ \mu_{i_1}(\widetilde{B})$.*
- (ii) *There is a graded realization $\widehat{\mathcal{C}}$ with the following properties:*
 - *$(\{M_i\}_{i \in I}, \widetilde{B})$ lifts to a quantum monoidal seed $(\{\widehat{M}_i\}_{i \in I}, \widetilde{B})$ which admits a mutation in direction k for all $k \in I_{ex}$.*
 - *The induced embedding $\mathcal{R}_q(L) \hookrightarrow K_0(\widehat{\mathcal{C}})$ extends to an embedding*

$$\mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm 1/2}]} \mathcal{A}_{(L, \widetilde{B})} \hookrightarrow \mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm 1/2}]} K_0(\widehat{\mathcal{C}}).$$

In this case, $(\{\widehat{M}_i\}_{i \in I}, \widetilde{B})$ admits the sequence $\mu_{i_n} \circ \dots \circ \mu_{i_1}$ of mutations as a quantum monoidal seed.

Corollary 4.26. *Let $(\{\widehat{M}_i\}_{i \in I}, \widetilde{B})$ be a quantum monoidal seed in $\widehat{\mathcal{C}}$ which admits a mutation in direction k for all $k \in I_{ex}$. If the induced embedding $\mathcal{R}_q(L) \hookrightarrow K_0(\widehat{\mathcal{C}})$ extends to an isomorphism $\mathcal{A}_{(L, \widetilde{B})} \cong K_0(\widehat{\mathcal{C}})$ (resp., $\mathcal{A}_{(L, \widetilde{B})}^{loc} \cong K_0(\widehat{\mathcal{C}})$), then $\widehat{\mathcal{C}}$ is a monoidal categorification of $\mathcal{A}_{(L, \widetilde{B})}$ (resp., $\mathcal{A}_{(L, \widetilde{B})}^{loc}$).*

Remark 4.27. Only case (ii) of Theorem 4.25 appears explicitly in [KKKO18], but case (i) is implicit in its proof. We highlight case (i) because we find it to be conceptually significant in relation to potentials, as discussed in the introduction. We have also removed the hypothesis that the objects in (ii) are self-dual, as the bar involution is a somewhat less elementary structure in geometric settings than algebraic ones (see Section 6.2) and the axiomatics we consider are kept more flexible if we do not assume its presence. The reader may check that with a few extra steps the proof in [KKKO18] can be made independent of any self-duality hypotheses.

5. CHIRAL CATEGORIES AND r -MATRICES

In this section we construct a system of renormalized r -matrices in the coherent Satake category using the Beilinson-Drinfeld Grassmannian. The construction is a meromorphic analogue of the Eckmann-Hilton argument, and is similar to its counterparts in the setting of KLR and quantum loop algebras. It can be carried out in any monoidal category \mathcal{C} which can be extracted from a fiber of a chiral category with some suitable compatibilities. At this level of generality our construction should find applications in other situations and is better illuminated by the physical perspective discussed in the introduction.

5.1. Definitions and background. The theory of chiral algebras is a geometric reformulation of the theory of vertex algebras [BD04]. In its equivalent formulation as a factorization algebra, a chiral algebra on a smooth complex curve X is, roughly speaking, a family of vector spaces $\mathcal{V}_{\{x_i\}}$ which are parametrized by all finite subsets $\{x_i\} \subset X$ and which behave well with respect to disjoint unions. Informally, a chiral category is the same notion but with categories in place of vector spaces.

Definition 5.1 ([Ras14]). A (unital) chiral category \mathcal{C}^{ch} on a curve X is a multiplicative sheaf of DG categories on the (unital) Ran space of X .

We unpack this definition as far as needed, referring to [Ras14] for details. In this section all DG categories are cocomplete, and we denote by $D_{qcoh}(X)$ the DG category of quasicoherent sheaves on X . A sheaf of DG categories on X is a module category \mathcal{C}_X of $D_{qcoh}(X)$. Given a morphism $f : Y \rightarrow X$ we have the pullback $f^*\mathcal{C}_X := D_{qcoh}(Y) \otimes_{D_{qcoh}(X)} \mathcal{C}_X$.

The **Ran space** Ran_X of X is an object that parametrizes nonempty finite subsets of X . If $f : J \rightarrow I$ is a surjection of finite sets, we write $\Delta_f : X^I \rightarrow X^J$ for the induced diagonal morphism. A sheaf of categories on Ran_X includes the data of a sheaf of categories $\mathcal{C}_{X^I}^{ch}$ on X^I for each finite set I . These are related by equivalences

$$(5.2) \quad \Delta_f^*(\mathcal{C}_{X^J}^{ch}) \cong \mathcal{C}_{X^I}^{ch}$$

for all surjections f , and these equivalences are related by natural compatibilities.

The operation of disjoint union of finite sets endows Ran_X with a partially-defined multiplication. Let us write $[X^I \times X^J]_{disj}$ for the variety parametrizing I -tuples and J -tuples of points with no overlap. In other words

$$[X^I \times X^J]_{disj} = \{(\underline{x}, \underline{x}') \in X^I \times X^J : x_i \neq x'_j \text{ for all } i \in I, j \in J\}.$$

A **multiplicative** sheaf of categories on Ran_X is one equipped with the additional structure of equivalences

$$(5.3) \quad \mathcal{C}_{X^I}^{ch} \boxtimes \mathcal{C}_{X^J}^{ch} |_{[X^I \times X^J]_{disj}} \cong \mathcal{C}_{X^{I \sqcup J}}^{ch} |_{[X^I \times X^J]_{disj}}$$

satisfying natural compatibilities (with respect both to iterated unions of the form $I \sqcup J \sqcup K$ and to the equivalence in (5.2)).

The equivalence in (5.3) is often called the factorization equivalence. In the simplest case, if we denote by $\Delta : X \rightarrow X^2$ the diagonal embedding and by $j : U \rightarrow X^2$ its complement, then the equivalences (5.2) and (5.3) say that

$$\Delta^*(\mathcal{C}_{X^2}^{ch}) \cong \mathcal{C}_X^{ch} \quad \text{and} \quad j^*(\mathcal{C}_{X^2}^{ch}) \cong j^*(\mathcal{C}_X^{ch} \boxtimes \mathcal{C}_X^{ch}).$$

The unital Ran space Ran_X^{un} parametrizes possibly empty subsets of X and carries information about not necessarily surjective maps between finite sets. In particular,

given a sheaf of categories \mathcal{C}^{ch} on Ran_X^{un} we have for each finite set I a unit functor $\eta : D_{qcoh}(X^I) \rightarrow \mathcal{C}_X^{ch}$. Thus, given an inclusion $I \hookrightarrow I \sqcup J$, the unit induces a functor $\eta : \mathcal{C}_X^{ch} \boxtimes D_{qcoh}(X^J) \rightarrow \mathcal{C}_{X^{I \sqcup J}}^{ch}$.

Specializing to $X = \mathbb{A}^1$, the \mathbb{G}_m action on \mathbb{A}^1 is naturally compatible with diagonal and unit maps, hence induces an action on Ran_X and Ran_X^{un} . This action preserves the disjoint locus, hence the multiplicative structures on Ran_X and Ran_X^{un} descend to $\text{Ran}_X/\mathbb{G}_m$ and $\text{Ran}_X^{un}/\mathbb{G}_m$.

Definition 5.4. A \mathbb{G}_m -equivariant (unital) chiral category \mathcal{C}^{ch} on $X = \mathbb{A}^1$ is a multiplicative sheaf of DG categories on $\text{Ran}_X/\mathbb{G}_m$ (resp., $\text{Ran}_X^{un}/\mathbb{G}_m$).

5.2. Renormalized r -matrices. From here on we let \mathcal{C}^{ch} be a unital chiral category over $X = \text{Spec } \mathbb{C}[t]$. We suppose that \mathcal{C}_X^{ch} and $\mathcal{C}_{X^2}^{ch}$ possess monoidal (i.e., \mathbb{E}_1) structures compatible with their $D_{qcoh}(X)$ - and $D_{qcoh}(X^2)$ -module structures. By restriction we obtain a monoidal structure on the fiber $\mathcal{C}_0 := i_0^*(\mathcal{C}_X^{ch}) = D_{qcoh}(pt) \otimes_{D_{qcoh}(X)} \mathcal{C}_X^{ch}$ of \mathcal{C}_X^{ch} at $0 \in X$. In natural examples \mathcal{C}^{ch} is a chiral \mathbb{E}_1 -category (in particular \mathcal{C}_{X^I} is monoidal for all I), but as our construction only requires a much smaller amount of structure we do not impose this as a hypothesis.

We assume \mathcal{C}^{ch} has the following properties:

- (1) The diagonal and factorization functors relating \mathcal{C}_X^{ch} and $\mathcal{C}_{X^2}^{ch}$ are monoidal.
- (2) The restriction functor $\mathcal{C}_X^{ch} \rightarrow \mathcal{C}_0$ admits a monoidal section $\mathcal{C}_0 \rightarrow \mathcal{C}_X^{ch}$ which preserves compact objects.
- (3) The unit functors relating $\mathcal{C}_X^{ch} \boxtimes D_{qcoh}(X)$ and $\mathcal{C}_{X^2}^{ch}$ are monoidal and preserve compact objects.
- (4) The monoidal structure on $\mathcal{C}_{X^2}^{ch}$ preserves compact objects.

Here in (3) we refer to the standard monoidal structure on $D_{qcoh}(X)$. For $M \in \mathcal{C}_0$ we denote by \widetilde{M} its image in \mathcal{C}_X^{ch} under the section in (2). Note that when \mathcal{C}_X^{ch} can be trivialized over X the object \widetilde{M} is a deformation of M , which could be taken to be trivial. Triviality of \mathcal{C}_X^{ch} , as well as all properties involving compactness, will be clear in our main example involving the BD Grassmannian.

Let $h\mathcal{C}_0$ denote the homotopy category of \mathcal{C}_0 , and let $\mathcal{C} \subset h\mathcal{C}_0$ be a full monoidal subcategory consisting of compact objects. Given the above data, we will say that the monoidal category \mathcal{C} is equipped with a **compatible chiral category**. We now explain how a compatible chiral category assigns a canonical map $\mathbf{r}_{M,N} : M * N \rightarrow N * M$ and an integer $\Lambda(M, N) \in \mathbb{Z} \cup \{-\infty\}$ to any two objects $M, N \in \mathcal{C}$.

Consider the two unit functors

$$\eta_1 : \mathcal{C}_X^{ch} \boxtimes D_{qcoh}(X) \rightarrow \mathcal{C}_{X^2}^{ch} \quad \text{and} \quad \eta_2 : D_{qcoh}(X) \boxtimes \mathcal{C}_X^{ch} \rightarrow \mathcal{C}_{X^2}^{ch}.$$

These correspond to the inclusions $i_1 : \{1\} \hookrightarrow \{1, 2\}$ and $i_2 : \{2\} \hookrightarrow \{1, 2\}$. On the other hand, the equivalence (5.2) applied to the two projections $p_1 : \{1, 2\} \twoheadrightarrow \{1\}$ and $p_2 : \{1, 2\} \twoheadrightarrow \{2\}$ gives us equivalences $\Delta_{p_1}^*(\mathcal{C}_{X^2}^{ch}) \cong \mathcal{C}_X^{ch} \cong \Delta_{p_2}^*(\mathcal{C}_{X^2}^{ch})$. Since $p_1 \circ i_1$ and $p_2 \circ i_2$ are identity maps it follows that these equivalences identify the restrictions of $\eta_1(\widetilde{M} \boxtimes \mathcal{O}_X)$ and $\eta_2(\mathcal{O}_X \boxtimes \widetilde{N})$ to the diagonal $\Delta \subset X^2$ with \widetilde{M} and \widetilde{N} , respectively.

Next, consider the two products

$$(5.5) \quad \eta_1(\widetilde{M} \boxtimes \mathcal{O}_X) * \eta_2(\mathcal{O}_X \boxtimes \widetilde{N}) \quad \text{and} \quad \eta_2(\mathcal{O}_X \boxtimes \widetilde{N}) * \eta_1(\widetilde{M} \boxtimes \mathcal{O}_X).$$

Since the factorization equivalence $\mathcal{C}_{X^2}^{ch}|_{[X^2]_{disj}} \cong (\mathcal{C}_X^{ch} \boxtimes \mathcal{C}_X^{ch})|_{[X^2]_{disj}}$ and unit functors are monoidal by conditions (1) and (3), and since \mathcal{O}_X is the monoidal unit

of $D_{qcoh}(X)$, the restrictions of the above two products to $[X^2]_{disj}$ are isomorphic (under the factorization equivalence) to $\widetilde{M} \boxtimes \widetilde{N} \in (\mathcal{C}_X^{ch} \boxtimes \mathcal{C}_X^{ch})|_{[X^2]_{disj}}$.

We will study the restrictions of these two products to $X \times \{0\} \subset X^2$. We denote these by $C_{M,N}$ and $C'_{M,N}$, respectively. If we let $\iota : \{0\} \rightarrow X$ denote the inclusion of the origin and $j : U \rightarrow X$ the inclusion of its complement, then following the above discussion we have

$$\begin{aligned} \iota^*(C_{M,N}) &\cong M * N, & \iota^*(C'_{N,M}) &\cong N * M, \\ j^*(C_{M,N}) &\cong j^*((\widetilde{M} \boxtimes \widetilde{N})|_{X \times \{0\}}) \cong j^*(C'_{N,M}). \end{aligned}$$

The identity map gives us a natural section

$$(5.6) \quad \mathcal{O}_U \rightarrow \text{Hom}(j^*C_{M,N}, j^*C'_{N,M}).$$

Combining conditions (2), (3), and (4) it follows that the two objects in (5.5) are compact. Moreover, the restriction functor to $X \times \{0\} \subset X^2$ has a right adjoint which commutes with colimits (cf. [Ras14, Remark 19.2.1]). This implies that $C_{M,N}$ and $C'_{N,M}$ are also compact. But we then have

$$\begin{aligned} \text{Hom}(j^*C_{M,N}, j^*C'_{N,M}) &\cong \text{Hom}(C_{M,N}, j_*j^*C'_{N,M}) \\ &\cong \text{Hom}(C_{M,N}, C'_{N,M} \otimes_{\mathbb{C}[t]} \mathbb{C}[t^\pm]) \\ &\cong \text{Hom}(C_{M,N}, C'_{N,M} \otimes_{\mathbb{C}[t]} \text{colim } t^{-k}\mathbb{C}[t]) \\ &\cong \text{Hom}(C_{M,N}, \text{colim}(C'_{N,M} \otimes_{\mathbb{C}[t]} t^{-k}\mathbb{C}[t])) \\ &\cong \text{colim}(\text{Hom}(C_{M,N}, C'_{N,M} \otimes_{\mathbb{C}[t]} t^{-k}\mathbb{C}[t])) \\ &\cong \text{Hom}(C_{M,N}, C'_{N,M} \otimes_{\mathbb{C}[t]} \mathbb{C}[t^\pm]), \end{aligned}$$

where compactness of $C_{M,N}$ is used to obtain the second-to-last isomorphism. It follows that (5.6) gives a map $\mathcal{O}_U \rightarrow j^*(V_{M,N})$, where $V_{M,N} := \text{Hom}(C_{M,N}, C'_{N,M})$.

Consider the composition

$$(5.7) \quad \alpha : \mathcal{O}_X \rightarrow j_*\mathcal{O}_U \rightarrow j_*j^*V_{M,N} \cong V_{M,N} \otimes_{\mathbb{C}[t]} \mathbb{C}[t^\pm].$$

The $\mathbb{C}[t]$ -module $\mathbb{C}[t^\pm]$ has a natural increasing filtration

$$\dots \subset F^{k-1} \subset F^k \subset F^{k+1} \subset \dots \subset \mathbb{C}[t^\pm],$$

where $F^k := t^{-k}\mathbb{C}[t]$. For a $\mathbb{C}[t]$ -module M we let $[M]^k$ denote the k th piece of the induced filtration on $M \otimes_{\mathbb{C}[t]} \mathbb{C}[t^\pm]$, letting $[M]^{-\infty}$ denote $\bigcap_{k \in \mathbb{Z}} [M]^k$.

Since \mathcal{O}_X is a compact object there exists some $k \in \mathbb{Z} \cup \{-\infty\}$ so that the map α factors through $[V_{M,N}]^k$. Given such a k we let $\alpha^k \in H^0([V_{M,N}]^k)$ denote the resulting section. We further let $\Lambda(M, N) \in 2\mathbb{Z} \cup \{-\infty\}$ be *twice* the minimal value such that α factors through $[V_{M,N}]^k$ (cf. Remark 5.13).

