

TOPOLOGICAL NOETHERIANITY OF POLYNOMIAL FUNCTORS

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1. INTRODUCTION

This paper is motivated by two recent developments in “asymptotic commutative algebra”. First, in [4], Ananyan-Hochster prove a conjecture due to Stillman [23, Problem 3.14], to the effect that the projective dimension of an ideal in a polynomial ring generated by a fixed number of homogeneous polynomials of prescribed degrees can be bounded independently of the number of variables. Second, in [9], Derksen-Eggermont-Snowden prove that the inverse limit over n of the space of cubic polynomials in n variables is topologically Noetherian up to linear coordinate transformations. These two theorems show striking similarities in content, and in [16], Erman-Sam-Snowden show that topological Noetherianity of a suitable space of tuples of homogeneous polynomials, together with Stillman’s conjecture, implies a generalisation of Stillman’s conjecture to other ideal invariants. In addition to being similar in content, the two questions have similar histories—e.g., both were first established for tuples of quadrics [3, 15]—but since [4] the Noetherianity problem has been lagging behind. The goal of this paper is to make it catch up.

1.1. Polynomial functors. Let K be an infinite field and let \mathbf{Vec} be the category of finite-dimensional vector spaces over K . We consider a covariant polynomial functor $P : \mathbf{Vec} \rightarrow \mathbf{Vec}$ of finite degree d . This means that for all V, W the map $P : \mathrm{Hom}_{\mathbf{Vec}}(V, W) \rightarrow \mathrm{Hom}_{\mathbf{Vec}}(P(V), P(W))$ is polynomial of degree at most d , with equality for at least some choice of V and W . The uniform upper bound d rules out examples like $V \mapsto \bigwedge^0 V \oplus \bigwedge^1 V \oplus \bigwedge^2 V \oplus \dots$

Then P splits as a direct sum $P = P_0 \oplus P_1 \oplus \dots \oplus P_d$ where

$$P_e(V) := \{p \in P(V) \mid P(t \cdot 1_V)p = t^e p \text{ for all } t \in K\};$$

see, e.g., [18]. For each e the map $\mathrm{Hom}(V, W) \rightarrow \mathrm{Hom}(P_e(V), P_e(W))$ is homogeneous of degree e , and we have $P_d(V) \neq 0$ for all V of sufficiently large dimension.

1.2. Topological spaces over a category. The polynomial functor P will also be interpreted as a functor to the category \mathbf{Top} of topological spaces. Here we equip the finite-dimensional vector space $P(V) = P_0(V) \oplus \dots \oplus P_d(V)$ with the Zariski-topology.

A general functor X from a category \mathbf{C} to \mathbf{Top} is called a \mathbf{C} -topological space, or \mathbf{C} -space for short. A second \mathbf{C} -space Y is called a \mathbf{C} -subspace of X if for each

Received by the editors May 11, 2017, and, in revised form, January 10, 2019.

2010 *Mathematics Subject Classification*. Primary 13E05, 14M12, 15A69, 20G05.

The author was partially supported by the NWO Vici grant entitled *Stabilisation in Algebra and Geometry*.

object A of \mathbf{C} the space $Y(A)$ is a subset of $X(A)$ with the induced topology, and if, moreover, for a morphism $f : A \rightarrow B$ the continuous map $Y(f) : Y(A) \rightarrow Y(B)$ is the restriction of $X(f)$ to $Y(A)$. We then write $Y \subseteq X$. The \mathbf{C} -subspace Y is called closed if $Y(A)$ is closed in $X(A)$ for each A ; we then also call Y a *closed \mathbf{C} -subset* of X . The \mathbf{C} -space X is called *Noetherian* if every descending chain of closed \mathbf{C} -subspaces stabilises.

Furthermore, a \mathbf{C} -continuous map from a \mathbf{C} -space X to a \mathbf{C} -space X' consists of a continuous map $\varphi_A : X(A) \rightarrow X'(A)$ for each object A , in such a way that for any morphism $f : A \rightarrow B$ we have $\varphi_B \circ X(f) = X'(f) \circ \varphi_A$. A \mathbf{C} -homeomorphism has the natural meaning.

1.3. The main theorem. We will establish the following fundamental theorem.

Theorem 1. *Let \mathbf{Vec} be the category of finite-dimensional vector spaces over an infinite field K , and let $P : \mathbf{Vec} \rightarrow \mathbf{Vec}$ be a finite-degree polynomial functor. Then P is Noetherian as a \mathbf{Vec} -topological space.*

Remark 2. The restriction to infinite K is crucial for the set-up and the proofs below—e.g., it is used in the decomposition of a polynomial functor into homogeneous parts and, implicitly, to argue that if an algebraic group $\mathrm{GL}_n(K)$ preserves an ideal, then so does its Lie algebra. In future work, we will pursue versions of Theorem 1 over \mathbb{Z} and possibly over finite fields.