Suppose $\Lambda(M, N) \in 2\mathbb{Z}$. We can tensor the exact sequence

$$0 \rightarrow F^{\frac{1}{2}\Lambda(M,N)-1} \rightarrow F^{\frac{1}{2}\Lambda(M,N)} \rightarrow \mathbb{C}_0 \otimes_{\mathbb{C}[t]} F^{\frac{1}{2}\Lambda(M,N)} \rightarrow 0$$

with $V_{M,N}$ to obtain a long exact sequence

$$\begin{aligned} \dots \rightarrow H^0([V_{M,N}]^{\frac{1}{2}\Lambda(M,N)-1}) &\rightarrow H^0([V_{M,N}]^{\frac{1}{2}\Lambda(M,N)}) \\ &\xrightarrow{\pi} H^0(V_{M,N}|_{\{0\}} \otimes_{\mathbb{C}[t]} F^{\frac{1}{2}\Lambda(M,N)}) \rightarrow \dots \end{aligned}$$

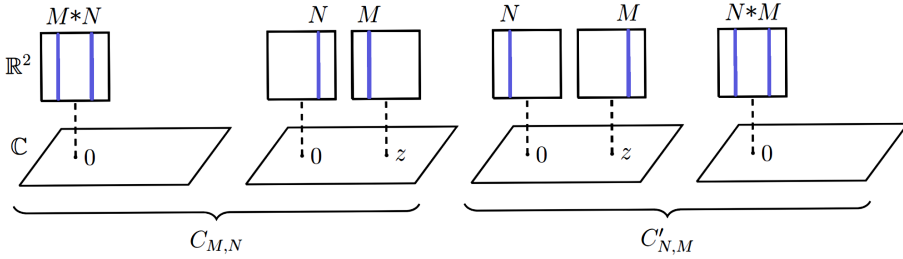


FIGURE 5.1

in cohomology. By construction we have $\pi(\alpha^{\frac{1}{2}\Lambda(M,N)}) \neq 0$. But

$$\begin{aligned} V_{M,N}|_{\{0\}} \otimes_{\mathbb{C}[t]} F^{\frac{1}{2}\Lambda(M,N)} &\cong V_{M,N}|_{\{0\}} \cong \text{Hom}(\iota^*(C_{M,N}), \iota^*(C'_{N,M})) \\ &\cong \text{Hom}(M * N, N * M), \end{aligned}$$

so $\pi(\alpha^{\frac{1}{2}\Lambda(M,N)})$ yields a nonzero map $\mathbf{r}_{M,N} \in H^0(\text{Hom}(M * N, N * M))$.

On the other hand, if $\Lambda(M, N) = -\infty$ we set $\mathbf{r}_{M,N} = 0$. Note this is just the result of applying the above construction to the exact sequence of complexes whose first two terms are $[V_{M,N}]^{\frac{1}{2}\Lambda(M,N)-1} = [V_{M,N}]^{\frac{1}{2}\Lambda(M,N)} = [V_{M,N}]^{-\infty}$ but whose last term is now zero.

Remark 5.8. The construction above uses the restriction to $X \times \{0\} \subset X^2$. Restriction to other lines through the origin will give an *a priori* different collection of r -maps. However, in the case that \mathcal{C}_X^{ch} is really a trivial family over X (as is the case with the Beilinson-Drinfeld construction in Section 5.3) the r -matrices will be independent of this choice.

Remark 5.9. Following the discussion in the introduction of line operators in 4d holomorphic-topological field theory it is instructive to visualize the present construction as in Figure 5.1. The objects of \mathcal{C}_0 are interpreted as being extended along lines in \mathbb{R}^2 with a fixed slope. Morphisms are interpreted as junctions between objects extended along different regions of a fixed line, while the monoidal structure is interpreted as collision of parallel lines. The structure of \mathcal{C}_0 is constrained by the existence of an extension to a family \mathcal{C}_X^{ch} over $X = \mathbb{C}$ in different ways depending on the nature of the dependence of \mathcal{C}_z on the point $z \in X$.

Were we considering a topological family, the usual Eckmann-Hilton argument depicted in the figure would force the monoidal structure on the homotopy category $h\mathcal{C}_0$, and interesting subcategories thereof, to be symmetric (\mathcal{C}_0 itself possessing an \mathbb{E}_3 structure). This is the case for the constructible Satake category, or more generally for line operators in a fully topological theory in 4d (e.g. topologically twisted $\mathcal{N} = 4$ gauge theory [KW07]).

On the other hand, the existence of a holomorphic or algebraic family \mathcal{C}_X^{ch} does not force $h\mathcal{C}_0$ to be symmetric. Instead, what one can salvage from the Eckmann-Hilton construction is exactly the pair $C_{M,N}, C'_{N,M}$ of sections of \mathcal{C}_X^{ch} which are isomorphic away from $z = 0$. The resulting constraint is that the subcategory of compact objects in $h\mathcal{C}_0$ possesses a system of renormalized r -matrices.

Theorem 5.10. *The maps $\mathbf{r}_{M,N}$ and elements $\Lambda(M, N) \in 2\mathbb{Z} \cup \{-\infty\}$ defined above form a system of renormalized r -matrices in \mathcal{C} .*

Proof. Properties (i) and (ii) of Definition 4.1 are clear. To check property (iii) consider $M, N_1, N_2 \in \mathcal{C}$ and let

$$\begin{aligned} C_{M,N_1,N_2} &:= [\eta_1(\widetilde{M} \boxtimes \mathcal{O}_X) * \eta_2(\mathcal{O}_X \boxtimes \widetilde{N}_1) * \eta_2(\mathcal{O}_X \boxtimes \widetilde{N}_2)]|_{X \times \{0\}}, \\ C'_{M,N_1,N_2} &:= [\eta_2(\mathcal{O}_X \boxtimes \widetilde{N}_1) * \eta_1(\widetilde{M} \boxtimes \mathcal{O}_X) * \eta_2(\mathcal{O}_X \boxtimes \widetilde{N}_2)]|_{X \times \{0\}}, \\ C''_{M,N_1,N_2} &:= [\eta_2(\mathcal{O}_X \boxtimes \widetilde{N}_1) * \eta_2(\mathcal{O}_X \boxtimes \widetilde{N}_2) * \eta_1(\widetilde{M} \boxtimes \mathcal{O}_X)]|_{X \times \{0\}}. \end{aligned}$$

As in the construction of $\mathbf{r}_{M,N}$, after restricting to $U \subset X$ the factorization equivalence identifies each of these with $\widetilde{M} \boxtimes \widetilde{N_1 * N_2}$. The identity of $\widetilde{M} \boxtimes \widetilde{N_1 * N_2}$ then induces maps

$$\begin{aligned} \alpha_1 : \mathcal{O}_X &\rightarrow \text{Hom}(C_{M,N_1,N_2}, C'_{M,N_1,N_2}) \otimes_{\mathbb{C}[t]} \mathbb{C}[t^{\pm 1}], \\ \alpha_2 : \mathcal{O}_X &\rightarrow \text{Hom}(C'_{M,N_1,N_2}, C''_{M,N_1,N_2}) \otimes_{\mathbb{C}[t]} \mathbb{C}[t^{\pm 1}] \end{aligned}$$

which factor through

$$[\text{Hom}(C_{M,N_1,N_2}, C'_{M,N_1,N_2})]^{\frac{1}{2}\Lambda(M,N_1)} \text{ and } [\text{Hom}(C'_{M,N_1,N_2}, C''_{M,N_1,N_2})]^{\frac{1}{2}\Lambda(M,N_2)},$$

respectively.

On the other hand,

$$\begin{aligned} C_{M,N_1,N_2} &\cong [\eta_1(\widetilde{M} \boxtimes \mathcal{O}_X) * \eta_2(\mathcal{O}_X \boxtimes \widetilde{N_1 * N_2})]|_{X \times \{0\}} \cong C_{M,N_1 * N_2}, \\ C''_{M,N_1,N_2} &\cong [\eta_2(\mathcal{O}_X \boxtimes \widetilde{N_1 * N_2}) * \eta_1(\widetilde{M} \boxtimes \mathcal{O}_X)]|_{X \times \{0\}} \cong C'_{N_1 * N_2, M}, \end{aligned}$$

where the right-hand sides refer to the notation used in the construction of $\mathbf{r}_{M,N_1 * N_2}$. Since they are both obtained by adjunction from the same identity section over U , we have an equality $\alpha_2 \circ \alpha_1 = \alpha$ of maps

$$\mathcal{O}_X \rightarrow \text{Hom}(C_{M,N_1 * N_2}, C'_{N_1 * N_2, M}) \otimes_{\mathbb{C}[t]} \mathbb{C}[t^{\pm 1}]$$

(where α is the map used to define $\mathbf{r}_{M,N_1 * N_2}$).

Since $\alpha_2 \circ \alpha_1$ factors through $[\text{Hom}(C_{M,N_1 * N_2}, C'_{N_1 * N_2, M})]^{\frac{1}{2}\Lambda(M,N_1) + \frac{1}{2}\Lambda(M,N_2)}$ it follows from the minimality of $\Lambda(M, N_1 * N_2)$ that

$$(5.11) \quad \Lambda(M, N_1 * N_2) \leq \Lambda(M, N_1) + \Lambda(M, N_2).$$

On the other hand, the final part of the constructions of $\mathbf{r}_{M,N_1 * N_2}$ applied to α_1 and α_2 produces the maps $\mathbf{r}_{M,N_1} * id_{N_2}$ and $id_{N_1} * \mathbf{r}_{M,N_2}$, respectively. If (5.11) is an equality it follows straightforwardly that the composition of these is equal to $\mathbf{r}_{M,N_1 * N_2}$. If instead the inequality is strict, the composition is zero since $\alpha^{\frac{1}{2}\Lambda(M,N_1) + \frac{1}{2}\Lambda(M,N_2)}$ is the image of $\alpha^{\frac{1}{2}\Lambda(M,N_1) + \frac{1}{2}\Lambda(M,N_2) - 1}$ in the relevant long exact sequence. This establishes property (iii).

To check property (iv) let

$$\alpha_1 : \mathcal{O}_X \rightarrow V_{M,N} \otimes_{\mathbb{C}[t]} \mathbb{C}[t^{\pm 1}], \quad \alpha_2 : \mathcal{O}_X \rightarrow V_{N,M} \otimes_{\mathbb{C}[t]} \mathbb{C}[t^{\pm 1}]$$

denote the maps appearing in the construction of $\mathbf{r}_{M,N}$ and $\mathbf{r}_{N,M}$, respectively. The composition $\alpha_2 \circ \alpha_1$ factors through $[\text{Hom}(C_{M,N}, C_{M,N})]^{\frac{1}{2}\Lambda(M,N) + \frac{1}{2}\Lambda(N,M)}$. On the other hand, it must also equal the map $\mathcal{O}_X \rightarrow \text{Hom}(C_{M,N}, C_{M,N}) \otimes_{\mathbb{C}[t]} \mathbb{C}[t^{\pm 1}]$ induced by the identity of $C_{M,N}$, which factors through $[\text{Hom}(C_{M,N}, C_{M,N})]^0$ but not $[\text{Hom}(C_{M,N}, C_{M,N})]^k$ for any $k < 0$. It follows as above that $\Lambda(M, N) + \Lambda(N, M) \geq 0$ with $\mathbf{r}_{N,M} \circ \mathbf{r}_{M,N}$ equal to $id_{M * N}$ if equality holds, zero otherwise. This proves condition (iv).

Finally, to check property (v) consider

$$C_{M,N_1} := [\eta_1(\widetilde{M} \boxtimes \mathcal{O}_X) * \eta_2(\mathcal{O}_X \boxtimes \widetilde{N}_1)]|_{X \times \{0\}},$$

$$C'_{N_2,M} := [\eta_2(\mathcal{O}_X \boxtimes \widetilde{N}_2) * \eta_1(\widetilde{M} \boxtimes \mathcal{O}_X)]|_{X \times \{0\}}.$$

These satisfy

$$i^*(C_{M,N_1}) \cong M * N_1, \quad i^*(C'_{N_2,M}) \cong N_2 * M,$$

$$j^*(C_{M,N_1}) \cong j^*(\widetilde{M} \boxtimes \widetilde{N}_1), \quad j^*(C'_{N_2,M}) \cong j^*(\widetilde{M} \boxtimes \widetilde{N}_2).$$

Letting $V_{M,N_\bullet} := \text{Hom}(C_{M,N_1}, C'_{N_2,M})$ we have a map

$$\alpha_f : \mathcal{O}_X \rightarrow V_{M,N_\bullet} \otimes_{\mathbb{C}[t]} \mathbb{C}[t^{\pm 1}]$$

obtained by adjunction from the morphism

$$id_{\widetilde{M}} \boxtimes \widetilde{f} : \widetilde{M} \boxtimes \widetilde{N}_1 \rightarrow \widetilde{M} \boxtimes \widetilde{N}_2.$$

We let $\Lambda(M, f) \in 2\mathbb{Z} \cup \{-\infty\}$ be the minimal value such that α_f factors through $[V_{M,N_\bullet}]^{\frac{1}{2}\Lambda(M,f)}$. As before we obtain a map $\mathbf{r}_{M,f} : M * N_1 \rightarrow N_2 * M$, where for example if $N_1 = N = N_2$ and $f = id$ we recover $\mathbf{r}_{M,N}$.

On the other hand, we also have

$$\beta_1 := [\eta_2(id_{\mathcal{O}_X} \boxtimes \widetilde{f}) * \eta_1(id_{\widetilde{M} \boxtimes \mathcal{O}_X})]|_{X \times \{0\}} \circ \alpha_1 : \mathcal{O}_X \rightarrow V_{M,N_\bullet} \otimes_{\mathbb{C}[t]} \mathbb{C}[t^{\pm 1}],$$

$$\beta_2 := \alpha_2 \circ [\eta_1(id_{\widetilde{M} \boxtimes \mathcal{O}_X}) * \eta_2(id_{\mathcal{O}_X} \boxtimes \widetilde{f})]|_{X \times \{0\}} : \mathcal{O}_X \rightarrow V_{M,N_\bullet} \otimes_{\mathbb{C}[t]} \mathbb{C}[t^{\pm 1}],$$

where α_1 and α_2 are the maps corresponding to \mathbf{r}_{M,N_1} and \mathbf{r}_{M,N_2} . These factor through $[V_{M,N_\bullet}]^{\frac{1}{2}\Lambda(M,N_1)}$ and $[V_{M,N_\bullet}]^{\frac{1}{2}\Lambda(M,N_2)}$, respectively. Considering the fiber over $0 \in X$, the usual construction applied to β_1, β_2 recovers

$$(f * id_M) \circ \mathbf{r}_{M,N_1} \quad \text{and} \quad \mathbf{r}_{M,N_2} \circ (id_M * f) \in \text{Hom}(M * N_1, N_2 * M).$$

From this we see that

- $\Lambda(M, f) \leq \Lambda(M, N_1)$ with $(f * id_M) \circ \mathbf{r}_{M,N_1}$ equal to $\mathbf{r}_{M,f}$ when equality holds and zero otherwise,
- $\Lambda(M, f) \leq \Lambda(M, N_2)$ with $\mathbf{r}_{M,N_2} \circ (id_M * f)$ equal to $\mathbf{r}_{M,f}$ when equality holds and zero otherwise.

Condition (v) now follows. Note that if $\Lambda(M, N_1) = \Lambda(M, N_2)$ but both are greater than $\Lambda(M, f)$, then the diagram in (v) commutes with both compositions being zero. □

Finally, suppose that \mathcal{C}^{ch} is pulled back from a \mathbb{G}_m -equivariant chiral category $\widehat{\mathcal{C}^{ch}}$ along the map $\text{Ran}_X^{un} \rightarrow \text{Ran}_X^{un}/\mathbb{G}_m$. Here we take \mathbb{G}_m to act on X with weight 4, i.e., we take the 4-fold cover of the standard action; this choice meshes most naturally with our general conventions about gradings. The fiber $\widehat{\mathcal{C}}_0$ of $\widehat{\mathcal{C}^{ch}}$ at $\{0\}/\mathbb{G}_m$ is the category of \mathbb{G}_m -equivariant objects in \mathcal{C}_0 . The full subcategory $\widehat{\mathcal{C}} \subset h\widehat{\mathcal{C}}_0$ consisting of objects whose image under $\widehat{\mathcal{C}} \rightarrow \mathcal{C}_0$ lies in \mathcal{C} is a graded realization in the sense of Section 4.3.

Proposition 5.12. *The system of renormalized r -matrices constructed in \mathcal{C} is compatible with the graded realization $\widehat{\mathcal{C}}$.*

Proof. As before we denote by $\{\cdot\}$ the \mathbb{G}_m -equivariance shift. Recall that

$$\pi(\alpha) \in H^0(V_{M,N}|_{\{0\}} \otimes_{\mathbb{C}[t]} F^k).$$

The result then follows since $F^k = t^{-k}\mathbb{C}[t] \xrightarrow{\sim} \mathbb{C}[t]\{2k\}$, where $t^{-k} \mapsto 1$ and $2k = \Lambda(M, N)$. □

Remark 5.13. The appearance of the shift $\{2k\}$ above explains why earlier we needed to define $\Lambda(M, N)$ as “twice the minimal value...”. It is possible to rescale the shifts so that this $\{2k\}$ becomes $\{k\}$ but this would clash with some common conventions used when dealing with equivariant shifts and the affine Grassmannian (particularly in Section 2).

5.3. The Beilinson-Drinfeld Grassmannian. The affine Grassmannian has the following moduli interpretation. Fix a smooth curve X and $x \in X$. Then Gr_G can be identified with the moduli of pairs (P, ϕ) , where P is a principal G -bundle on X and

$$\phi : P_0|_{X \setminus \{x\}} \xrightarrow{\sim} P|_{X \setminus \{x\}}$$

is an isomorphism (here P_0 denotes the trivial G -bundle on X). The twisted product $\text{Gr}_G \widetilde{\times} \text{Gr}_G = G(\mathcal{K}) \times_{G(\mathcal{O})} \text{Gr}_G$ also has a moduli interpretation as

$$\{(P, P', \phi, \phi') : \phi : P_0|_{X \setminus \{x\}} \xrightarrow{\sim} P|_{X \setminus \{x\}}, \phi' : P|_{X \setminus \{x\}} \xrightarrow{\sim} P'|_{X \setminus \{x\}}\}.$$

Allowing $x \in X$ to move we obtain a global version of Gr_G called the Beilinson-Drinfeld (BD) Grassmannian [BD04]. Consider a tuple $\underline{x} \in X^I$ of points labeled by a finite set I . Then the BD Grassmannian Gr_{G, X^I} parametrizes

$$\{(\underline{x}, P, \phi) : \underline{x} \in X^I, \phi : P_0|_{X \setminus \{\underline{x}\}} \xrightarrow{\sim} P|_{X \setminus \{\underline{x}\}}\},$$

where again P is a principal G -bundle. Remembering only $\underline{x} \in X^I$ gives us a map $\text{Gr}_{G, X^I} \rightarrow X^I$.

The case most relevant to us is $|I| = 2$. In this case the fibers over a point $(x_1, x_2) \in X^2$ are given by

$$(\text{Gr}_{G, X^2})_{(x_1, x_2)} \cong \begin{cases} \text{Gr}_G \times \text{Gr}_G & \text{if } x_1 \neq x_2, \\ \text{Gr}_G & \text{if } x_1 = x_2. \end{cases}$$

Taken together over all finite sets I , the BD Grassmannians Gr_{G, X^I} provide the prototypical example of a factorization space. That is, we have compatibilities along diagonal maps and disjoint unions generalizing the ones described above for $|I| = 2$.

There is also a global analogue $G_{\mathcal{O}, X^I}$ of $G(\mathcal{O})$. It parametrizes automorphisms

$$\{(\underline{x}, \gamma) : \underline{x} \in X^I, \gamma : P_0|_{\widehat{X \setminus \{\underline{x}\}}} \xrightarrow{\sim} P_0|_{\widehat{X \setminus \{\underline{x}\}}}\},$$

where $\widehat{X \setminus \{\underline{x}\}}$ denotes the formal completion of X along $\{\underline{x}\}$. One can show that $G_{\mathcal{O}, X^I}$ acts on Gr_{G, X^I} and that it has the usual naturality properties with respect to restriction to diagonals and disjoint unions. Subsequently, the DG categories $\text{IndCoh}^{G_{\mathcal{O}, X^I}}(\text{Gr}_{G, X^I})$ of equivariant ind-coherent sheaves give us a chiral category \mathcal{C}_G^{ch} on X . The section $X^I \rightarrow \text{Gr}_{G, X^I}$ which takes $\underline{x} \mapsto \{(\underline{x}, P_0, id)\}$ equips it with a unital structure.

We also have global analogues of the convolution product on Gr_G . First consider the space parametrizing tuples

$$\{(\underline{x}, \underline{x}', P, P', \phi, \phi') \mid \underline{x} \in X^I, \underline{x}' \in X^J, \phi : P_0|_{X \setminus \{\underline{x}\}} \xrightarrow{\sim} P|_{X \setminus \{\underline{x}\}}, \phi' : P|_{X \setminus \{\underline{x}'\}} \xrightarrow{\sim} P'|_{X \setminus \{\underline{x}'\}}\}.$$

This has the structure of a twisted product $\text{Gr}_{G, X^I} \widetilde{\times} \text{Gr}_{G, X^J}$ (cf. [Zhu17, Eq. 3.1.22]). More precisely, there exists a $G_{\mathcal{O}, X^J}$ -torsor over $\text{Gr}_{G, X^I} \times X^J$ classifying

$$\mathbf{E} := \{(\underline{x}, P, \phi, \underline{x}', \gamma) : \underline{x} \in X^I, \underline{x}' \in X^J, \phi : P_0|_{X \setminus \{\underline{x}\}} \xrightarrow{\sim} P|_{X \setminus \{\underline{x}\}}, \gamma : P|_{\widehat{\{\underline{x}'\}}} \xrightarrow{\sim} P_0|_{\widehat{\{\underline{x}'\}}}\}.$$

Then $\text{Gr}_{G, X^I} \widetilde{\times} \text{Gr}_{G, X^J}$ is the fiber product $\mathbf{E} \times_{G_{\mathcal{O}, X^J}} \text{Gr}_{G, X^J}$.

As in the local case, given $\mathcal{F} \in \text{IndCoh}^{G_{\mathcal{O}, X^I}}(\text{Gr}_{G, X^I})$, $\mathcal{G} \in \text{IndCoh}^{G_{\mathcal{O}, X^J}}(\text{Gr}_{G, X^J})$ we can form the twisted product $\mathcal{F} \widetilde{\boxtimes} \mathcal{G} \in \text{IndCoh}^{G_{\mathcal{O}, X^I}}(\text{Gr}_{G, X^I} \widetilde{\times} \text{Gr}_{G, X^J})$. We then push forward along the generalized multiplication map

$$\begin{aligned} \overline{m} : \text{Gr}_{G, X^I} \widetilde{\times} \text{Gr}_{G, X^J} &\rightarrow \text{Gr}_{G, X^{I \sqcup J}} \\ (\underline{x}, \underline{x}', P, P', \phi, \phi') &\mapsto ((\underline{x}, \underline{x}'), P', \phi' \circ \phi) \end{aligned}$$

to obtain an object in $\text{IndCoh}^{G_{\mathcal{O}, X^I}}(\text{Gr}_{G, X^{I \sqcup J}})$. If $I = J$ we can restrict this object to the diagonal $X^I \subset X^{I \sqcup I}$ to obtain an object $\mathcal{F} * \mathcal{G} \in \text{IndCoh}^{G_{\mathcal{O}, X^I}}(\text{Gr}_{G, X^I})$. This defines a monoidal structure on each $(\mathcal{C}_G^{ch})_{X^I}$.

Finally, if X has a \mathbb{G}_m -action, then the construction above can be done \mathbb{G}_m -equivariantly to obtain a \mathbb{G}_m -equivariant chiral category $\widehat{\mathcal{C}}_G^{ch}$. Specifically we consider the case $X = \mathbb{A}^1$ where \mathbb{G}_m acts with weight 4.

Theorem 5.14. *The category $D_{coh}^{G(\mathcal{O})}(\text{Gr}_G)$ possesses a system of renormalized r -matrices compatible with the graded realization $D_{coh}^{G(\mathcal{O}) \rtimes \mathbb{G}_m}(\text{Gr}_G)$. By restriction, the coherent Satake category $\mathcal{P}_{coh}^{G(\mathcal{O})}(\text{Gr}_G)$ inherits a system of renormalized r -matrices compatible with $\mathcal{P}_{coh}^{G(\mathcal{O}) \rtimes \mathbb{G}_m}(\text{Gr}_G)$, and in which all r -matrices are nonzero.*

Proof. Using Theorem 5.10, we can construct a system of renormalized r -matrices in $D_{coh}^{G(\mathcal{O})}(\text{Gr}_G)$ by applying the construction of Section 5.2 to the chiral category \mathcal{C}_G^{ch} over \mathbb{A}^1 . Note that the subcategory of compact objects in $\text{IndCoh}^{G(\mathcal{O})}(\text{Gr}_G)$ is $D_{coh}^{G(\mathcal{O})}(\text{Gr}_G)$. The fact that the functors from conditions (2), (3), and (4) of Section 5.2 preserve compact objects is immediate since they all send coherent sheaves to coherent sheaves.

Compatibility with the graded realizations follows from applying Proposition 5.12 to the \mathbb{G}_m -equivariant chiral category $\widehat{\mathcal{C}}_G^{ch}$ over \mathbb{A}^1 . The fact that the r -matrices in $\mathcal{P}_{coh}^{G(\mathcal{O})}(\text{Gr}_G)$ are nonzero follows from Corollaries 3.2 and 4.7. \square

Note that in general $\text{Gr}_{G, X}$ is not isomorphic to the trivial product $\text{Gr}_X \times X$ (it is a twisted product $\text{Gr}_X \widetilde{\times} X$). But in the case $X = \mathbb{A}^1$ this twist is indeed trivial, justifying Remark 5.8.

From Theorem 5.14 we immediately obtain a number of results on the convolution structure of $\mathcal{P}_{coh}^{G(\mathcal{O}) \rtimes \mathbb{G}_m}(\text{Gr}_G)$.

Theorem 5.15. *Let $\mathcal{F}, \mathcal{G} \in \mathcal{P}_{coh}^{G(\mathcal{O}) \times \mathbb{G}_m}(\text{Gr}_G)$ be simple perverse coherent sheaves and suppose either \mathcal{F} or \mathcal{G} is real. Then*

- (1) $\mathcal{F} * \mathcal{G}$ has a simple head and a simple socle,
- (2) the head and socle of $\mathcal{F} * \mathcal{G}$ each appear exactly once as a factor in any composition series of $\mathcal{F} * \mathcal{G}$, and
- (3) if $[\mathcal{F}]$ and $[\mathcal{G}]$ q -commute in $K^{G(\mathcal{O}) \times \mathbb{G}_m}(\text{Gr}_G)$, then $\mathcal{F} * \mathcal{G}$ is simple and isomorphic to $\mathcal{G} * \mathcal{F} \{ \Lambda(\mathcal{F}, \mathcal{G}) \}$.

Proof. The three claims follow from Proposition 4.12, Proposition 4.15, and Theorem 4.19, respectively. □

We note that the second claim says that the basis of $K^{G(\mathcal{O}) \times \mathbb{G}_m}(\text{Gr}_G)$ formed by classes of simple perverse coherent sheaves satisfies a partial analogue of Leclerc’s conjecture [Lec03] on the multiplicative structure of the dual canonical (or upper global) basis of a quantized enveloping algebra $\mathcal{U}_q(\mathfrak{n})$.

6. MONOIDAL CLUSTER CATEGORIFICATION

We are now ready to combine the results of the previous sections and prove that the coherent Satake category of GL_n is a monoidal cluster categorification. We then discuss the bar involution and quantum twist from this point of view, as well as the connection to Q -systems.

6.1. A finite generating set. To begin we define an index set $I = I_{ex} \sqcup I_{fr}$ by

$$I_{ex} = \{(k, 0), (k, 1)\}_{1 \leq k < n}, \quad I_{fr} = \{(n, 0), (n, 1)\}.$$

We let $\tilde{B}_n = (b_{ij})$ and $L_n = (\lambda_{ij})$ be the $I_{ex} \times I$ and $I \times I$ matrices given by

$$\begin{aligned} b_{(k_1, \ell_1), (k_2, \ell_2)} &= (\ell_1 - \ell_2) a_{(k_1, k_2)}, \\ \lambda_{(k_1, \ell_1), (k_2, \ell_2)} &= 2(\ell_2 - \ell_1) \min\{k_1, k_2\}, \end{aligned}$$

where $A = (a_{ij})$ is the Cartan matrix of type A_{n+1} . These form a compatible pair with the principal part of $L_n \tilde{B}_n$ equal to $2\text{Id}_{I_{ex} \times I_{ex}}$.

Example 6.1. In the case $n = 3$, we have

$$L_3 = \begin{pmatrix} 0 & 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 2 & 4 & 4 \\ 0 & 0 & 0 & 2 & 4 & 6 \\ -2 & -2 & -2 & 0 & 0 & 0 \\ -2 & -4 & -4 & 0 & 0 & 0 \\ -2 & -4 & -6 & 0 & 0 & 0 \end{pmatrix}, \quad \tilde{B}_3 = \begin{pmatrix} 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \\ 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Here the rows of \tilde{B}_3 and L_3 correspond in order to the indices $(1, 0), (2, 0), (3, 0), (1, 1), (2, 1), (3, 1)$. Note that \tilde{B}_3 is the adjacency matrix of the quiver \tilde{Q}_3 of Figure 1.1.

Proposition 6.2. *The pair $(\{\mathcal{P}_{k, \ell}\}_{(k, \ell) \in I}, \tilde{B}_n)$ is a quantum monoidal seed with coefficient matrix L_n , and admits a mutation in all exchangeable directions. Its mutation in direction $(k, 1)$ is $(\{\mathcal{P}_{k, \ell}\}_{(k, \ell) \in I \setminus \{(k, 1)\}} \sqcup \{\mathcal{P}_{k, -1}\}, \mu_{(k, 1)}(\tilde{B}_n))$, and its mutation in direction $(k, 0)$ is $(\{\mathcal{P}_{k, \ell}\}_{(k, \ell) \in I \setminus \{(k, 0)\}} \sqcup \{\mathcal{P}_{k, 2}\}, \mu_{(k, 0)}(\tilde{B}_n))$.*

Proof. We know that the objects $\{\mathcal{P}_{k,\ell}\}_{(k,\ell)\in I}$ are simple and real (cf. Lemma 2.6). The fact that they q -commute with powers of q prescribed by L_n is the content of Proposition 2.9. It is clear from Proposition 2.5 that the normalized monomials in $\{\mathcal{P}_{k,\ell}\}_{(k,\ell)\in I}$ are pairwise nonisomorphic.

Consider the first of the stated mutations, the other being similar. We need to check that the first exact sequence of Proposition 2.16 coincides with the sequence in (4.24). The agreement of the outer terms follows by inspection of \tilde{B}_n . The agreement of the grading shifts of the middle terms follows from

$$\frac{1}{2}\Lambda(\mathcal{P}_{k,1}, \mathcal{P}_{k,-1}) = \frac{1}{2}\Lambda(\mathcal{P}_{k,1}, \mathcal{P}_{k,1} \nabla \mathcal{P}_{k,-1}) = \frac{1}{2}\Lambda(\mathcal{P}_{k,1}, \mathcal{P}_{k,0}^2) = 2k.$$

Finally, given that the matrices L_n and \tilde{B}_n are easily checked to be compatible, it follows from Proposition 4.23 that $(\{\mathcal{P}_{k,\ell}\}_{(k,\ell)\in I \setminus \{(k,1)\}} \sqcup \{\mathcal{P}_{k,-1}\}, \mu_{(k,1)}(\tilde{B}_n))$ is indeed a quantum monoidal seed. \square

To show that $(\{\mathcal{P}_{k,\ell}\}_{(k,\ell)\in I}, \tilde{B}_n)$ admits arbitrary mutation sequences using Theorem 4.25, we must show that the classes $\{\mathcal{P}_{k,\ell}\}_{(k,\ell)\in I}$ induce an embedding $\mathcal{A}_{(L,\tilde{B})} \hookrightarrow K^{GL_n(\mathcal{O}) \times \mathbb{G}_m}(\text{Gr}_{GL_n})$ (at least after extending scalars to $\mathbb{Q}(q^{1/2})$). We do this by interpreting $\mathcal{A}_{(L_n,\tilde{B}_n)}$ as the quantized coordinate ring of a unipotent subgroup of LSL_2 . This applies the main results of [GLS13], which we briefly recall.

Let $A = (a_{ij})$ be a symmetrizable Cartan matrix, P its weight lattice, w an element of its Weyl group, and $\mathbf{i} = (i_1, \dots, i_r)$ a list of indices such that $s_{i_r} \cdots s_{i_1}$ is a reduced word for w . We will associate to this data a compatible pair $(L(\mathbf{i}) = (\lambda(\mathbf{i})_{k\ell}), \tilde{B}(\mathbf{i}) = (b(\mathbf{i})_{k\ell}))$ as follows. For $1 \leq k \leq r$ we write

$$k^- := \max(\{0\} \cup \{1 \leq \ell < k \mid i_k = i_\ell\}),$$

$$k^+ := \min(\{k < \ell \leq r \mid i_k = i_\ell\} \cup \{r + 1\})$$

and set $\beta_k := \omega_{i_k} - w_{\leq k}^{-1}\omega_{i_k}$, where $w_{\leq k}^{-1} := s_{i_1} \cdots s_{i_k}$. The rows of $\tilde{B}(\mathbf{i})$ will be indexed by $I(\mathbf{i}) := [1, r]$ and the frozen indices will be exactly those $k \in I(\mathbf{i})$ such that $k^+ = r + 1$. We then set

$$b(\mathbf{i})_{k\ell} = \begin{cases} -a_{i_k i_\ell}, & k < \ell < k^+ \leq \ell^+, \\ a_{i_k i_\ell}, & \ell < k < \ell^+ \leq k^+, \\ -1, & k = \ell^-, \\ 1, & k^- = \ell, \\ 0, & \text{else,} \end{cases} \quad \lambda(\mathbf{i})_{k\ell} = \begin{cases} (\beta_k | 2\omega_{i_\ell} - \beta_\ell), & k < \ell, \\ -(\beta_\ell | 2\omega_{i_k} - \beta_k), & k > \ell, \\ 0, & k = \ell, \end{cases}$$

where $(-|-) : P \times P \rightarrow \mathbb{Z}$ denotes a W -invariant symmetric form with $(\alpha_i | \alpha_j) = a_{ij}$.

We let $A_q(\mathfrak{n}(w))$ denote the quantized coordinate ring of the unipotent group $N_+(w)$. We will be interested in the quantum unipotent minors $D[b, d] := D_{w_{<b}^{-1}\omega_{i_b}, w_{\leq d}^{-1}\omega_{i_d}}$. Here $1 \leq b \leq d \leq r$ are indices with $i_b = i_d$, and $w_{<k}^{-1} := s_{i_1} \cdots s_{i_{k-1}}$. We note that the minor we denote by $D[b, d]$ is denoted by $D(b^-, d)$ in [GLS13].

Theorem 6.3 ([GLS13]). *There is a $\mathbb{Q}(q^{1/2})$ -algebra isomorphism*

$$\mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm 1}]} A_q(\mathfrak{n}(w)) \cong \mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm 1/2}]} \mathcal{A}_{(L(\mathbf{i}), \tilde{B}(\mathbf{i}))}$$

which takes the minors $\{D[b, d]\}_{1 \leq b \leq d \leq r}$ to quantum cluster variables rescaled by powers of $q^{1/2}$. Since $\mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm 1}]} A_q(\mathfrak{n}(w))$ is an iterated skew polynomial ring in

the dual PBW generators $\{D[b, b]\}_{1 \leq b \leq r}$, it follows that $\mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm 1/2}]} \mathcal{A}_{(L(\mathbf{i}), \tilde{B}(\mathbf{i}))}$ is generated by the images of these elements.

From now on we fix $\mathbf{i} = (0, 1, \dots, 0, 1)$, which is associated to the element $(s_1 s_0)^n$ of the Weyl group of type $A_1^{(1)}$. It is easy to check that $L(\mathbf{i})$ is as follows.