Theorem 1 will be useful in different contexts where finiteness results are sought. In the remainder of this section we discuss several such consequences; since an earlier version of this paper appeared on the arXiv, several other ramifications have appeared, e.g., in [16].

1.4. Equivariant Noethericity of limits. The polynomial functor P gives rise to a topological space $P_\infty := \lim_{\leftarrow n} P(K^n)$, the projective limit along the linear maps $P(\pi_n) : P(K^{n+1}) \rightarrow P(K^n)$ where π_n is the projection $K^{n+1} \rightarrow K^n$ forgetting the last coordinate. By functoriality, each $P(K^n)$ is acted upon by the general linear group GL_n , the map $P(\pi_n)$ is GL_n -equivariant if we embed GL_n into GL_{n+1} via $g \mapsto \mathrm{diag}(g, 1)$, and hence P_∞ is acted upon by the direct limit $\mathrm{GL}_\infty := \bigcup_n \mathrm{GL}_n$, the group of all invertible $\mathbb{N} \times \mathbb{N}$ -matrices which in all but finitely many entries equal the infinite identity matrix.

Given a closed \mathbf{Vec} -subset X of P , the inverse limit $X_\infty := \lim_{\leftarrow n} X(K^n)$ is a closed, GL_∞ -stable subset of P_∞ , and using embeddings $K^n \rightarrow K^{n+1}$ appending a zero coordinate, one finds that X_∞ surjects onto each $X(K^n)$. Conversely, given a closed, GL_∞ -stable subset Y of P_∞ , then for any finite-dimensional vector space V and any linear isomorphism $\varphi : K^n \rightarrow V$, we set $X(V) := P(\varphi)(Y_n)$, where Y_n is the image of Y in $P(K^n)$. One can check that $V \mapsto X(V)$ is a closed \mathbf{Vec} -subset of P and that this construction $Y \mapsto X$ is inverse to the construction $X \mapsto X_\infty$ above. Thus the theorem is equivalent to the following corollary.

Corollary 3. *Let \mathbf{Vec} be the category of finite-dimensional vector spaces over an infinite field K , let $P : \mathbf{Vec} \rightarrow \mathbf{Vec}$ be a finite-degree polynomial functor, and equip $P_\infty := \lim_{\leftarrow n} P(K^n)$ with the inverse-limit topology of the Zariski topologies on the $P(K^n)$. Then P_∞ is GL_∞ -Noetherian, i.e., every chain $P_\infty \supseteq Y_1 \supseteq Y_2 \supseteq \dots$ of GL_∞ -stable closed subsets of P_∞ is eventually constant. Equivalently, every GL_∞ -stable closed subset Y of P_∞ is the set of common zeros of finitely many GL_∞ -orbits of polynomial equations.*

Example 4. The paper [9] concerns the case where $P = S^3$, the third symmetric power. In this case, P_∞ is the space of infinite cubics $\sum_{1 \leq i \leq j \leq k} c_{ijk} x_i x_j x_k$, and GL_∞ acts by linear transformations that affect only finitely many of the variables x_i . ♣

Remark 5. The proofs below could have been formulated directly in this infinite-dimensional setting, rather than the finite-dimensional, functorial setting. However, one of the key techniques, namely, shifting a polynomial functor by a constant vector space, is best expressed in the functorial language. Moreover, the functorial language allows us to stay in the more familiar realm of finite-dimensional algebraic geometry.

1.5. Generalisations of Stillman’s conjecture. In [16], Erman, Sam and Snowden use the following special case of Theorem 1.

Corollary 6. *Let K be an infinite field, fix natural numbers d_1, \dots, d_k , and consider the polynomial functor $P : V \mapsto S^{d_1}V \oplus \dots \oplus S^{d_k}V$. Then P is a Noetherian \mathbf{Vec} -topological space, and hence its limit P_∞ is GL_∞ -Noetherian.*

Let μ be a function that associates a number $\mu(I) \in \mathbb{Z} \cup \{\infty\}$ to any homogeneous ideal I in a symmetric algebra SV on $V \in \mathbf{Vec}$, in such a way that $\mu(\langle S(\varphi)I \rangle) = \mu(I)$ for any injective linear map $\varphi : V \rightarrow W$ with induced homomorphism $S(\varphi) : SV \rightarrow SW$, and such that μ is upper semicontinuous in flat families. In [16] the following is proved.

Theorem 7 ([16]). *Corollary 6 implies that for any ideal invariant μ with the properties above there exists a number N such that for all $V \in \mathbf{Vec}$, any ideal $I \subseteq SV$ generated by k homogeneous polynomials of degrees d_1, \dots, d_k either has $\mu(I) \leq N$ or $\mu(I) = \infty$.*

The crucial point here is that N does not depend on $\dim V$. Stillman’s conjecture [23, Problem 3.14] is this statement for μ equal to the projective dimension, and it is used in the proof of the generalisation just stated. However, in a follow-up paper [17], the same authors give two new proofs of Stillman’s conjecture, one of which uses Corollary 6. An algorithmic variant of this latter proof is presented in [13].

1.6. Twisted commutative algebras. For $K = \mathbb{C}$, the algebra of polynomial functions on P_∞ , i.e., the direct limit of the symmetric algebras on $P_n(K^n)^*$, is a *twisted commutative algebra* in one of its incarnations [31, 32]. In this context, Theorem 1 says the following.

Corollary 8. *Any finitely generated twisted commutative algebra over \mathbb{C} is topologically Noetherian.*

1.7. Functors from \mathbf{Vec}^ℓ . Theorem 1 has the following generalisation to functors that take several distinct vector spaces as input.

Corollary 9. *Let K be an infinite field, let \mathbf{Vec} be the category of finite-dimensional vector spaces over K , let ℓ be a positive integer, and let P be a functor from \mathbf{Vec}^ℓ to \mathbf{Vec} such that for any $\underline{V}, \underline{W} \in \mathbf{Vec}^\ell$ the map $\text{Hom}_{\mathbf{Vec}^\ell}(\underline{V}, \underline{W}) \rightarrow \text{Hom}_{\mathbf{Vec}}(P(\underline{V}), P(\underline{W}))$ is polynomial of uniformly bounded degree. Then P is a Noetherian \mathbf{Vec}^ℓ -topological space.*

Note that the group of automorphisms of (V_1, \dots, V_ℓ) is $\prod_i \text{GL}(V_i)$, which when the V_i are all equal contains a diagonal copy of $\text{GL}(V)$. This suggests that Theorem 1 for $V \mapsto P(V, \dots, V)$ is in fact stronger than this corollary, as we prove now.

Proof of Corollary 9 from Theorem 1. Let $X_1 \supseteq X_2 \supseteq \dots$ be a chain of closed \mathbf{Vec}^ℓ -subsets of P . Let $Q : \mathbf{Vec} \rightarrow \mathbf{Vec}$ be the functor that sends V to $P(V, \dots, V)$, and set $Y_n(V) := X_n(V, \dots, V)$, a closed \mathbf{Vec} -subset of the polynomial functor Q . By Theorem 1, the sequence $(Y_n)_n$ is constant for n at least some n_0 . We claim that so is $(X_n)_n$. Indeed, let $\underline{V} = (V_1, \dots, V_\ell) \in \mathbf{Vec}^\ell$. Choose a $V \in \mathbf{Vec}$ with $\dim V \geq \dim V_i$ for each i , and choose surjections $\pi_i : V \rightarrow V_i$ and injections $\iota_i : V_i \rightarrow V$ such that $\pi_i \circ \iota_i = 1_{V_i}$. Then for $n \geq n_0$ we have

$$\begin{aligned} X_n(V_1, \dots, V_\ell) &= P(\pi_1, \dots, \pi_\ell)P(\iota_1, \dots, \iota_\ell)X_n(V_1, \dots, V_\ell) \\ &\subseteq P(\pi_1, \dots, \pi_\ell)X_n(V, \dots, V) \\ &= P(\pi_1, \dots, \pi_\ell)X_{n+1}(V, \dots, V) \subseteq X_{n+1}(V_1, \dots, V_\ell), \end{aligned}$$

as desired. □

1.8. Slice rank. Taking $P(V_1, \dots, V_\ell) = V_1 \otimes \dots \otimes V_\ell$, a tensor in $P(\underline{V})$ is said to have slice rank 1 if it is nonzero and of the form $v \otimes A$ for a v in one of the V_i and an $A \in \bigotimes_{j \neq i} V_j$. A tensor has slice rank at most k if it is a sum of at most k tensors of slice rank 1; in this sum the slice index i may vary through $\{1, \dots, d\}$. Being of slice rank at most a fixed number k is a Zariski-closed condition (see Tao and Sawin’s blog post [33]) and preserved under tensor products of linear maps. Corollary 9 implies the following.