Lemma 6.4. *For $1 \leq k \leq n$ we have*

$$\beta_{2k} = k(k + 1)\alpha_0 + k^2\alpha_1, \quad \beta_{2k-1} = k^2\alpha_0 + k(k - 1)\alpha_1.$$

For $1 \leq k, l \leq n$ we have

$$\begin{aligned} (\beta_{2k} | 2\omega_1 - \beta_{2\ell}) &= 2k(k - \ell), & (\beta_{2k} | 2\omega_0 - \beta_{2\ell-1}) &= 2k(k + 1 - \ell), \\ (\beta_{2k-1} | 2\omega_1 - \beta_{2\ell}) &= 2k(k - \ell - 1), & (\beta_{2k-1} | 2\omega_0 - \beta_{2\ell-1}) &= 2k(k - \ell). \end{aligned}$$

To identify $\mathcal{A}_{(L(\mathbf{i}), \tilde{B}(\mathbf{i}))}$ and $\mathcal{A}_{(L_n, \tilde{B}_n)}$, we define a larger set $\{X_{k,\ell}\}_{(k,\ell) \in [1,n-1] \times \mathbb{Z}}$ of quantum cluster variables in $\mathcal{A}_{(L_n, \tilde{B}_n)}$ extending $\{X_{k,\ell}\}_{(k,\ell) \in I_{ex}}$ as follows.

First set $X_{k,2} := \mu_{(k,0)} X_{k,0}$ for any $k \in [1, n - 1]$. Note that the mutations in directions $(k, 0)$ for different k commute with each other. The result of performing these mutations in any order on \tilde{B}_n is an exchange matrix whose principal part is identified with that of \tilde{B}_n upon permuting the index set I_{ex} by the involution $(k, 0) \leftrightarrow (k, 1)$. For $\ell \geq 1$ we inductively define $X_{k,\ell+1}$ as the variable introduced by mutating the cluster $\{X_{k,\ell}\}_{(k,\ell) \in [1,n-1] \times [\ell-1,\ell]} \sqcup \{X_{n,0}, X_{n,1}\}$ in direction $(k, \ell - 1)$. For $\ell < 0$ we define $X_{k,\ell}$ similarly, starting with $X_{k,-1} := \mu_{(k,1)} X_{k,1}$.

Now let μ be the sequence of mutations of (L_n, \tilde{B}_n) defined inductively as the result of the following $n - 1$ steps. At the beginning of the k th step, the associated cluster in $\mathcal{A}_{(L_n, \tilde{B}_n)}$ contains the $n - k$ cluster variables $X_{1,k-1}, \dots, X_{n-k,k-1}$, the mutations at these variables all commute, and the k th step consists of mutating at them in any order.

Proposition 6.5. *The compatible pairs $(\mu(L_n), \mu(\tilde{B}_n))$ and $(L(\mathbf{i}), \tilde{B}(\mathbf{i}))$ coincide for a unique identification of their index sets. Under the induced isomorphism $\mathcal{A}_{(L(\mathbf{i}), \tilde{B}(\mathbf{i}))} \cong \mathcal{A}_{(L_n, \tilde{B}_n)}$ the initial cluster $\{X_i\}_{i \in [1,2n]}$ of $\mathcal{A}_{(L(\mathbf{i}), \tilde{B}(\mathbf{i}))}$ is identified with the cluster $\{X_{k,\ell}\}_{k+\ell \in [n,n+1]}$ via*

$$X_{2k} \leftrightarrow X_{k,n-k}, \quad X_{2k-1} \leftrightarrow X_{k,n-k+1}.$$

Proof. It is easy to check pictorially using quivers that $\mu(\tilde{B}_n)$ is equal to $\tilde{B}(\mathbf{i})$ via the indicated correspondence of indices (for example, as illustrated in Figure 6.1). To show that $\mu(L_n)$ is equal to $L(\mathbf{i})$, we must show that the exponents appearing in the q -commutation relations of the cluster $\{X_i\}_{i \in [1,2n]}$ in $\mathcal{A}_{(L(\mathbf{i}), \tilde{B}(\mathbf{i}))}$ coincide with those of the cluster $\{X_{k,\ell}\}_{k+\ell \in [n,n+1]}$ in $\mathcal{A}_{(L_n, \tilde{B}_n)}$. Given q -commuting elements X, Y of a $\mathbb{Z}[q^{\pm 1/2}]$ -algebra, we let $\Lambda(X, Y)$ denote the exponent appearing in the identity $q^{\Lambda(X,Y)} XY = YX$.

Recall that the product $L_n \tilde{B}_n$ is invariant under mutation. Since $L_n \tilde{B}_n = L(\mathbf{i}) \tilde{B}(\mathbf{i})$ and the principal part of $\tilde{B}(\mathbf{i})$ has full rank, it suffices to show that the $I \times I_{fr}$ submatrices of $\mu(L_n)$ and $L(\mathbf{i})$ are equal. Thus we only need to check that for $1 \leq k \leq n$,

$$\begin{aligned} \lambda(\mathbf{i})_{2k,2n} &= -\Lambda(X_{k,n-k}, X_{n,0}), & \lambda(\mathbf{i})_{2k,2n-1} &= -\Lambda(X_{k,n-k}, X_{n,1}), \\ \lambda(\mathbf{i})_{2k-1,2n} &= -\Lambda(X_{k,n-k+1}, X_{n,0}), & \lambda(\mathbf{i})_{2k-1,2n-1} &= -\Lambda(X_{k,n-k+1}, X_{n,1}). \end{aligned}$$

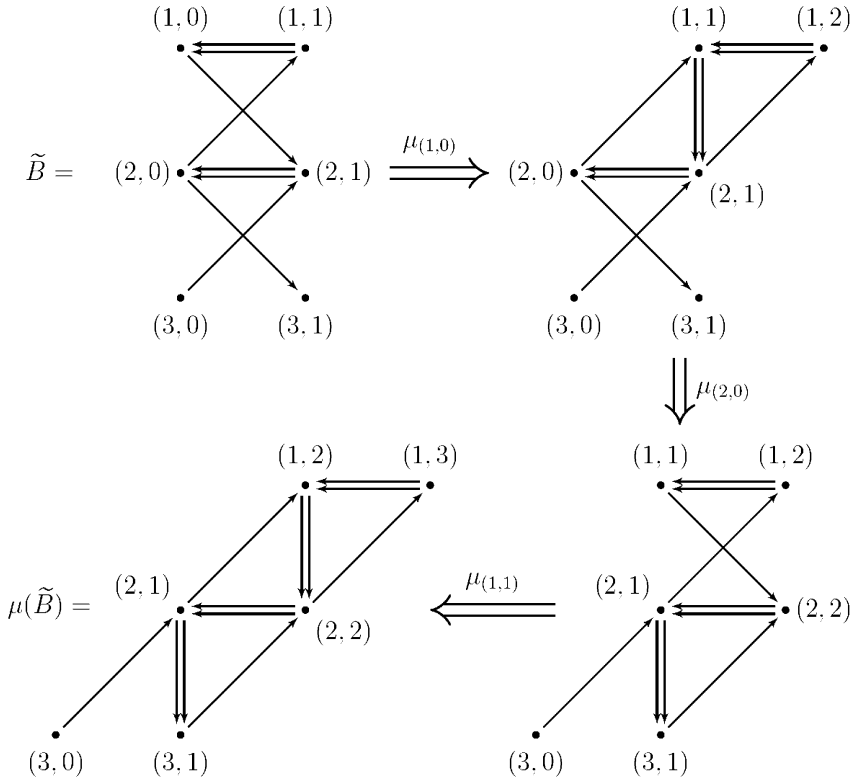


FIGURE 6.1

Since for any $X_{k,m} \in \mathcal{A}_{(L_n, \tilde{B}_n)}$ we have a relation of the form

$$X_{k,m-1}X_{k,m+1} = X_{k,m}^2 + \text{another monomial},$$

these q -commutation exponents satisfy the recurrence

$$\Lambda(X_{k,m+1}, X_{n,0}) = 2\Lambda(X_{k,m}, X_{n,0}) - \Lambda(X_{k,m-1}, X_{n,0}),$$

as well as the recurrence obtained by writing $X_{n,1}$ in place of $X_{n,0}$. Starting from the values of $\Lambda(X_{k,m}, X_{n,0})$ and $\Lambda(X_{k,m}, X_{n,1})$ for $m \in \{0, 1\}$ recorded in the matrix L , it follows by induction that

$$\Lambda(X_{k,m}, X_{n,0}) = 2km, \quad \Lambda(X_{k,m}, X_{n,1}) = 2k(m - 1).$$

One can now check directly using Lemma 6.4 that the remaining entries of $\mu(L_n)$ and $L(\mathbf{i})$ agree. For example, by the lemma and the above computation we obtain

$$\Lambda(X_{2k}, X_{2n}) = 2k(n - k) = \Lambda(X_{k,n-k}, X_{n,0})$$

as desired. □

Example 6.6. In the case $n = 2$, μ consists of a single mutation. In the case $n = 3$, μ consists of three mutations, illustrated in Figure 6.1.

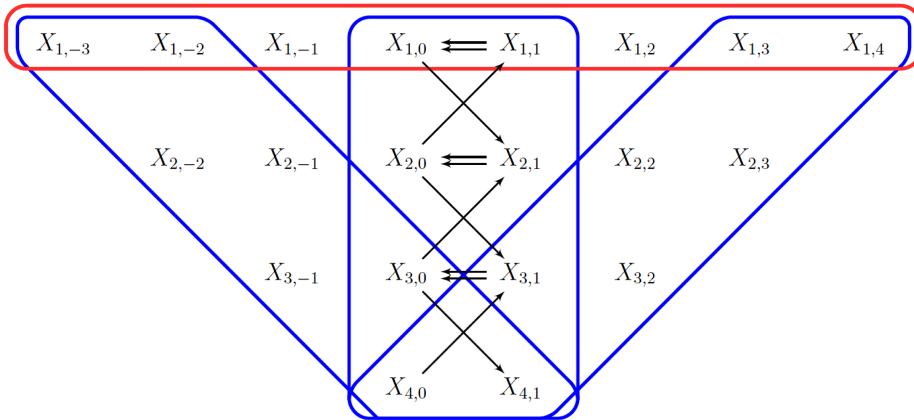


FIGURE 6.2

Consider now the composition

$$\begin{aligned} \mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm 1}]} A_q(\mathbf{n}((s_1 s_0)^n)) &\cong \mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm 1/2}]} \mathcal{A}_{(L(\mathbf{i}), \tilde{B}(\mathbf{i}))} \\ &\cong \mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm 1/2}]} \mathcal{A}_{(L_n, \tilde{B}_n)} \end{aligned}$$

of the isomorphisms of Theorem 6.3 and Proposition 6.5. Up to a power of $q^{1/2}$, the first isomorphism identifies the initial cluster variable $X_k \in \mathcal{A}_{(L(\mathbf{i}), \tilde{B}(\mathbf{i}))}$ with $D[k_{min}, k]$, where $k_{min} = \min\{1 \leq s \leq r \mid i_s = i_k\}$. The remaining $D[b, d]$ are obtained along the following distinguished mutation sequence of [GLS11].

For $1 \leq k \leq r$ and j a row of the Cartan matrix A , let

$$k[j] = |\{1 \leq s \leq k - 1 \mid i_s = j\}|,$$

so for example $t_j := (r + 1)[j]$ is the number of appearances of j in \mathbf{i} . The distinguished sequence consists of r steps, where at step k we perform mutations in increasing order at the lowest $t_{i_k} - 1 - k[i_k]$ indices s such that $i_s = i_k$. Up to powers of q , each of these mutations replaces a minor $D[b, d]$ with the minor $D[b^+, d^+]$. Transporting this mutation sequence across the second isomorphism and recalling from Proposition 6.5 that $\mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm 1}]} A_q(\mathbf{n}((s_1 s_0)^n))$ is generated by $\{D[b, b]\}_{1 \leq b \leq r}$, we obtain the following claim.

Lemma 6.7. *Up to a power of $q^{1/2}$, the composition*

$$\begin{aligned} \mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm 1}]} A_q(\mathbf{n}((s_1 s_0)^n)) &\cong \mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm 1/2}]} \mathcal{A}_{(L(\mathbf{i}), \tilde{B}(\mathbf{i}))} \\ &\cong \mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm 1/2}]} \mathcal{A}_{(L_n, \tilde{B}_n)} \end{aligned}$$

identifies the unipotent minor $D[b, d]$ with the quantum cluster variable $X_{k,\ell}$ via the correspondences

$$k \leftrightarrow 1 + \frac{1}{2}(d - b), \quad \ell \leftrightarrow n + 1 - \frac{1}{2}(b + d).$$

In particular, $\mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm 1/2}]} \mathcal{A}_{(L_n, \tilde{B}_n)}$ is generated by the quantum cluster variables $\{X_{1,\ell}\}_{\ell \in [1-n, n]}$.

Example 6.8. In Figure 6.2 we diagram the quantum cluster variables of $\mathcal{A}_{(L_4, \tilde{B}_4)}$ corresponding to the unipotent minors $D[b, d]$ for $n = 4$. The three collections circled in blue are clusters. The middle is our initial cluster with its quiver superimposed. The right is the result of the mutation sequence μ , identified with the initial cluster of $\mathcal{A}_{(L_{(i)}, \tilde{B}_{(i)})}$ by Proposition 6.5. The left is the result of applying the distinguished mutation sequence of [GLS11] to the right cluster, and as discussed in the next section its exchangeable variables are related to those of the right cluster by the quantum twist. The top collection circled in red is the generating set corresponding to the dual PBW generators of $A_q(\mathfrak{n}((s_1 s_0)^n))$.

We now return to the quantum monoidal seed $(\{\mathcal{P}_{k,\ell}\}_{(k,\ell) \in I}, \tilde{B}_n)$ from the beginning of the section. Recall that we denote by $\mathcal{R}_q(L_n) \subset \mathcal{T}_q(L_n)$ the quantum affine space with q -commutation relations given by L_n , and that the classes of the $\{\mathcal{P}_{k,\ell}\}_{(k,\ell) \in I}$ generate an embedding of $\mathcal{R}_q(L_n)$ into $K^{GL_n(\mathcal{O}) \rtimes \mathbb{G}_m}(\text{Gr}_{GL_n})$.

Lemma 6.9. *For any reductive G we can embed $K^{G(\mathcal{O}) \rtimes \mathbb{G}_m}(\text{Gr}_G)$ into a division ring.*

Proof. Since $K^{G(\mathcal{O}) \rtimes \mathbb{G}_m}(\text{Gr}_G)$ is free as a $\mathbb{Z}[q^{\pm 1/2}]$ -module it embeds into $\mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm 1/2}]} K^{G(\mathcal{O}) \rtimes \mathbb{G}_m}(\text{Gr}_G)$, hence it suffices to show the claim for $\mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm 1/2}]} K^{G(\mathcal{O}) \rtimes \mathbb{G}_m}(\text{Gr}_G)$. Consider the filtration of $\mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm 1/2}]} K^{G(\mathcal{O}) \rtimes \mathbb{G}_m}(\text{Gr}_G)$ given by dimension of support. It suffices to show the associated graded algebra is an Ore domain [Coh61].

We can now define a filtration on $\text{gr } \mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm 1/2}]} K^{G(\mathcal{O}) \rtimes \mathbb{G}_m}(\text{Gr}_G)$ where the r th filtered piece is spanned over $\mathbb{Q}(q^{1/2})$ by the $\mathcal{P}_{\lambda^\vee, \mu}$ with $\|\lambda^\vee\|^2 + \|\mu\|^2 \leq r^2$ for an arbitrary fixed W -invariant quadratic form $\|\cdot\|^2$. By Proposition 2.5 multiplication respects this filtration. Moreover, as a $\mathbb{Q}(q^{1/2})$ -algebra filtration it is of polynomial growth, which implies that $\text{gr } \mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm 1/2}]} K^{G(\mathcal{O}) \rtimes \mathbb{G}_m}(\text{Gr}_G)$ is Ore (cf. [BZ05, Proposition 11.1]). \square

Proposition 6.10. *The embedding $\mathcal{R}_q(L_n) \hookrightarrow K^{GL_n(\mathcal{O}) \rtimes \mathbb{G}_m}(\text{Gr}_{GL_n})$ generated by the classes $\{\mathcal{P}_{k,\ell}\}_{(k,\ell) \in I}$ extends to an embedding*

$$(6.11) \quad \mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm 1/2}]} \mathcal{A}_{(L_n, \tilde{B}_n)}^{loc} \hookrightarrow \mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm 1/2}]} K^{GL_n(\mathcal{O}) \rtimes \mathbb{G}_m}(\text{Gr}_{GL_n})$$

of $\mathbb{Q}(q^{1/2})$ -algebras. This embedding has the property that if $k + \ell \leq n + 1$ and $k - \ell \leq n$, $[\mathcal{P}_{k,\ell}]$ maps to the quantum cluster variable $X_{k,\ell}$.

Proof. Following Lemma 6.9 above, let \mathcal{F} denote a division ring containing $K^{GL_n(\mathcal{O}) \rtimes \mathbb{G}_m}(\text{Gr}_{GL_n})$. It follows that the embedding

$$\mathcal{R}_q(L_n) \hookrightarrow K^{GL_n(\mathcal{O}) \rtimes \mathbb{G}_m}(\text{Gr}_{GL_n})$$

extends to an embedding $\mathcal{F}_q(L_n) \hookrightarrow \mathcal{F}$, where $\mathcal{F}_q(L_n)$ is the division ring of fractions of $\mathcal{R}_q(L_n)$.