Corollary 10. *Let \mathbf{Vec} be the category of finite-dimensional vector spaces over an infinite field K . For fixed ℓ and k , there exists a tuple $\underline{W} \in \mathbf{Vec}^\ell$ such that for all $\underline{V} \in \mathbf{Vec}^\ell$ a tensor $T \in V_1 \otimes \dots \otimes V_\ell$ has slice rank at most k if and only if for all ℓ -tuples of linear maps $\varphi_i : V_i \rightarrow W_i$ the tensor $(\varphi_1 \otimes \dots \otimes \varphi_\ell)T$ has slice rank at most k .*

Equivalently, in the space of ℓ -way $\infty \times \infty \times \dots \times \infty$ -tensors, the variety of slice rank at most k tensors is defined by finitely many GL_∞^ℓ -orbits (and even finitely many GL_∞ -orbits, acting diagonally) of polynomial equations. A more in-depth study of the algebraic geometry of slice rank is a forthcoming work with Oosterhof.

1.9. Related work. Theorem 1 fits in a trend at the interface between representation theory, algebraic geometry, commutative algebra, and applications, which studies algebraic structures over some base category and aims to establish stabilisation results. Recent examples, in addition to those referenced above, include the theory of modules over the category \mathbf{FI} of finite sets with injective maps [7]; Gröbner techniques [29] for modules over more general combinatorial categories that, among other things, led to a resolution of the Artinian conjecture [24] and to a resolution of a conjecture by Rauh-Sullivant [26] on iterated toric fibre products [14]; and finiteness results for secant varieties of Segre and Segre-Veronese embeddings; see [21, 25, 27, 28] and the notion of inheritance in [20]. The current paper, while logically independent of these results, was very much influenced by the categorical viewpoint developed in these papers.

2. PROOF OF THE MAIN THEOREM

2.1. Overview of the proof. The proof of Theorem 1 is a double induction. The *outer induction* is on the polynomial functor P via a (“lexicographic”) partial order \prec on the class of polynomial functors introduced in §2.2. Using classical work by Friedlander-Suslin, we prove that this is a well-founded order. Any degree-zero polynomial functor, i.e., a constant vector space U independent of the input V , is smaller than all polynomial functors of positive degree, and Hilbert’s basis theorem yields the base case of the induction.

So when we want to prove the theorem for P , we may assume that it holds for all polynomial functors smaller than P . We then show that every closed **Vec**-subset X of P is Noetherian, by the *inner induction* on the smallest degree of a nonzero equation $f \in K[P(U)]$ vanishing on $X(U)$ for some $U \in \mathbf{Vec}$. Roughly, this works as follows (see the next paragraph for subtleties): fix an irreducible component R of the highest-degree part P_d of P , and find a direction $r_0 \in R(U)$ such that the directional derivative $h := \partial f / \partial r_0$ is not identically zero. Let Y be the largest **Vec**-closed subset of P on which h vanishes identically. Since h has a lower degree than f , Y is Noetherian by the inner induction hypothesis. On the other hand, set $P'(V) := P(U \oplus V)$ and $Q'(V) := P'(V)/R(V)$, so that $Q' \prec P$; this is discussed in §2.4. In §2.9 we show that the complement Z of Y has a closed embedding into a basic open subset of Q' , so Z Noetherian by the outer induction hypothesis. Hence X , the union of two Noetherian spaces, is Noetherian.

There are four subtleties: First, f may not depend on the coordinates on $R(U)$, so that $\partial f / \partial r_0 = 0$ for all $r_0 \in R(U)$. We therefore need to look for f in the ideal of X that is nonzero even after modding out the ideal of the projection of X in $Q := P/R$; see §2.5. Second, in positive characteristic, directional derivatives (linearisations) do not necessarily behave well; we replace these by additive polynomials in §2.6. Third, the closed embedding is in fact a Zariski homeomorphism to a closed subset; see §2.7 for the relevant lemma. Fourth, it is not quite Z that embeds into a basic open subset of Q' — Z is not functorial in V —but rather the locus $Z'(V)$ in $X'(V) := X(U \oplus V)$ where h (which involves only coordinates on the constant vector space U) is nonzero. The closed embedding is then just the restriction of the projection $P'(V) \rightarrow Q'(V)$ along $R(V)$. Smearing around $Z'(V)$ by $\mathrm{GL}(U \oplus V)$, we obtain $Z(U \oplus V)$, and in §2.9 we show that this is good enough.

Example 11. As a running example to illustrate the proof, we assume that $\mathrm{char} K \neq 2$, $P(V) = V \otimes V = Q(V) \oplus R(V)$ where $Q(V)$ is the space of symmetric two-tensors (matrices) and $R(V)$ is the space of skew-symmetric tensors. Let $(x_{ij})_{ij}$ be the standard coordinates on $P(K^n)$, let $(y_{ij})_{i \leq j}$ be the standard coordinates on $Q(K^n)$ extended to $P(K^n)$ by declaring them zero on $R(K^n)$, and let $(z_{ij})_{i < j}$ be the standard coordinates on $R(K^n)$, similarly extended to $P(K^n)$.