The exact sequences of Proposition 2.16 imply that for the indicated values of k, ℓ the classes $[\mathcal{P}_{k,\ell}] \in K^{GL_n(\mathcal{O}) \rtimes \mathbb{G}_m}(\text{Gr}_{GL_n}) \subset \mathcal{F}$ are contained in $\mathcal{F}_q(L_n) \subset \mathcal{F}$. As in the proof of Proposition 6.2, comparison with the quantum exchange relations of the pair (L_n, \tilde{B}_n) shows that $[\mathcal{P}_{k,\ell}]$ coincides with the quantum cluster variable $X_{k,\ell} \in \mathcal{A}_{(L_n, \tilde{B}_n)}^{loc} \subset \mathcal{F}_q(L_n)$ (the indicated values of k, ℓ are exactly those such that $[\mathcal{P}_{k,\ell}]$ can be computed using only instances of the exact sequences not involving $\mathcal{P}_{n,\ell}$ for $\ell \notin \{0, 1\}$). It follows that $K^{GL_n(\mathcal{O}) \rtimes \mathbb{G}_m}(\text{Gr}_{GL_n}) \subset \mathcal{F}$ contains

the subalgebra of $\mathcal{A}_{(L_n, \tilde{B}_n)} \subset \mathcal{F}_q(L_n)$ generated by these quantum cluster variables. But by Lemma 6.7 $\mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm 1/2}]} \mathcal{A}_{(L_n, \tilde{B}_n)}$ is generated as a $\mathbb{Q}(q^{1/2})$ -algebra by the subset $\{X_{1,\ell}\}_{\ell \in [1-n,n]}$ of these variables. Since the frozen quantum cluster variables are the classes of the invertible objects $\mathcal{P}_{n,0}$ and $\mathcal{P}_{n,1}$, the claim follows. \square

Corollary 6.12. *The quantum monoidal seed $(\{\mathcal{P}_{k,\ell}\}_{(k,\ell) \in I}, \tilde{B}_n)$ admits arbitrary sequences of mutations in $\mathcal{P}_{coh}^{GL_n(\mathcal{O}) \rtimes \mathbb{G}_m}(\mathrm{Gr}_{GL_n})$.*

Proof. The proof follows immediately by combining Proposition 6.10 with Theorems 4.25 and 5.14. \square

As a consequence, we see that the images of all quantum cluster variables under the embedding (6.11) are classes of simple perverse coherent sheaves. We immediately obtain the following.

Corollary 6.13. *The embedding (6.11) is induced from an embedding*

$$(6.14) \quad \mathcal{A}_{(L_n, \tilde{B}_n)}^{loc} \hookrightarrow K^{GL_n(\mathcal{O}) \rtimes \mathbb{G}_m}(\mathrm{Gr}_{GL_n})$$

of $\mathbb{Z}[q^{\pm 1/2}]$ -algebras.

To conclude the proof that $\mathcal{P}_{coh}^{GL_n(\mathcal{O}) \rtimes \mathbb{G}_m}(\mathrm{Gr}_{GL_n})$ is a monoidal cluster categorification it remains to show that (6.14) is an isomorphism.

Remark 6.15. The existence of a Poisson isomorphism

$$\mathbb{C} \otimes \mathcal{A}_{\tilde{B}(\mathbf{i})}^{loc} \cong \mathbb{C} \otimes K^{GL_n(\mathcal{O})}(\mathrm{Gr}_{GL_n}),$$

where $\mathcal{A}_{\tilde{B}(\mathbf{i})}^{loc}$ is equipped with the Poisson structure induced by $L(\mathbf{i})$, is a special case of a result of [FKRD18]. It passes through the complexified Poisson version of the isomorphism $\mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm 1/2}]} \mathcal{A}_{(L(\mathbf{i}), \tilde{B}(\mathbf{i}))} \cong \mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm 1}]} A_q(\mathfrak{n}((s_1 s_0)^n))$ we have been using. Given Theorem 6.18 it would follow from a proof of [GLS13, Conjecture 12.7] that $K^{GL_n(\mathcal{O}) \rtimes \mathbb{G}_m}(\mathrm{Gr}_{GL_n})$ is isomorphic as a $\mathbb{Z}[q^{\pm 1/2}]$ -algebra to $\mathbb{Z}[q^{\pm 1/2}] \otimes_{\mathbb{Z}[q^{\pm 1}]} A_q(N^{(s_1 s_0)^n})$, where $A_q(N^{(s_1 s_0)^n})$ is the quantized unipotent cell obtained by localizing $A_q(\mathfrak{n}((s_1 s_0)^n))$ appropriately.

Given Corollary 2.20, it suffices to show for any $\ell \in \mathbb{Z}$ that $[\mathcal{P}_{1,\ell}]$ is contained in the image of $\mathcal{A}_{(L_n, \tilde{B}_n)}^{loc}$, since $[\mathcal{P}_{n,0}]^{-1}$ manifestly is. In fact, for any $k \in [1, n - 1]$ we will show that $[\mathcal{P}_{k,\ell}]$ is the product of the quantum cluster variable $X_{k,\ell}$ and a Laurent monomial in the frozen variables, up to a power of q . Note that the identity implied upon specializing q and the frozen variables to 1 follows trivially from the exact sequences of Proposition 2.16 and the definition of the $X_{k,\ell}$ —the necessary work amounts to bookkeeping of frozen variables and powers of q .

Recall the quantum monoidal seed $(\{X_{k,\ell}\}_{k+\ell \in [n,n+1]}, \mu(\tilde{B}_n))$ resulting from the mutation sequence μ used in Proposition 6.5. Consider now the following infinite mutation sequence μ' which starts at this seed.

We define μ' inductively as a series of steps consisting of $n - 1$ mutations each. At the j th step we mutate in order at $X_{n-1,j}, X_{n-2,j+1}, \dots, X_{2,j+n-3}, X_{1,j+n-2}$. The unfrozen variables in the cluster resulting from the j th iteration are $\{X_{k,\ell}\}_{k+\ell \in [n+j,n+j+1]}$.

The sequence μ' is periodic on the principal parts of the exchange matrices that appear: after applying the j th step the resulting exchange matrix has the same

principal part as $\mu(\tilde{B}_n)$. The key to organizing the computation of the exchange relations we must understand is to have an inductive description of how μ' interacts with the frozen rows of these exchange matrices.

Lemma 6.16. *Let $(\{X_{k,\ell}\}_{(k,\ell)\in S}, \tilde{B}'_n)$ be a seed occurring after applying a nonzero number of mutations in the sequence μ' . Given $(k_1, \ell_1), (k_2, \ell_2) \in S$, we write $b'_{(k_1, \ell_1), (k_2, \ell_2)}$ for the corresponding entry of \tilde{B}'_n . Then the nonzero entries in the frozen rows of \tilde{B}'_n are exactly those prescribed by the following conditions:*

- (1) *If $(n-1, j)$ and $(n-1, j+1)$ are both in S for some $j \in \mathbb{Z}$, then $b'_{(n,1), (n-1, j+1)} = -j$ and $b'_{(n,1), (n-1, j)} = j + 1$.*
- (2) *If $(1, j)$ and $(1, j+1)$ are both in S for some $j \geq n$, then $b'_{(n,0), (1, j)} = j - n$ and $b'_{(n,0), (1, j+1)} = 1 + n - j$.*
- (3) *If $(j, n-j)$ and $(j+1, n-j+1)$ are both in S for some $1 \leq j \leq n-2$, then $b'_{(n,0), (j, n-j)} = -1$ and $b'_{(n,0), (j+1, n-j+1)} = 1$.*

The last situation does not occur after the first $n-1$ mutations in μ' . In particular, after the first $n-1$ mutations the quiver of \tilde{B}'_n has no arrows between the $(n, 1)$ th vertex and the (k, ℓ) th vertex unless $k = n-1$, and no arrows between the $(n, 0)$ th vertex and the (k, ℓ) th vertex unless $k = 1$.

Proof. As the claims are easy to check pictorially using the relevant quivers, we omit the details. The key inductive reason the first two claims hold is the following. Suppose one has a quiver with vertices i, j, k , and with two arrows from i to j , $m+1$ arrows from j to k , and m arrows from k to i . Then after mutating at vertex j there are two arrows from j to i , $m+2$ arrows from i to k , and $m+1$ arrows from k to j . □

As before given q -commuting elements X, Y of a $\mathbb{Z}[q^{\pm 1/2}]$ -algebra, we define $\Lambda(X, Y)$ by $q^{\Lambda(X, Y)}XY = YX$. As in Section 4.4, if \mathcal{F} and \mathcal{G} are objects of $\mathcal{P}_{coh}^{GL_n(\mathcal{O}) \rtimes \mathbb{G}_m}(\text{Gr}_{GL_n})$ we set $\mathcal{F} \odot \mathcal{G} := (\mathcal{F} * \mathcal{G})\{-\frac{1}{2}\Lambda(\mathcal{F}, \mathcal{G})\}$. Similarly, given q -commuting elements X, Y of a $\mathbb{Z}[q^{\pm 1/2}]$ -algebra we write $X \odot Y := q^{\frac{1}{2}\Lambda(X, Y)}XY$.

Lemma 6.17. *For any $k \in [1, n-1], \ell \in \mathbb{Z}$ we have*

$$X_{k,\ell} = \begin{cases} [\mathcal{P}_{k,\ell} \odot \mathcal{P}_{n,0}^{\ell+k-n-1}], & \ell + k > n + 1, \\ [\mathcal{P}_{k,\ell} \odot \mathcal{P}_{n,1}^{k-\ell-n}], & k - \ell > n, \\ [\mathcal{P}_{k,\ell}], & \text{otherwise,} \end{cases}$$

in $\mathcal{A}_{(L_n, \tilde{B}_n)}^{loc} \subset K^{GL_n(\mathcal{O}) \rtimes \mathbb{G}_m}(\text{Gr}_{GL_n})$.

Proof. The last case is part of Proposition 6.10. We consider the first case explicitly, the second being similar.

We prove the claim inductively by considering the $X_{k,\ell}$ as they appear along the mutation sequence μ' , using Lemma 6.16 to identify the relevant exchange relations and compare them with the exact sequences of Proposition 2.16. The first exchange relation is

$$q^{\frac{1}{2}\Lambda(X_{n-1,1}, X_{n-2,2} X_{n,1}^2)} X_{n-1,1} X_{n-1,3} = q X_{n-2,2} \odot X_{n,1}^2 + X_{n-1,2}^2 \odot X_{n,0},$$

which we compare to

$$q^{\frac{1}{2}\Lambda(\mathcal{P}_{n-1,1}, \mathcal{P}_{n-2,2} \mathcal{P}_{n,2})} [\mathcal{P}_{n-1,1}] [\mathcal{P}_{n-1,3}] = q [\mathcal{P}_{n-2,2} \odot \mathcal{P}_{n,2}] + [\mathcal{P}_{n-1,2}]^2.$$

Multiplying by $X_{n,0} = [\mathcal{P}_{n,0}]$ and normalizing as suitable, this becomes

$$\begin{aligned}
 & q^{\frac{1}{2}\Lambda(\mathcal{P}_{n-1,1}, \mathcal{P}_{n-2,2}, \mathcal{P}_{n,0}) - \frac{1}{2}\Lambda(\mathcal{P}_{n-1,3}, \mathcal{P}_{n,0})} [\mathcal{P}_{n-1,1}] [\mathcal{P}_{n-1,3} \odot \mathcal{P}_{n,0}] \\
 &= q^{1 - \frac{1}{2}\Lambda(\mathcal{P}_{n-2,2} \odot \mathcal{P}_{n,2}, \mathcal{P}_{n,0})} [\mathcal{P}_{n-2,2} \odot \mathcal{P}_{n,2} \odot \mathcal{P}_{n,0}] + q^{-\frac{1}{2}\Lambda(\mathcal{P}_{n-1,2}^2, \mathcal{P}_{n,0})} [\mathcal{P}_{n-1,2}^2 \odot \mathcal{P}_{n,0}].
 \end{aligned}$$

Substituting in the quantum cluster variables already identified with the specified classes and dividing the right-hand side by $q^{-\frac{1}{2}\Lambda(\mathcal{P}_{n-1,2}^2, \mathcal{P}_{n,0})}$, we obtain the right-hand side of the above exchange relation. Here use the fact that $\mathcal{P}_{n,1}^2 \cong \mathcal{P}_{n,2} \odot \mathcal{P}_{n,0}$ by Remark 2.15 and that $\Lambda(\mathcal{P}_{n-2,2} \odot \mathcal{P}_{n,2}, \mathcal{P}_{n,0}) = \Lambda(\mathcal{P}_{n-1,2}^2, \mathcal{P}_{n,0})$ by Remark 2.13. Dividing the left-hand sides by $X_{n-1,1} = [\mathcal{P}_{n-1,1}]$ and inspecting powers of q , the identity $X_{n-1,3} = [\mathcal{P}_{n-1,3} \odot \mathcal{P}_{n,0}]$ now follows since

$$\begin{aligned}
 & \Lambda(\mathcal{P}_{n-1,1}, \mathcal{P}_{n-2,2} * \mathcal{P}_{n,1}^2) - \Lambda(\mathcal{P}_{n-1,1}, \mathcal{P}_{n-2,2} * \mathcal{P}_{n,2}) \\
 & \quad + \Lambda(\mathcal{P}_{n-1,3}, \mathcal{P}_{n,0}) - \Lambda(\mathcal{P}_{n-2,2} * \mathcal{P}_{n,2}, \mathcal{P}_{n,0}) \\
 &= \Lambda(\mathcal{P}_{n-1,1}, \mathcal{P}_{n,0}) + \Lambda(\mathcal{P}_{n-1,3}, \mathcal{P}_{n,0}) - \Lambda(\mathcal{P}_{n-2,2} * \mathcal{P}_{n,2}, \mathcal{P}_{n,0}) = 0.
 \end{aligned}$$

The next $n - 2$ mutations in μ are of the form

$$\begin{aligned}
 & q^{\frac{1}{2}\Lambda(X_{k,n-k}, X_{k+1,n-k+1}, X_{k-1,n-k+1})} X_{k,n-k} X_{k,n-k+2} \\
 &= q X_{k+1,n-k+1} \odot X_{k-1,n-k+1} + X_{k,n-k+1}^2 \odot X_{n,0}
 \end{aligned}$$

for k descending from $n - 2$ to 1. Here by convention $X_{0,\ell} := 1$ for $\ell \in \mathbb{Z}$. Thereafter, each of the length $n - 1$ steps of μ takes a uniform form: in the k th step the first exchange relation is

$$q^{\frac{1}{2}\Lambda(X_{n-1,k}, X_{n-2,k+1}, X_{n,1}^{k+1})} X_{n-1,k} X_{n-1,k+2} = q X_{n-2,k+1} \odot X_{n,1}^{k+1} + X_{n-1,k+1}^2,$$

the next $n - 3$ exchange relations are of the form

$$\begin{aligned}
 & q^{\frac{1}{2}\Lambda(X_{n-j-1}, X_{n-j,j+k+1}, X_{n-j-2,j+k+1})} X_{n-j-1,j+k} X_{n-j-1,j+k+2} \\
 &= q X_{n-j,j+k+1} \odot X_{n-j-2,j+k+1} + X_{n-j-1,j+k+1}^2
 \end{aligned}$$

for j increasing from 1 to $n - 3$, and the last exchange relation is

$$q^{\frac{1}{2}\Lambda(X_{1,n+k-2}, X_{2,n+k-1}, X_{n,0}^{k-2})} X_{1,n+k-2} X_{1,n+k} = q X_{2,n+k-1} \odot X_{n,0}^{k-2} + X_{1,n+k-1}^2.$$

With these inductive expressions in hand the claim may be checked by a finite list of calculations resembling the one above, and whose details we omit. □

Theorem 6.18. *The coherent Satake category $\mathcal{P}_{coh}^{GL_n(\mathcal{O}) \rtimes \mathbb{G}_m}(\mathrm{Gr}_{GL_n})$ is a monoidal categorification of the quantum cluster algebra $\mathcal{A}_{(L_n, \tilde{B}_n)}^{loc}$.*

Proof. As indicated in the discussion preceding Lemma 6.16, given Corollary 2.20 it follows from Lemma 6.17 that the embedding of Corollary 6.13 is in fact an isomorphism. The result then follows since by Corollary 6.12 all quantum cluster monomials are classes of simple perverse coherent sheaves. □

Remark 6.19. Following [HL10] one anticipates that all cluster variables should be classes of prime objects, i.e., objects that cannot be expressed as a product of two objects which are not invertible. While we do not know if this property holds for all perverse coherent sheaves whose classes are quantum cluster variables, it is clear from Proposition 2.5 and Lemma 6.17 that the $X_{k,\ell}$ are classes of prime objects.

6.2. Categorification of the bar involution. Recall that the bar involution of a quantum cluster algebra $\mathcal{A}_{(\bar{B}, L)}$ is the restriction of the antiautomorphism of $\mathcal{F}(L)$ that preserves X^v for $v \in \mathbb{Z}^I$ and which exchanges q and q^{-1} [BZ05]. We now explain how to lift the bar involution of $\mathcal{A}_{(\bar{B}_n, L_n)}^{loc}$ from K-theory to the category $D_{coh}^{GL_n(\mathcal{O}) \rtimes \mathbb{G}_m}(\mathrm{Gr}_{GL_n})$.

Consider the bijection $\pi_0 \mathrm{Gr}_{GL_n} \xrightarrow{\sim} \mathbb{Z}$ realized at the level of orbits by the assignment $\lambda^\vee \mapsto \langle \lambda^\vee, \omega_n \rangle$. Under this map the component containing $\mathrm{Gr}_{GL_n}^k$ is labeled by k . We also define $\mathbf{n} : P^\vee \rightarrow \mathbb{Z}$ by $\mathbf{n}(\lambda^\vee) := \langle \lambda^\vee, \omega_n \rangle (n - \langle \lambda^\vee, \omega_n \rangle)$.

Lemma 6.20. *The canonical sheaf of $\overline{\mathrm{Gr}}_{GL_n}^{\lambda^\vee}$ is isomorphic to*

$$(6.21) \quad \mathcal{O}_{\overline{\mathrm{Gr}}_{GL_n}^{\lambda^\vee}}(-n) \otimes \det(L_0/tL_0)^{\langle \lambda^\vee, \omega_n \rangle} \{ \mathbf{n}(\lambda^\vee) - \dim \mathrm{Gr}_{GL_n}^{\lambda^\vee} \}.$$

Proof. If $\lambda^\vee = \omega_k^\vee$ the result follows by the standard calculation of the canonical sheaf of the finite Grassmannian (cf. (3.9)). More generally, if $\lambda^\vee = \omega_{k_1}^\vee + \dots + \omega_{k_m}^\vee$ we have a resolution

$$\overline{m} : \mathrm{Gr}_{GL_n}^k := \mathrm{Gr}_{GL_n}^{k_1} \tilde{\times} \dots \tilde{\times} \mathrm{Gr}_{GL_n}^{k_m} \rightarrow \overline{\mathrm{Gr}}_{GL_n}^{\sum_i \omega_{k_i}^\vee}.$$

The canonical sheaf of $\mathrm{Gr}_{GL_n}^k$ is

$$\left(\mathcal{O}_{\mathrm{Gr}_{GL_n}^{k_1}}(-n) \otimes \det(L_0/tL_0)^{k_1} \right) \boxtimes \dots \boxtimes \left(\mathcal{O}_{\mathrm{Gr}_{GL_n}^{k_m}}(-n) \otimes \det(L_{m-1}/tL_{m-1})^{k_m} \right).$$

Using that $\det(L_i/tL_i) \cong \det(L_{i-1}/tL_{i-1})\{-2k_{i-1}\}$ repeatedly reduces this expression to

$$\mathcal{O}_{\mathrm{Gr}_{GL_n}^k}(-n) \otimes \det(L_0/tL_0)^{\sum_i k_i} \{ -2 \sum_{i \neq j} k_i k_j \}.$$

Applying \overline{m}_* recovers (6.21). The general case of λ follows by also considering the convolution with \mathcal{P}_n^{-1} . □

Notice that apart from the $\{-\dim \mathrm{Gr}_{GL_n}^{\lambda^\vee}\}$ shift the canonical sheaf is global. We capture this by defining \mathbb{L} to be the automorphism of $D_{coh}^{GL_n(\mathcal{O}) \rtimes \mathbb{G}_m}(\mathrm{Gr}_{GL_n})$ which, on the k th component of Gr_{GL_n} , acts by tensoring with

$$\mathcal{O}_{\mathrm{Gr}_{GL_n}}(-n) \otimes \det(L_0/tL_0)^k \{ k(n - k) \}.$$

Letting g^T denote the transpose of $g \in GL_n$, we consider the involution of Gr_{GL_n} given by

$$[g(t)] \mapsto [(g(t)^T)^{-1}].$$

Twisting the action of $GL_n(\mathcal{O}) \rtimes \mathbb{G}_m$ by $(g(t), a) \mapsto ((g(t)^T)^{-1}, a)$ this defines an involution of $D_{coh}^{GL_n(\mathcal{O}) \rtimes \mathbb{G}_m}(\mathrm{Gr}_{GL_n})$ which we denote by $\mathcal{F} \mapsto \mathcal{F}'$. It is covariant with respect to both convolution and Hom, and commutes with $\mathcal{F} \mapsto \mathcal{F}^*$.

Lemma 6.22. *The involutions $\mathbb{D}, \mathbb{L}, \mathcal{F} \mapsto \mathcal{F}^*$, and $\mathcal{F} \mapsto \mathcal{F}'$ of $D_{coh}^{GL_n(\mathcal{O}) \rtimes \mathbb{G}_m}(\mathrm{Gr}_{GL_n})$ take perverse sheaves to perverse sheaves. Moreover, we have*

$$(\mathbb{L}^{-1} \circ \mathbb{D})\mathcal{P}_{\lambda^\vee, \mu} \cong \mathcal{P}_{\lambda^\vee, -\mu}, \quad \mathcal{P}_{\lambda^\vee, \mu}^* \cong \mathcal{P}_{-\lambda^\vee, \mu}, \quad \mathcal{P}'_{\lambda^\vee, \mu} \cong \mathcal{P}_{-\lambda^\vee, -\mu}.$$

Proof. That \mathbb{D} and $\mathcal{F} \mapsto \mathcal{F}^*$ preserve perversity has been discussed (see Corollary 3.2). The case of $\mathcal{F} \mapsto \mathcal{F}'$ may be treated similarly to that of $\mathcal{F} \mapsto \mathcal{F}^*$, and the case of \mathbb{L} is obvious.

Given that these functors restrict to involutions of $\mathcal{P}_{coh}^{GL_n(\mathcal{O}) \times \mathbb{G}_m}(\mathrm{Gr}_{GL_n})$, it follows that they must permute simple perverse coherent sheaves up to isomorphism. The transformation rules for λ^\vee simply record the support of the image of $\mathcal{P}_{\lambda^\vee, \mu}$ under the relevant functor, and are clearly as stated.

In each case it then remains to identify the transformation rules for μ . It suffices to identify the restriction of the image of $\mathcal{P}_{\lambda^\vee, \mu}$ to the dense $GL_n(\mathcal{O})$ -orbit in its support, which up to a cohomological degree shift is a simple equivariant vector bundle. Such a bundle is determined by the $P_{\lambda^\vee} \times \mathbb{G}_m$ action on the fiber over $[t^{\lambda^\vee}]$ (or $[t^{-\lambda^\vee}]$ in the last two cases). In each case this action is the result of an elementary computation similar to that of Proposition 2.5. \square

Remark 6.23. Note that the transformation rules in Lemma 6.22 depend on the precise choice of \mathbb{G}_m -equivariant structure on $\mathcal{P}_{\lambda^\vee, \mu}$ made in Section 2.1. A different choice would require an additional equivariant shift in the formulas above.

We now define an involution $\mathcal{F} \mapsto \overline{\mathcal{F}}$ of $D_{coh}^{GL_n(\mathcal{O}) \times \mathbb{G}_m}(\mathrm{Gr}_{GL_n})$ as the composition

$$\overline{\mathcal{F}} := \mathbb{L}^{-1} \circ \mathbb{D}((\mathcal{F}')^*) \cong \mathbb{D} \circ \mathbb{L}((\mathcal{F}')^*).$$

By Lemma 6.22 we have the following result.

Corollary 6.24. *The involution $\mathcal{F} \mapsto \overline{\mathcal{F}}$ is contravariant with respect to both convolution and Hom. We have $\overline{\mathcal{F}\{\ell\}} \cong \overline{\mathcal{F}\{-\ell\}}$ for any $\ell \in \mathbb{Z}$ and $\overline{\mathcal{P}_{\lambda^\vee, \mu}} \cong \mathcal{P}_{\lambda^\vee, \mu}$ for any $\lambda^\vee \in P^\vee, \mu \in P$.*

It follows that $\mathcal{F} \mapsto \overline{\mathcal{F}}$ induces an antiautomorphism of $K^{GL_n(\mathcal{O}) \times \mathbb{G}_m}(\mathrm{Gr}_{GL_n})$ which exchanges q and q^{-1} . Since the isomorphism $K^{GL_n(\mathcal{O}) \times \mathbb{G}_m}(\mathrm{Gr}_{GL_n}) \cong \mathcal{A}_{(\overline{B}_n, L_n)}^{loc}$ identifies the initial quantum cluster variables with classes of sheaves of the form $\mathcal{P}_{\omega_k^\vee, \ell\omega_k}$, we further obtain the following result.

Corollary 6.25. *The isomorphism $K^{GL_n(\mathcal{O}) \times \mathbb{G}_m}(\mathrm{Gr}_{GL_n}) \cong \mathcal{A}_{(\overline{B}_n, L_n)}^{loc}$ identifies the antiautomorphism $[\mathcal{F}] \mapsto [\overline{\mathcal{F}}]$ with the bar involution of $\mathcal{A}_{(\overline{B}_n, L_n)}^{loc}$.*

Remark 6.26. The above discussion should extend with minor modification to the case of general G ; see in particular [BD96, Section 4.6] in comparison with Lemma 6.20.

Remark 6.27. The K-theoretic description of the bar involution given above is formally in keeping with analogous results in the setting of the Steinberg variety and nilpotent cone [Lus98, Ost00].

6.3. Categorification of the quantum twist. We now interpret the quantum twist in terms of the rigidity of $\mathcal{P}_{coh}^{GL_n(\mathcal{O}) \times \mathbb{G}_m}(\mathrm{Gr}_{GL_n})$. Quantum twists were introduced for general double Bruhat cells in [GY16], and we follow the reformulation of [KO17] in the unipotent case. We then review the implied connection to maximal green sequences and generalized DT invariants following [Kel11].

First recall a few conventions from [GLS11] on preprojective algebras. The quantum unipotent minors $D[b, d] \in \mathcal{A}_{(\overline{B}, L)}$ recalled in the previous section are associated to representations $M[b, d]$ of the preprojective algebra Λ of the 2-Kronecker quiver. We write V_k for the representation associated to $D[k_{min}, k]$; it has dimension vector β_k . The $M[b, d]$ lie in the subcategory \mathcal{C}_w of Λ -mod attached to w in [BIRS09], and V_{2n-1}, V_{2n} are the projective-injectives of \mathcal{C}_w .

Associated to any $M \in \mathcal{C}_w$ are elements $X_M, Y_M \in \mathcal{A}_{(\tilde{B}, L)}$ related by $Y_M = q^{\|\underline{\dim} M\|^2/4} X_M$ [KO17]. Here the dimension vector $\underline{\dim} M$ is considered as an element of the $A_1^{(1)}$ root lattice and we fix $\|\alpha\| := (\alpha|\alpha)$, where the latter denotes the bilinear form characterized by $(\alpha_i|\alpha_j) = a_{ij}$ as in Section 6.1. If M is rigid, indecomposable, and reachable from the $M[b, d]$ by a sequence of mutations, then X_M is a quantum cluster variable. All quantum cluster variables are of this form.

Given $M \in \mathcal{C}_w$, we let $I(M)$ denote the injective hull of M in \mathcal{C}_w , m_i the multiplicity in $I(M)$ of the injective hull of the simple whose dimension vector is the i th simple root, and $\Omega_w^{-1}(M)$ the last term of

$$0 \rightarrow M \rightarrow I(M) \rightarrow \Omega_w^{-1}(M) \rightarrow 0.$$

As formulated in [KO17], the quantum twist η_w of $A_q(N^w)$ is the $\mathbb{Z}[q^{\pm 1}]$ -algebra automorphism characterized by

$$\eta_w(Y_M) = q^{\sum_i m_i (\underline{\dim} M)_i} Y_{I(M)}^{-1} Y_{\Omega_w^{-1}(M)}.$$

For consistency with the previous section we follow the conventions of [GLS13], which are transpose to those of [KO17] as discussed in Remark 6.10 of loc. cit. The classical limit of η_w was introduced in [BZ97], for which the categorical description above was discovered in [GLS12]. When w is the square of a Coxeter element, for example as in our $n = 2$ case, quantizations of the classical twist were first studied in [BR15].

Theorem 6.28. *The isomorphism*

$$\begin{aligned} \mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm 1/2}]} K^{GL_n(\mathcal{O}) \rtimes G_m}(\mathrm{Gr}_{GL_n}) &\cong \mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm 1/2}]} \mathcal{A}_{(\tilde{B}_n, L_n)} \\ &\cong \mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm 1}]} A_q(N^{(s_1 s_0)^n}) \end{aligned}$$

identifies the antiautomorphism $[\mathcal{F}] \mapsto [\mathcal{F}^L]$ with the composition $X \mapsto \overline{\eta_{(s_1 s_0)^n}(X)}$ of the bar involution and the quantum twist.

Proof. For $j < 2n - 1$ there is an exact sequence

$$(6.29) \quad 0 \rightarrow V_j \rightarrow V_{j_{max}} \rightarrow T_j \rightarrow 0,$$

where $j_{max} = \max\{1 \leq s \leq r | i_s = i_j\}$ and $T_j = M[j^+, j_{max}]$ [GLS11, Section 9.8]. From the above description of $\eta_{(s_1 s_0)^n}$ we obtain

$$\eta_{(s_1 s_0)^n}(q^{\|\underline{\dim} V_j\|^2/4} X_{V_j}) = q^{(\underline{\dim} V_j)_{i_j}} (q^{\|\underline{\dim} V_{j_{max}}\|^2/4} X_{V_{j_{max}}})^{-1} (q^{\|\underline{\dim} T_j\|^2/4} X_{T_j}).$$

It follows from (6.29) and the calculation of β_j in Lemma 6.4 that if k is chosen so that j is equal to $2k - 1$ or $2k$, then

$$\begin{aligned} (\underline{\dim} V_j)_{i_j} &= k^2, \quad \|\underline{\dim} V_j\|^2 = 2k^2, \\ \|\underline{\dim} V_{j_{max}}\|^2 &= 2n^2, \quad \|\underline{\dim} T_j\|^2 = 2(n - k)^2. \end{aligned}$$

Simplifying and applying the bar involution we then obtain

$$\overline{\eta_{(s_1 s_0)^n}(X_{V_j})} = q^{k(n-k)} X_{T_j} X_{V_{j_{max}}}^{-1}.$$

On the other hand, by Lemma 6.7 and Proposition 6.10 we have $X_{T_{2k}} = [\mathcal{P}_{n-k, -k}]$ and $X_{T_{2k-1}} = [\mathcal{P}_{n-k, 1-k}]$. By Remark 2.13 we see that whether j is $2k$ or $2k - 1$ we have

$$X_{T_j} X_{V_{j_{max}}}^{-1} = q^{-2k(n-k)} X_{V_{j_{max}}}^{-1} X_{T_j}.$$

Comparing with Proposition 3.10 we thus have

$$\overline{\eta_{(s_1 s_0)^n}([\mathcal{P}_{n-k,-k}])} = [\mathcal{P}_{n-k,-k}^L], \quad \overline{\eta_{(s_1 s_0)^n}([\mathcal{P}_{n-k,1-k}])} = [\mathcal{P}_{n-k,1-k}^L].$$

Now consider the frozen variables $X_{V_{2n}} = [\mathcal{P}_{n,0}]$ and $X_{V_{2n-1}} = [\mathcal{P}_{n,1}]$. Using the calculation of β_{2n} and β_{2n-1} one checks as above that $\eta_{(s_1 s_0)^n}(X_{V_{2n}}) = X_{V_{2n}}^{-1}$ and $\eta_{(s_1 s_0)^n}(X_{V_{2n-1}}) = X_{V_{2n-1}}^{-1}$. On the other hand, $[\mathcal{P}_{n,0}^L] = [\mathcal{P}_{n,0}]^{-1}$ and $[\mathcal{P}_{n,1}^L] = [\mathcal{P}_{n,1}]^{-1}$ as these objects are invertible.

We have thus verified that the antiautomorphisms being compared agree on the elements $X_{V_1}, \dots, X_{V_{2n}}$. Since these form a cluster they generate the division ring of fractions of $\mathcal{A}_{(\tilde{B},L)}$, and the claim follows. \square

The sequence of mutations taking the cluster $\{X_{V_i}\}$ to $\{X_{T_i}\}$ is an example of a maximal green sequence [Kel11]. A maximal green sequence defines an automorphism of a cluster algebra called the total DT invariant in [Kel11] in reference to its role in [KS08], although as discussed in the introduction a variety of terms appear in the literature.

From the categorical description of η_w it follows from [KR07] that after specializing frozen variables (hence necessarily q) to 1, the twist coincides with the total DT invariant. Note that a priori the total DT invariant is only a notion defined for quivers without frozen variables, so we would not know how to formulate a statement relating it to the twist without this specialization.

Corollary 6.30. *After specializing the frozen variables and q to 1, the isomorphism $\mathcal{A}_{(\tilde{B}_n, L_n)}^{loc} \cong K^{GL_n(\mathcal{O}) \rtimes \mathbb{G}_m}(\text{Gr}_{GL_n})$ identifies the automorphism $[\mathcal{F}] \mapsto [\mathcal{F}^L]$ with the total DT invariant of $\mathcal{A}_{\tilde{B}}^{loc}$.*

Finally, we note that in addition to the maximal green sequence of [GLS11] which starts at the cluster $X_{T_1}, \dots, X_{T_{2n-2}}, X_{V_{2n-1}}, X_{V_{2n}}$, there is another simple maximal green sequence starting at the cluster $\{X_{k,\ell}\}_{(k,\ell) \in I}$ [ACC+14]. This consists of mutating inductively at the $X_{k,0}$ in any order, then the $X_{k,1}$ in any order and so on, until one reaches the cluster $\{X_{k,\ell}\}_{(k,\ell) \in [1,n-1] \times [n,n+1]} \sqcup \{X_{n,0}, X_{n,1}\}$. It is this maximal green sequence which also appears in [Ked08], where it describes the time evolution of the $A_n^{(1)}$ Q -system.

6.4. Q -systems and global sections. Recall that the Q -system is a recurrence relation describing characters of Kirillov-Reshetikhin (KR) modules of the Yangian of a simple Lie algebra [KR87, HKO+99]. In [Ked08] it was shown that the relations of the $A_{n-1}^{(1)}$ Q -system can be normalized to coincide with exchange relations in the cluster algebra \mathcal{A}_{B_n} , where B_n denotes the principal part of \tilde{B}_n . In this section we explain how this emerges from our main result by passing to the global sections of the sheaves $\mathcal{P}_{k,\ell}$.