We then have

$$x_{ij} = \begin{cases} y_{ij} + z_{ij} & \text{if } i < j; \\ y_{ji} - z_{ji} & \text{if } j < i; \text{ and} \\ y_{ii} & \text{if } i = j. \end{cases}$$

Take $X(V) = \{v \otimes w \mid v, w \in V\}$, the variety of rank 1 tensors. Then $X(K^1)$ is the entire space $P(K^1)$, but $X(K^2) \subsetneq P(K^2)$, so we may take $U = K^2$ and for f the determinant

$$f = x_{11}x_{22} - x_{12}x_{21} = y_{11}y_{22} - (y_{12} + z_{12})(y_{12} - z_{12}) = y_{11}y_{22} - y_{12}^2 + z_{12}^2.$$

We take $r_0 := e_1 \otimes e_2 - e_2 \otimes e_1$ and find

$$h = \partial f / \partial r_0 = \partial f / \partial z_{12} = 2z_{12}.$$

In this case, for $n \geq 2$, $Y(K^n)$ is the subvariety of $X(K^n)$ on which the GL_n -orbit of z_{12} vanishes identically, i.e., $Y(V)$ is the set of rank 1 tensors in $Q(V)$. This is a coincidence; in the general setting of the proof, $Y(V)$ does not embed into $Q(V)$, but it always has a lower-degree polynomial vanishing on it. We discuss the complement $Z(V)$ in Example 26. ♣

2.2. A well-founded order on polynomial functors. We will prove Theorem 1 by induction on the polynomial functor, along a partial order that we introduce now. Define a relation \prec on polynomial functors of finite degree by $Q \prec P$ if $Q \not\cong P$ and moreover if e is the highest degree with $Q_e \not\cong P_e$, then Q_e is a homomorphic image of P_e ; this is a partial order on (isomorphism classes of) polynomial functors.

Lemma 12. *The relation \prec is a well-founded order on polynomial functors of finite degree.*

Proof. It suffices to prove that this order is well-founded when restricted to polynomial functors of degree at most a fixed number d . By [18, Lemma 3.4], if V is any vector space of dimension at least d , then the map $P \mapsto P(V)$ is an equivalence of abelian categories from polynomial functors of degree at most d and finite-dimensional polynomial $\text{GL}(V)$ -representations of degree at most d . Hence $Q \prec P$ implies that the sequence $(\dim Q_e(V))_{e=1}^d$ is strictly smaller than the sequence $(\dim P_e(V))_{e=1}^d$ in the lexicographic order where position e is more significant than position $e - 1$. Since this lexicographic order is a well-order, \prec is well-founded. □

2.3. Vec-varieties and their ideals. Write $K[P]$ for the contravariant functor from **Vec** to K -algebras that assigns to V the coordinate ring $K[P(V)]$. A closed **Vec**-subset X of P will be called a **Vec**-variety in P , and denoted $X \subseteq P$. Its ideal is a contravariant functor that sends V to the ideal of $X(V)$ inside $K[P(V)]$.

Using scalar multiples of the identity $V \rightarrow V$ and the fact that K is infinite, one finds that the ideal of $X(V)$ is homogeneous with respect to the $\mathbb{Z}_{\geq 0}$ -grading that assigns to the coordinates on $K[P_e(V)]$ the degree e . The degree function deg on $K[P(V)]$ and its quotients by homogeneous ideals is defined relative to this grading.

Example 13. In our running Example 11, f, h have degrees 4, 2, respectively. ♣

2.4. The shift functor. Fixing a $U \in \mathbf{Vec}$, we let $\text{Sh}_U : \mathbf{Vec} \rightarrow \mathbf{Vec}$ be the *shift functor* that sends $V \mapsto U \oplus V$ and the homomorphism $\varphi : V \rightarrow W$ to the homomorphism $\text{Sh}_U(\varphi) := \text{id}_U \oplus \varphi : \text{Sh}_U(V) \rightarrow \text{Sh}_U(W)$.

Lemma 14. *For any polynomial functor P of degree d , $P \circ \text{Sh}_U$ is a polynomial functor of degree d whose degree- d homogeneous part is canonically isomorphic to that of P .*

Proof. Set $P' := P \circ \text{Sh}_U$. For $V, W \in \mathbf{Vec}$ the map

$$\text{Hom}(V, W) \rightarrow \text{Hom}(P'(V), P'(W))$$

given by $\varphi \mapsto P'(\varphi)$ is the composition of the affine-linear map $\varphi \mapsto 1_U \oplus \varphi$ and the polynomial map $\psi \mapsto P(\psi)$ of degree at most d , so P' is a polynomial functor of degree at most d .