First, we pass to the classical limits of KR modules, representations of the current algebra rather than the Yangian. The Q -system describes their characters as representations of the underlying simple Lie algebra \mathfrak{g} . For $\mathfrak{g} = \mathfrak{sl}_n$ (but not for other types) they are irreducible as \mathfrak{sl}_n -representations, and are exactly the representations $V_{\ell\omega_k}$ with highest weight $\ell\omega_k$ for $\ell \in \mathbb{N}$ and ω_k a fundamental weight.

Proposition 6.31. *For $\ell \geq 0$ we have isomorphisms*

$$\begin{aligned} \Gamma(\mathcal{P}_{k-1,\ell} * \mathcal{P}_{k+1,\ell}) &\cong V_{\ell\omega_{k-1}} \otimes V_{\ell\omega_{k+1}} [k(n-k) - 1], \\ \Gamma(\mathcal{P}_{k,\ell+1} * \mathcal{P}_{k,\ell-1}) &\cong V_{(\ell+1)\omega_k} \otimes V_{(\ell-1)\omega_k} [k(n-k)], \\ \Gamma(\mathcal{P}_{k,\ell} * \mathcal{P}_{k,\ell}) &\cong V_{\ell\omega_k} \otimes V_{\ell\omega_k} [k(n-k)] \end{aligned}$$

of GL_n representations.

Proof. Let us begin with the first identity. The statement reduces to showing that

$$\mathcal{F} := [\mathcal{O}_{\text{Gr}_{GL_n}^{k-1}} \otimes \det(L_0/L_1)^\ell] * [\mathcal{O}_{\text{Gr}_{GL_n}^{k+1}} \otimes \det(L_1/L_2)^\ell]$$

has global sections $V_{\ell\omega_{k-1}} \otimes V_{\ell\omega_{k+1}}$ and has no higher cohomology. Consider the map

$$\overline{m} : \text{Gr}_{GL_n}^{(k-1,k+1)} \rightarrow \overline{\text{Gr}}_{GL_n}^{\omega^\vee},$$

where $\omega^\vee := \omega_{k-1}^\vee + \omega_{k+1}^\vee$. Then

$$\Gamma(\mathcal{F}) \cong \Gamma(\overline{m}^*(\mathcal{O}_{\overline{\text{Gr}}_{GL_n}^{\omega^\vee}} \otimes \det(L_0/L_2)^\ell)) \cong \Gamma(\mathcal{O}_{\overline{\text{Gr}}_{GL_n}^{\omega^\vee}} \otimes \det(L_0/L_2)^\ell).$$

The higher cohomology vanishes since the canonical sheaf of $\overline{\text{Gr}}_{GL_n}^{\omega^\vee}$ is $\det(L_0/L_2)^{-n} \otimes \det(L_0/tL_0)^{2k}$ (up to an equivariant shift) and $\det(L_0/L_2)$ is ample.

To compute $H^0(\mathcal{F})$ consider the BD deformation of $\text{Gr}_{GL_n}^{(k-1,k+1)}$ over \mathbb{A}^1 . Its general fiber (away from $0 \in \mathbb{A}^1$) is isomorphic to the product $\text{Gr}_{GL_n}^{\omega_{k-1}^\vee} \times \text{Gr}_{GL_n}^{\omega_{k+1}^\vee}$. The line bundle $\det(L_0/L_1)^\ell \otimes \det(L_1/L_2)^\ell$ deforms to the exterior product $\det(L_0/L_1)^\ell \boxtimes \det(L_0/L_1)^\ell$ whose space of global sections is $V_{\ell\omega_{k-1}} \otimes V_{\ell\omega_{k+1}}$. Since \mathcal{F} has no higher cohomology it then follows (by semicontinuity) that $H^0(\mathcal{F})$ is also $V_{\ell\omega_{k-1}} \otimes V_{\ell\omega_{k+1}}$.

A similar argument also proves the third identity. For the second identity we must be a little more careful because the line bundle is *not* a pullback from $\overline{\text{Gr}}_{GL_n}^{2\omega_k^\vee}$. Let us write

$$\mathcal{F} := [\mathcal{O}_{\text{Gr}_{GL_n}^k} \otimes \det(L_0/L_1)^{\ell+1}] * [\mathcal{O}_{\text{Gr}_{GL_n}^k} \otimes \det(L_1/L_2)^{\ell-1}].$$

Then $\mathcal{F} \cong \overline{m}_*(\mathcal{O}_{\text{Gr}_{GL_n}^{(k,k)}} \otimes \mathcal{L} \otimes \det(L_0/L_2)^{-n})$, where

$$\mathcal{L} := \det(L_0/L_2)^{\ell-1+n} \otimes \det(L_0/L_1)^2$$

and $\overline{m} : \text{Gr}_{GL_n}^{(k,k)} \rightarrow \text{Gr}_{GL_n}^{2\omega_k^\vee}$. Now, $\det(L_0/L_2)^{\ell-1+n}$ is ample and $\det(L_0/L_1)^2$ is nef (since it is the pullback via $\text{Gr}_{GL_n}^{(k,k)} \rightarrow \text{Gr}_{GL_n}^k$ of the ample line bundle $\det(L_0/L_1)^2$). It follows that \mathcal{L} is nef. On the other hand, the line bundle \mathcal{L} deforms to $\tilde{\mathcal{L}} := \det(L_0/L_1)^{\ell+1+n} \boxtimes \det(L_0/L_1)^{\ell-1+n}$ on the general fiber $\text{Gr}_{GL_n}^k \times \text{Gr}_{GL_n}^k$ of the BD deformation. Since $\tilde{\mathcal{L}}$ is ample (and hence big) it follows that \mathcal{L} is also big.

Thus \mathcal{L} is both nef and big. The vanishing of the higher cohomology of \mathcal{F} now follows by Kawamata-Viehweg vanishing. Finally, $H^0(\mathcal{F})$ can be computed as before (using the BD deformation). \square

Now consider the first short exact sequence in Proposition 2.16. Taking global sections and using Proposition 6.31 we obtain (from the associated long exact sequence) a short exact sequence

$$0 \rightarrow V_{(\ell+1)\omega_k} \otimes V_{(\ell-1)\omega_k} \rightarrow V_{\ell\omega_k} \otimes V_{\ell\omega_k} \rightarrow V_{\ell\omega_{k-1}} \otimes V_{\ell\omega_{k+1}} \rightarrow 0$$

for any $\ell \geq 0$. Passing to characters these exact sequences result exactly in the relations that comprise the Q -system.

APPENDIX A. UNBOUNDED DERIVED CATEGORIES OF QUASICOHERENT SHEAVES

A.1. Preliminaries. We gather here some general results on functors between unbounded derived categories of quasicoherent sheaves (denoted $D_{qcoh}(\cdot)$). For our purposes we assume all schemes are Noetherian and all maps proper.

Given a map $f : X \rightarrow Y$ we have functors $f_* = f_!, f^*, f^!$ going between $D_{qcoh}(X)$ and $D_{qcoh}(Y)$. Here f^* is the left adjoint to f_* and $f^!$ the right adjoint of f_* . We also have bifunctors $\mathcal{H}om(\bullet, \bullet)$ and \otimes with $\bullet \otimes \mathcal{A}$ being the left adjoint of $\mathcal{H}om(\mathcal{A}, \bullet)$. In particular, this last adjunction gives us an isomorphism (cf. [Lip09, Ex. 3.5.3(e)])

$$(A.1) \quad \mathcal{H}om(\mathcal{A} \otimes \mathcal{B}, \mathcal{C}) \xrightarrow{\sim} \mathcal{H}om(\mathcal{A}, \mathcal{H}om(\mathcal{B}, \mathcal{C})).$$

For $\mathcal{B}, \mathcal{B}' \in D_{qcoh}(Y)$ we have a functorial map (cf. [Lip09, Eq. 3.5.4.5])

$$(A.2) \quad f^* \mathcal{H}om(\mathcal{B}, \mathcal{B}') \rightarrow \mathcal{H}om(f^* \mathcal{B}, f^* \mathcal{B}').$$

If $\mathcal{A} \in D_{qcoh}(X)$ we have an isomorphism

$$(A.3) \quad \alpha : \mathcal{H}om(\mathcal{B}, f_* \mathcal{A}) \xrightarrow{\sim} f_* \mathcal{H}om(f^* \mathcal{B}, \mathcal{A})$$

which can be written as the composition (cf. [Lip09, p. 105])

$$(A.4) \quad \begin{aligned} \mathcal{H}om(\mathcal{B}, f_* \mathcal{A}) &\xrightarrow{\text{adj}} f_* f^* \mathcal{H}om(\mathcal{B}, f_* \mathcal{A}) \\ &\rightarrow f_* \mathcal{H}om(f^* \mathcal{B}, f^* f_* \mathcal{A}) \xrightarrow{\text{adj}} f_* \mathcal{H}om(f^* \mathcal{B}, \mathcal{A}). \end{aligned}$$

Next, we have an isomorphism (cf. [Lip09, Ex. 4.2.3(d)])

$$(A.5) \quad \beta : f_* \mathcal{H}om(\mathcal{A}, f^! \mathcal{B}) \xrightarrow{\sim} \mathcal{H}om(f_* \mathcal{A}, \mathcal{B})$$

which can be written as the composition (cf. [Lip09, Theorem 4.2])

$$(A.6) \quad \begin{aligned} f_* \mathcal{H}om(\mathcal{A}, f^! \mathcal{B}) &\xrightarrow{\text{adj}} f_* \mathcal{H}om(f^* f_* \mathcal{A}, f^! \mathcal{B}) \\ &\xrightarrow{\alpha^{-1}} \mathcal{H}om(f_* \mathcal{A}, f_* f^! \mathcal{B}) \xrightarrow{\text{adj}} \mathcal{H}om(f_* \mathcal{A}, \mathcal{B}). \end{aligned}$$

We also have a functorial isomorphism (cf. [Lip09, Ex. 4.2.3(e)])

$$(A.7) \quad \gamma : \mathcal{H}om(f^* \mathcal{B}, f^! \mathcal{B}') \xrightarrow{\sim} f^! \mathcal{H}om(\mathcal{B}, \mathcal{B}')$$

which can be written as the composition (cf. [Lip09, Ex. 4.2.3(f)])

$$(A.8) \quad \begin{aligned} \mathcal{H}om(f^* \mathcal{B}, f^! \mathcal{B}') &\xrightarrow{\text{adj}} f^! f_* \mathcal{H}om(f^* \mathcal{B}, f^! \mathcal{B}') \\ &\xrightarrow{\alpha^{-1}} f^! \mathcal{H}om(\mathcal{B}, f_* f^! \mathcal{B}') \xrightarrow{\text{adj}} f^! \mathcal{H}om(\mathcal{B}, \mathcal{B}'). \end{aligned}$$

Next suppose all our schemes X have a dualizing complex $\omega_X = \pi^! \mathcal{O}_{pt}$, where $\pi : X \rightarrow pt$ is the projection to a point. We denote by $\mathbb{D}(\bullet) = \mathcal{H}om(\bullet, \omega_X)$ the dualizing functor. The adjunction properties of $\mathcal{H}om$ and \otimes give us a natural transformation $\tau : id \rightarrow \mathbb{D}\mathbb{D}$. This map is an isomorphism when restricted to the subcategory of complexes with coherent cohomology (but is not an isomorphism in general).

Taking $\mathcal{B} = \omega_Y$ in (A.5) we obtain an isomorphism $f_*\mathbb{D} \xrightarrow{\sim} \mathbb{D}f_*$. On the other hand, taking \mathcal{B} arbitrary and $\mathcal{B}' = \omega_Y$ in (A.7) gives us an isomorphism $\eta : \mathbb{D}f^* \xrightarrow{\sim} f^!\mathbb{D}$ which can be written as the composition

$$\eta : \mathbb{D}f^* \xrightarrow{adj} f^!f_*\mathbb{D}f^* \xrightarrow{\sim} f^!\mathbb{D}f_*f^* \xrightarrow{adj} f^!\mathbb{D}.$$

Finally, we denote by $\theta : f^! \rightarrow \mathbb{D}f^*\mathbb{D}$ the composition

$$f^! \xrightarrow{\tau} f^!\mathbb{D}\mathbb{D} \xrightarrow{\eta^{-1}} \mathbb{D}f^*\mathbb{D},$$

where the first map is induced by the natural map $\tau : id \rightarrow \mathbb{D}\mathbb{D}$. Notice that θ is generally not an isomorphism since τ is generally not an isomorphism.

Lemma A.9. *For a proper map $f : X \rightarrow Y$ the following diagrams commute:*

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{adj} & \mathbb{D}f^*f_* \\ \downarrow adj & & \downarrow \eta \\ f^!f_*\mathbb{D} & \xrightarrow{\sim} & f^!\mathbb{D}f_* \end{array} \qquad \begin{array}{ccc} f_*\mathbb{D}f^* & \xrightarrow{\eta} & f_*f^!\mathbb{D} \\ \downarrow \sim & & \downarrow adj \\ \mathbb{D}f_*f^* & \xrightarrow{adj} & \mathbb{D} \end{array}$$

Proof. The composition $\mathbb{D} \rightarrow f^!f_*\mathbb{D} \xrightarrow{\sim} f^!\mathbb{D}f_*$ can be written as

$$\begin{aligned} \mathcal{H}om(\mathcal{A}, f^!\omega_Y) &\xrightarrow{adj} f^!f_*\mathcal{H}om(\mathcal{A}, f^!\omega_Y) \xrightarrow{adj} f^!f_*\mathcal{H}om(f^*f_*\mathcal{A}, f^!\omega_Y) \\ &\xrightarrow{\alpha^{-1}} f^!\mathcal{H}om(f_*\mathcal{A}, f_*f^!\omega_Y) \xrightarrow{adj} f^!\mathcal{H}om(f_*\mathcal{A}, \omega_Y). \end{aligned}$$

Likewise the composition $\mathbb{D} \xrightarrow{adj} \mathbb{D}f^*f_* \xrightarrow{\eta} f^!\mathbb{D}f_*$ can be written as

$$\begin{aligned} \mathcal{H}om(\mathcal{A}, f^!\omega_*) &\xrightarrow{adj} \mathcal{H}om(f^*f_*\mathcal{A}, f^!\omega_Y) \xrightarrow{adj} f^!f_*\mathcal{H}om(f^*f_*\mathcal{A}, f^!\omega_Y) \\ &\xrightarrow{\alpha^{-1}} f^!\mathcal{H}om(f_*\mathcal{A}, f_*f^!\omega_Y) \xrightarrow{adj} f^!\mathcal{H}om(f_*\mathcal{A}, \omega_Y). \end{aligned}$$

These two compositions are clearly equivalent which proves that the first square commutes. The second square commutes by a similar argument. \square

The following is an easy corollary of Lemma A.9.

Corollary A.10. *For a proper map $f : X \rightarrow Y$ the following diagrams commute:*

$$\begin{array}{ccc} id & \xrightarrow{\tau} & \mathbb{D}\mathbb{D} \xrightarrow{adj} \mathbb{D}f^*f_*\mathbb{D} \\ \downarrow adj & & \downarrow \sim \\ f^!f_* & \xrightarrow{\theta} & \mathbb{D}f^*\mathbb{D}f_* \end{array} \qquad \begin{array}{ccc} f_*f^! & \xrightarrow{\theta} & f_*\mathbb{D}f^*\mathbb{D} \xrightarrow{\sim} \mathbb{D}f_*f^*\mathbb{D} \\ \downarrow adj & & \downarrow adj \\ id & \xrightarrow{\tau} & \mathbb{D}\mathbb{D} \end{array}$$

A.2. The natural transformation ϕ .

Definition A.11 ([Lip09]). We say that a fiber product diagram

$$(A.12) \quad \begin{array}{ccc} Z & \xrightarrow{j_1} & X' \\ j_2 \downarrow & & \downarrow i_2 \\ X & \xrightarrow{i_1} & Y \end{array}$$

is *independent* if the composition

$$(A.13) \quad i_2^* i_{1*} \rightarrow i_2^* i_{1*} (j_{2*} j_2^*) = (i_2^* i_{2*}) j_{1*} j_2^* \rightarrow j_{1*} j_2^*$$

is an isomorphism.

Remark A.14. One can check [Lip09, Prop. 3.7.2] that the composition in (A.13) is the same as the composition

$$i_2^* i_{1*} \xrightarrow{adj} (j_{1*} j_1^*) i_2^* i_{1*} = j_{1*} j_2^* (i_1^* i_{1*}) \xrightarrow{adj} j_{1*} j_2^*.$$

Condition (A.13) is satisfied, for instance, if X, X', Y are smooth, i_1, i_2 are inclusions, and X, X' intersect in the expected dimension inside Y . Although this condition *a priori* seems to depend on an orientation of the square the following lemma shows otherwise.

Lemma A.15. *The following four conditions are equivalent:*

- (1) *The square in (A.12) is independent.*
- (2) *The composition $i_1^* i_{2*} \xrightarrow{adj} (j_{2*} j_2^*) i_1^* i_{2*} = j_{2*} j_1^* (i_2^* i_{2*}) \xrightarrow{adj} j_{2*} j_1^*$ is an isomorphism.*
- (3) *The composition $j_{1*} j_2^! \xrightarrow{adj} j_{1*} j_2^! (i_1^! i_{1*}) = (j_{1*} j_1^!) i_2^! i_{1*} \xrightarrow{adj} i_2^! i_{1*}$ is an isomorphism.*
- (4) *The composition $j_{2*} j_1^! \xrightarrow{adj} j_{2*} j_1^! (i_2^! i_{2*}) = (j_{2*} j_2^!) i_1^! i_{2*} \xrightarrow{adj} i_1^! i_{2*}$ is an isomorphism.*

Proof. This follows from [Lip09, Theorem 3.10.3] and [Lip09, Ex. 3.10.4]. □

Given an independent square as in (A.12) we obtain a natural transformation $\phi : j_2^* i_1^! \rightarrow j_1^! i_2^*$ as the composition

$$(A.16) \quad j_2^* i_1^! \xrightarrow{adj} (j_1^! j_{1*}) j_2^* i_1^! \xrightarrow{\sim} j_1^! i_2^* (i_{1*} i_1^!) \xrightarrow{adj} j_1^! i_2^*,$$

where the middle isomorphism is the inverse of $i_2^* i_{1*} \xrightarrow{\sim} j_{1*} j_2^*$ from (A.13). The map ϕ appears in [Lip09, Remark 3.10.2.1(c)] where it is called β_σ .