For $V \in \mathbf{Vec}$ let ι_V be the embedding $V \rightarrow U \oplus V$, $v \mapsto (0, v)$ and let π_V be the projection $U \oplus V \rightarrow V$, $(u, v) \mapsto v$. These give rise to morphisms of polynomial functors $\alpha : P \rightarrow P'$ and $\beta : P' \rightarrow P$ given by $\alpha(V) := P(\iota_V) : P(V) \rightarrow P(U \oplus V)$ and $\beta(V) := P(\pi_V)$. Straightforward computations show that α and β map each homogeneous part P_e into P'_e and vice versa, and that $\beta \circ \alpha$ is the identity. Conversely, for q in the highest-degree part $P'_d(V)$ we have $P(1_U \oplus t1_V)q = t^d q$ for all $t \in K$. The coefficient of t^d in the left-hand side equals $P(0 \oplus 1_V)q$, so we have $P(0 \oplus 1_V)q = q$ and therefore

$$\alpha(V)\beta(V)q = P(\iota_V)P(\pi_V)q = P(0 \oplus 1_V)q = q,$$

which proves that $\beta : P'_d(V) \rightarrow P_d(V)$ is indeed a linear isomorphism. □

Example 15. If $P(V) = S^d V$, then $(P \circ \text{Sh}_U)(V) = S^d(U \oplus V) \cong \bigoplus_{e=0}^d S^{d-e}U \otimes S^e V$, so $P \circ \text{Sh}_U$ equals P plus a polynomial functor of degree $d - 1$. ♣

Example 16. In our running Example 11, $P(V) = V \otimes V = Q(V) \oplus R(V)$, $U = K^2$ and

$$\begin{aligned} P \circ \text{Sh}_U(V) &= (U \oplus V) \otimes (U \oplus V) = U \otimes U + U \otimes V + V \otimes U + V \otimes V \\ &\cong K^4 + V^4 + Q(V) + R(V); \end{aligned}$$

note that the degree-two part of $(P \circ \text{Sh}_U)/R$ is Q , so that $(P \circ \text{Sh}_U)/R \prec P$. ♣

2.5. Splitting off a term of highest degree. Assume that P is a polynomial functor of degree $d > 0$. Let R be any irreducible sub-polynomial functor of the highest-degree part P_d of P , define $Q := P/R$, and let $\pi : P \rightarrow Q$ be the natural projection. Then $K[Q]$ embeds into $K[P]$ via the pull-back of π . If X is a \mathbf{Vec} -variety in P , then let X_Q be the \mathbf{Vec} -variety in Q defined by setting $X_Q(V)$ equal to the Zariski-closure in $Q(V)$ of $\pi(V)(X(V))$.

We will think of X as a \mathbf{Vec} -variety over X_Q . Accordingly, we write \mathcal{I}_X for the contravariant functor that assigns to V the ideal of $X(V)$ in $K[\pi(V)^{-1}(X_Q(V))]$, the quotient of $K[P(V)]$ by the ideal in $K[P(V)]$ generated by the ideal of $X_Q(V)$ in $K[Q(V)]$.

In particular, we have $\mathcal{I}_X = 0$ if and only if for all V we have $X(V) = \pi^{-1}(X_Q(V))$. We write $\delta_X \in \{1, 2, \dots, \infty\}$ for the minimal degree of a nonzero homogeneous polynomial $f \in \mathcal{I}_X(V)$ as V runs over \mathbf{Vec} . Note that any polynomial in $K[P(V)]$ of degree zero is contained in $K[P_0(V)] \subseteq K[Q(V)]$; here we use that $d > 0$. So if a degree-zero polynomial vanishes on $X(V)$, then it is an equation for $X_Q(V)$ and has already been modded out in the definition of \mathcal{I}_X . This explains why $\delta_X \geq 1$. Furthermore, note that $\delta_X = \infty$ if and only if $\mathcal{I}_X = 0$.

Example 17. In Examples 11,13, $\delta_X = \deg f = 4$. ♣

2.6. Additive polynomials as directional derivatives. For a finite-dimensional vector space W over the infinite field K , we write $\text{Add}(W)$ for the subset of $K[W]$ consisting of polynomials f such that $f(v + w) = f(v) + f(w)$ for all $v, w \in W$. Then $\text{Add}(W)$ is a K -subspace of $K[W]$, and equal to W^* when $\text{char } K = 0$. In general, if we let p be the characteristic exponent of K —so $p = 1$ if $\text{char } K = 0$ and $p = \text{char } K$ otherwise—and if we choose a basis x_1, \dots, x_n of W^* , then $\text{Add}(W)$ has as a basis the polynomials $x_i^{p^e}$ where i runs through $\{1, \dots, n\}$ and e through $\mathbb{Z}_{\geq 0}$ if $p > 1$ and through $\{0\}$ if $p = 1$. The span of these for fixed e is denoted $\text{Add}(W)_e$.