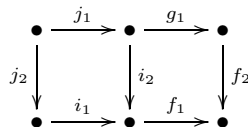
Lemma A.17 ([Lip09, Ex. 3.10.4b]). *The map ϕ is equivalently given by the composition*

$$(A.18) \quad j_2^* i_1^! \xrightarrow{adj} j_2^* i_1^! (i_{2*} i_2^*) \xrightarrow{\sim} (j_2^* j_{2*}) j_1^! i_2^* \xrightarrow{adj} j_1^! i_2^*,$$

where the middle isomorphism is the inverse of $j_{2*} j_1^! \xrightarrow{\sim} i_1^! i_{2*}$.

A.3. Some properties of ϕ .

Lemma A.19. *The map ϕ is natural in the following sense. Suppose one has the following two independent fiber product diagrams:*



Then the fiber product diagram

$$(A.20) \quad \begin{array}{ccc} \bullet & \xrightarrow{g_1 \circ j_1} & \bullet \\ j_2 \downarrow & & \downarrow f_2 \\ \bullet & \xrightarrow{f_1 \circ i_1} & \bullet \end{array}$$

is also independent and the corresponding natural map

$$(A.21) \quad j_2^*(f_1 \circ i_1)! \xrightarrow{\phi} (g_1 \circ f_1)! f_2^*$$

is equal to the composition

$$(A.22) \quad j_2^* i_1^! f_1^! \xrightarrow{\phi_1 \circ id} j_1^! i_2^* f_1^! \xrightarrow{id \circ \phi_2} j_1^! g_1^! f_2^*.$$

Proof. The first part of this lemma appears as [Lip09, Lemma 3.10.3.2] while the second part is proved in [Lip09, Prop. 4.6.8]. \square

Proposition A.23 ([Lip09, Ex. 4.2.3i]). *Suppose the square in (A.12) is independent and take $\mathcal{B}, \mathcal{B}' \in D_{qcoh}(Y)$. Then the diagram*

$$(A.24) \quad \begin{array}{ccccc} j_2^* i_1^! \mathcal{H}om(\mathcal{B}, \mathcal{B}') & \xrightarrow{\gamma^{-1}} & j_2^* \mathcal{H}om(i_1^* \mathcal{B}, i_1^! \mathcal{B}') & \longrightarrow & \mathcal{H}om(j_2^* i_1^* \mathcal{B}, j_2^* i_1^! \mathcal{B}') \\ \downarrow \phi & & & & \downarrow \phi \\ j_1^! i_2^* \mathcal{H}om(\mathcal{B}, \mathcal{B}) & \longrightarrow & j_1^! \mathcal{H}om(i_2^* \mathcal{B}, i_2^! \mathcal{B}') & \xrightarrow{\gamma^{-1}} & \mathcal{H}om(j_1^* i_2^* \mathcal{B}, j_1^! i_2^! \mathcal{B}') \end{array}$$

commutes, where in the right-hand vertical map we use ϕ and that $j_2^* i_1^*(\mathcal{B}) = j_1^* i_2^*(\mathcal{B})$.

Proposition A.25. *Suppose the square in (A.12) is independent and consider maps ρ, ρ' given as the compositions*

$$(A.26) \quad j_1^* \mathbb{D} i_2^* \mathbb{D} \xrightarrow{\theta^{-1}} j_1^* i_2^! \xrightarrow{\phi} j_2^! i_1^* \xrightarrow{\theta} \mathbb{D} j_2^* \mathbb{D} i_1^*,$$

$$(A.27) \quad j_2^* \mathbb{D} i_1^* \mathbb{D} \xrightarrow{\theta^{-1}} j_2^* i_1^! \xrightarrow{\phi} j_1^! i_2^* \xrightarrow{\theta} \mathbb{D} j_1^* \mathbb{D} i_2^*$$

of functors $D_{coh}(Y) \rightarrow D_{qcoh}(Z)$. Then ρ' is equal to the composition

$$j_2^* \mathbb{D} i_1^* \mathbb{D} \xrightarrow{\tau} \mathbb{D} \mathbb{D} j_2^* \mathbb{D} i_1^* \mathbb{D} \xrightarrow{\mathbb{D} \rho \mathbb{D}} \mathbb{D} j_1^* \mathbb{D} i_2^* \mathbb{D} \mathbb{D} \xrightarrow{\tau^{-1}} \mathbb{D} j_1^* \mathbb{D} i_2^*.$$

Notice that the τ^{-1} exists since the domain is $D_{coh}(Y)$.

Proof. Starting with $\mathbb{D} \rho$ and writing out ϕ leaves us with

$$(A.28) \quad \mathbb{D} \mathbb{D} j_2^* \mathbb{D} i_1^* \xrightarrow{\theta} \mathbb{D} j_2^! i_1^* \xrightarrow{adj} \mathbb{D} j_1^* j_{1*} j_2^! i_1^* \xrightarrow{\sim} \mathbb{D} j_1^* i_2^! i_{1*} i_1^* \xrightarrow{adj} \mathbb{D} j_1^* i_2^! \xrightarrow{\theta^{-1}} \mathbb{D} j_1^* \mathbb{D} i_2^* \mathbb{D}.$$

By Lemma A.30 one has the following commutative diagram:

$$\begin{array}{ccccccc} \mathbb{D} j_1^* (j_{1*} j_2^!) i_1^* & \xrightarrow{\sim} & \mathbb{D} j_1^* (i_2^! i_{1*}) i_1^* & \xrightarrow{adj} & \mathbb{D} j_1^* i_2^! \\ \uparrow \theta & & \uparrow \theta & & \uparrow \theta \\ \mathbb{D} j_1^* j_{1*} \mathbb{D} j_2^* \mathbb{D} i_1^* & \xrightarrow{\sim} & \mathbb{D} j_1^* \mathbb{D} j_{1*} j_2^* \mathbb{D} i_1^* & \xrightarrow{\sim} & \mathbb{D} j_1^* \mathbb{D} i_2^* i_{1*} \mathbb{D} i_1^* & \xrightarrow{\sim} & \mathbb{D} j_1^* \mathbb{D} i_2^* \mathbb{D} i_{1*} i_1^* & \xrightarrow{adj} & \mathbb{D} j_1^* \mathbb{D} i_2^* \mathbb{D} \end{array}$$

This means that we can rewrite (A.28) as the following composition:

$$\begin{aligned} \mathbb{D}\mathbb{D}j_2^* \mathbb{D}i_1^* &\xrightarrow{adj} \mathbb{D}j_1^* j_{1*} \mathbb{D}j_2^* \mathbb{D}i_1^* \xrightarrow{\sim} \mathbb{D}j_1^* \mathbb{D}j_{1*} j_2^* \mathbb{D}i_1^* \\ &\xrightarrow{\sim} \mathbb{D}j_1^* \mathbb{D}i_2^* i_{1*} \mathbb{D}i_1^* \xrightarrow{\sim} \mathbb{D}j_1^* \mathbb{D}i_2^* \mathbb{D}i_{1*} i_1^* \xrightarrow{adj} \mathbb{D}j_1^* \mathbb{D}i_2^* \mathbb{D}. \end{aligned}$$

Precomposing with $j_2^* \mathbb{D}i_1^* \rightarrow \mathbb{D}\mathbb{D}j_2^* \mathbb{D}i_1^*$ and using the left-hand diagram in Corollary A.10 this composition becomes

$$\begin{aligned} j_2^* \mathbb{D}i_1^* &\xrightarrow{adj} j_1^! j_{1*} j_2^* \mathbb{D}i_1^* \xrightarrow{\theta} \mathbb{D}j_1^* \mathbb{D}j_{1*} j_2^* \mathbb{D}i_1^* \\ &\xrightarrow{\sim} \mathbb{D}j_1^* \mathbb{D}i_2^* i_{1*} \mathbb{D}i_1^* \xrightarrow{\sim} \mathbb{D}j_1^* \mathbb{D}i_2^* \mathbb{D}i_{1*} i_1^* \xrightarrow{adj} \mathbb{D}j_1^* \mathbb{D}i_2^* \mathbb{D} \end{aligned}$$

which can easily be rewritten as

$$j_2^* \mathbb{D}i_1^* \xrightarrow{adj} j_1^! j_{1*} j_2^* \mathbb{D}i_1^* \xrightarrow{\sim} j_1^! i_2^* i_{1*} \mathbb{D}i_1^* \xrightarrow{\sim} j_1^! i_2^* \mathbb{D}i_{1*} i_1^* \xrightarrow{adj} j_1^! i_2^* \mathbb{D} \xrightarrow{\theta} \mathbb{D}j_1^* \mathbb{D}i_2^* \mathbb{D}.$$

Now compose everything on the right with \mathbb{D} and then precompose with

$$j_2^* \mathbb{D}i_1^* \mathbb{D} \xrightarrow{\theta^{-1}} j_2^* i_1^! \xrightarrow{\theta} j_2^* \mathbb{D}i_1^* \mathbb{D}.$$

We can rearrange this to give

$$\begin{aligned} j_2^* \mathbb{D}i_1^* \mathbb{D} &\xrightarrow{\theta^{-1}} j_2^* i_1^! \xrightarrow{adj} j_1^! j_{1*} j_2^* i_1^! \\ &\xrightarrow{\sim} j_1^! i_2^* i_{1*} i_1^! \xrightarrow{\theta} j_1^! i_2^* i_{1*} \mathbb{D}i_1^* \mathbb{D} \xrightarrow{\sim} j_1^! i_2^* \mathbb{D}i_{1*} i_1^* \mathbb{D} \xrightarrow{adj} j_1^! i_2^* \mathbb{D} \mathbb{D} \xrightarrow{\theta} \mathbb{D}j_1^* \mathbb{D}i_2^* \mathbb{D} \mathbb{D}. \end{aligned}$$

Using the right-hand diagram in Corollary A.10 this is equal to

$$j_2^* \mathbb{D}i_1^* \mathbb{D} \xrightarrow{\theta^{-1}} j_2^* i_1^! \xrightarrow{adj} j_1^! j_{1*} j_2^* i_1^! \xrightarrow{\sim} j_1^! i_2^* i_{1*} i_1^! \xrightarrow{adj} j_1^! i_2^* \xrightarrow{\theta} \mathbb{D}j_1^* \mathbb{D}i_2^* \rightarrow \mathbb{D}j_1^* \mathbb{D}i_2^* \mathbb{D} \mathbb{D}$$

which is just

$$(A.29) \quad j_2^* \mathbb{D}i_1^* \mathbb{D} \xrightarrow{\theta^{-1}} j_2^* i_1^! \xrightarrow{\phi} j_1^! i_2^* \xrightarrow{\theta} \mathbb{D}j_1^* \mathbb{D}i_2^* \xrightarrow{\tau} \mathbb{D}j_1^* \mathbb{D}i_2^* \mathbb{D} \mathbb{D}.$$

The result follows since this shows that (A.29) is equal to the composition

$$j_2^* \mathbb{D}i_1^* \mathbb{D} \xrightarrow{\tau} \mathbb{D}\mathbb{D}j_2^* \mathbb{D}i_1^* \mathbb{D} \xrightarrow{\mathbb{D}\rho\mathbb{D}} \mathbb{D}j_1^* \mathbb{D}i_2^* \mathbb{D} \mathbb{D}.$$

□

Lemma A.30. *Assuming the square in (A.12) is independent, the following diagram commutes:*

$$(A.31) \quad \begin{array}{ccc} i_2^! i_{1*} & \xrightarrow{\sim} & j_{1*} j_2^! \\ \downarrow \theta & & \downarrow \theta \\ \mathbb{D}i_2^* \mathbb{D}i_{1*} & \xrightarrow{\sim} \mathbb{D}i_2^* i_{1*} \mathbb{D} \xrightarrow{\sim} \mathbb{D}j_{1*} j_2^* \mathbb{D} \xrightarrow{\sim} & j_{1*} \mathbb{D}j_2^* \mathbb{D} \end{array}$$

Proof. Notice that the more natural morphism is $j_{1*} j_2^! \xrightarrow{\sim} i_2^! i_{1*}$ while the one appearing above is its inverse. So what we need to show is that the following diagram commutes:

$$(A.32) \quad \begin{array}{ccccccc} j_{1*} j_2^! & \xrightarrow{adj} & j_{1*} j_2^! i_{1*} & \xrightarrow{=} & j_{1*} j_2^! i_2^! i_{1*} & \xrightarrow{adj} & i_2^! i_{1*} \\ \downarrow \theta & & & & & & \downarrow \theta \\ j_{1*} \mathbb{D}j_2^* \mathbb{D} & \xrightarrow{\sim} & \mathbb{D}j_{1*} j_2^* \mathbb{D} & \xrightarrow{adj} & \mathbb{D}j_{1*} j_2^* i_1^* i_{1*} \mathbb{D} & \xrightarrow{=} & \mathbb{D}j_{1*} j_2^* i_2^* i_{1*} \mathbb{D} \xrightarrow{adj} \mathbb{D}i_2^* i_{1*} \mathbb{D} \xrightarrow{\sim} \mathbb{D}i_2^* \mathbb{D}i_{1*} \end{array}$$

We can rewrite this diagram as follows:

(A.33)

$$\begin{array}{ccccccc}
 j_{1*}j_2^! & \xrightarrow{\text{adj}} & j_{1*}j_2^!i_1^!i_{1*} & \xrightarrow{=} & j_{1*}j_1^!i_2^!i_{1*} & \xrightarrow{\text{adj}} & i_2^!i_{1*} \\
 \downarrow \theta & & \downarrow \theta & & \downarrow \theta & & \downarrow \theta \\
 & & j_{1*}j_2^!\mathbb{D}i_1^*\mathbb{D}i_{1*} & & j_{1*}j_1^!\mathbb{D}i_2^*\mathbb{D}i_{1*} & & \\
 & & \downarrow \eta^{-1} & & \downarrow \eta^{-1} & & \\
 j_{1*}\mathbb{D}j_2^*\mathbb{D} & \xrightarrow{\text{adj}} & j_{1*}\mathbb{D}j_2^*i_1^*i_{1*}\mathbb{D} & \xrightarrow{\sim} & j_{1*}\mathbb{D}j_2^*i_1^*\mathbb{D}i_{1*} & \xrightarrow{=} & j_{1*}\mathbb{D}j_1^*i_2^*\mathbb{D}i_{1*} & \xrightarrow{\sim} & \mathbb{D}j_{1*}j_1^*i_2^*\mathbb{D}i_{1*} & \xrightarrow{\text{adj}} & \mathbb{D}i_2^*\mathbb{D}i_{1*}
 \end{array}$$

It is not hard to check that the compositions along the bottom row in (A.32) and (A.33) agree. Now consider the following diagram:

$$\begin{array}{ccccc}
 j_2^! & \xrightarrow{\text{adj}} & j_2^!i_1^!i_{1*} & & \\
 \downarrow \tau & & \downarrow \theta & & \\
 j_2^!\mathbb{D}\mathbb{D} & \xrightarrow{\text{adj}} & j_2^!\mathbb{D}i_1^*i_{1*}\mathbb{D} & \xrightarrow{\sim} & j_2^!\mathbb{D}i_1^*\mathbb{D}i_{1*} \\
 \downarrow \eta^{-1} & & \downarrow \eta^{-1} & & \downarrow \eta^{-1} \\
 \mathbb{D}j_2^*\mathbb{D} & \xrightarrow{\text{adj}} & \mathbb{D}j_2^*i_1^*i_{1*}\mathbb{D} & \xrightarrow{\sim} & \mathbb{D}j_2^*i_1^*\mathbb{D}i_{1*}
 \end{array}$$

The top rectangle commutes by the left diagram in Corollary A.10 while the other two squares commute for obvious reasons. This proves the commutativity of the left-hand square in (A.33). Similarly, using the right diagram in Corollary A.10, one can prove the commutativity of the right-hand square in (A.33). Finally, to see that the middle rectangle commutes note that the composition

$$j_2^!i_1^! \xrightarrow{\theta} j_2^!\mathbb{D}i_1^*\mathbb{D} \xrightarrow{\eta^{-1}} \mathbb{D}j_2^*i_1^*\mathbb{D}$$

is by definition

$$j_2^!i_1^! \xrightarrow{\tau} j_2^!i_1^!\mathbb{D}\mathbb{D} \xrightarrow{\eta^{-1}} j_2^!\mathbb{D}i_1^*\mathbb{D} \xrightarrow{\eta^{-1}} \mathbb{D}j_2^*i_1^*\mathbb{D},$$

which is nothing but $(i_1 \circ j_2)^! \xrightarrow{\theta} \mathbb{D}(i_1 \circ j_2)^*\mathbb{D}$. Likewise the composition

$$j_1^!i_2^! \xrightarrow{\theta} j_1^!\mathbb{D}i_2^*\mathbb{D} \xrightarrow{\eta^{-1}} \mathbb{D}j_1^*i_2^*\mathbb{D}$$

is just $(i_2 \circ j_1)^! \xrightarrow{\theta} \mathbb{D}(i_2 \circ j_1)^*\mathbb{D}$. The commutation of (A.33) (and hence of (A.32)) follows. \square

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