Lemma 18. *Let $W' \supseteq W$ be finite-dimensional vector spaces over the infinite field K , let f be a polynomial on W' , and consider the expression $f(w' + tw)$, a polynomial function of the triple $(w', t, w) \in W' \times K \times W$. Then one of the following hold:*

- (1) *$f(w' + tw)$ is independent of t ; this happens if and only if f factors through the projection $\pi_{W'/W} : W' \rightarrow W'/W$; or*
- (2) *the nonzero part of $f(w' + tw)$ of lowest degree in t is of the form $t^{p^e} h(w', w)$ for a unique e (taken 0 if $\text{char } K = 0$). Then for each fixed $w' \in W'$ the map $w \mapsto h(w', w)$ is in $\text{Add}(W)_e$.*

Proof. For $w \in W$, let $D_w^{(r)} : K[W'] \rightarrow K[W']$ denote the r th Hasse directional derivative in the direction w . This linear map is defined in characteristic 0 by $D_w^{(r)} g = \frac{1}{r!} (\frac{\partial}{\partial w})^r g$ and in arbitrary characteristic by realising that the latter expression actually has integer coefficients relative to any monomial basis of $K[W']$. Explicitly: let x_1, \dots, x_{n-1} be a basis of (W'/Kw) and let $x_n \in (W')^*$ with $x_n(w) = 1$. Then

$$D_w^{(r)} x_1^{a_1} \cdots x_n^{a_n} := \binom{a_n}{r} x_1^{a_1} \cdots x_{n-1}^{a_{n-1}} x_n^{a_n-r};$$

in particular, if $r = a_n$, then this is nonzero as a polynomial over K , even when $a_n \cdots (a_n - r + 1)$ is zero. A straightforward check shows that this is independent of the choice of basis, and that $D_{cw}^{(r)} = c^r D_w^{(r)}$.

Taylor’s formula in arbitrary characteristic reads

$$f(w' + tw) = \sum_{r \geq 0} (D_w^{(r)} f)(w') t^r.$$

If $D_w^{(r)} f = 0$ for all $r > 0$, then we see from the above that f does not involve the variable x_n , i.e., f factors through $W'/\langle w \rangle$. Similarly, if $D_w^{(r)} f = 0$ for all $w \in W$ and all $r > 0$, then f factors through W'/W .

Suppose that there exist $r > 0$ and $w \in W$ such that $D_w^{(r)} f \neq 0$; take such a pair (r, w) with r minimal. Then, in the coordinates above, f contains a monomial $x_1^{a_1} \cdots x_n^{a_n}$ for which $\binom{a_n}{r}$ is nonzero in K , but $\binom{a_n}{r'} = 0$ in K for all r' with $0 < r' < r$. By Lucas’s theorem on binomial coefficients, r is a power of p , a_n is divisible by r , and $\binom{a_n}{r} = a_n/r$ in K . Since in fact $\binom{a'_n}{r} = 0$ holds for all $r' < r$ and *all* monomials $x_1^{a'_1} \cdots x_n^{a'_n}$ in f , a'_n is a multiple of r , and hence $f = g(x_1, \dots, x_{n-1}, x_n^r)$ for a unique polynomial $g(x_1, \dots, x_{n-1}, y_n)$. Moreover, $D_w^{(r)} f = (\partial g / \partial y_n)(x_1, \dots, x_{n-1}, x_n^r)$.

More generally, let x_1, \dots, x_k be a basis of $(W'/W)^*$ and extend to a basis x_1, \dots, x_n of $(W')^*$.

We then find, by minimality of r , that $f = g(x_1, \dots, x_k, x_{k+1}^r, \dots, x_n^r)$ for a unique polynomial $g(x_1, \dots, x_n, y_{k+1}, \dots, y_n)$, and for all $u \in W$ we have

$$D_u^{(r)} f = \sum_{i=k+1}^n (x_i(u))^r (\partial g / \partial y_i)(x_1, \dots, x_k, x_{k+1}^r, \dots, x_n^r).$$

Since $r = p^e$ for some e , the right-hand side is additive in u , which concludes the proof of the lemma. □

For $W' \supseteq W$ and $f \in K[W']$ and $h \in K[W'] \otimes K[W]$ as in the second case of the lemma and for $w \in W$ we write $\partial_w f \in K[W']$ for the polynomial $w' \mapsto h(w', w)$, and call this the *directional derivative* of f in the direction w . This polynomial

has degree less than $\deg f$, and agrees with the usual directional derivative for $\text{char } K = 0$. Note that $\partial_w f$ depends on the choice of W inside W' , not just on w : if $f(w' + tw)$ depends on t but vanishes to a higher degree at $t = 0$ for a specific $w \in W$ than it does for general $w \in W$, then we have $\partial_w f = 0$.

Also in the first case of the lemma we write $\partial_w f := 0$ for all $w \in W$. We extend the notation to rational functions with nonzero denominator $h \in K[W'/W]$ by $\partial_w(f/h) := (\partial_w f)/h$. The following lemma is immediate from Lemma 18.

Lemma 19. *For $W' \supseteq W$, $f \in K[W']$, $e \in \mathbb{Z}_{\geq 0}$ as in Lemma 18, $h \in K[W'/W] \setminus \{0\}$, we have $\partial_{v+w}(f/h) = \partial_v(f/h) + \partial_w(f/h)$ and $\partial_{cw}(f/h) = c^{p^e} \partial_w(f/h)$ for $c \in K$ and $w \in W$. □*

Example 20. Assume that $p > 2$. Let $W' = K^3$ with standard coordinates x, y, z , let W be the span of the first two standard basis vectors, and let $f = y^{p^2} z^2 + x^{2p} y^{p^2} z$. Then

$$\begin{aligned} f((x, y, z) + t(a, b, 0)) &= (y + tb)^{p^2} z^2 + (x + ta)^{2p} (y + tb)^{p^2} z \\ &= f(x, y, z) + t^p (2a^p x^p y^{p^2} z) + \dots \end{aligned}$$

where the remaining terms are divisible by t^{2p} . Hence $\partial_{(a,b,0)} f = 2a^p x^p y^{p^2} z$. ♣

2.7. A closed embedding. Retaining the notation in the previous section, let B be a basic open subset in W'/W defined by the nonvanishing of some polynomial $h \in K[W'/W] \setminus \{0\}$ (we allow $h = 1$, in which case $B = W'/W$). Let Z be a Zariski-closed subset of $A := \pi_{W'/W}^{-1}(B) \subseteq W'$ and let J be the ideal of Z inside $K[A] = K[W'][1/h]$. Fix a number $e \in \mathbb{Z}_{\geq 0}$, equal to 0 if $\text{char } K = 0$, and let J_e be the set of elements $k \in J$ such that $k(a + tw) = k(a) + t^{p^e} (\partial_w k)(a)$ for all $a \in A, t \in K, w \in W$ (so k is *affine-additive* in W with additive part of degree p^e). Note that via the pull-back $\pi_{W'/W}^* : K[B] \rightarrow K[A]$, J_e is a $K[B]$ -submodule of $K[A]$.

Lemma 21. *Assume that K is algebraically closed and suppose that for each $a \in A$ the map $J_e \rightarrow \text{Add}(W)_e, k \mapsto (w \mapsto (\partial_w k)(a))$ is surjective. Then $\pi_{W'/W}$ restricts to a Zariski-homeomorphism from Z to a closed subset of B .*

Proof. Fix any tuple $x_1, \dots, x_n \in (W')^*$ whose restrictions to W form a basis of W^* . Then the natural map $K[B][x_1, \dots, x_n] \rightarrow K[A]$ is an isomorphism by which we identify the two algebras. Under this identification, each element of J_e can be written uniquely as $k_0 + \sum_{j=1}^n k_j x_j^{p^e}$ for suitable elements $k_0, k_1, \dots, k_n \in K[B]$. Let $k^{(1)}, \dots, k^{(m)}$ be $K[B]$ -module generators of J_e , and let $k_j^{(i)} \in K[B]$ be the coefficient of $x_j^{p^e}$ in $k^{(i)}$. Let $M \in K[B]^{m \times n}$ be the matrix whose (i, j) entry equals $k_j^{(i)}$. The condition in the lemma says that $M(b)$ has rank n for all $b \in B$.

Since K is algebraically closed, the Nullstellensatz implies that 1 lies in the ideal generated by the nonzero $n \times n$ -subdeterminants $\Delta_1, \dots, \Delta_N \in K[B]$ of M : $1 = \sum_{l=1}^N f_l \Delta_l$ for suitable $f_l \in K[B]$.

Fix a $j \in \{1, \dots, n\}$. For each $l \in \{1, \dots, N\}$ we can write $e_j^T = v_{jl}^T M$, where e_j is the j th standard basis vector in $K[B]^n$ and v_{jl} is a vector in $K(B)^m$ (supported only on the positions corresponding to Δ_l) satisfying $\Delta_l v_j \in K[B]^m$; this is just

Cramer’s rule. Now

$$e_j = 1e_j = \sum_{l=1}^N f_l \Delta_l e_j = \sum_{l=1}^N f_l (\Delta_l v_j^T) M,$$

and we conclude that J_e contains an element of the form $k_{0,j} + x_j^{p_e}$ with $k_{0,j} \in K[B]$.

Consider the morphism $\varphi : A \rightarrow A$ dual to the homomorphism $K[A] \rightarrow K[A]$ that restricts to the identity on $K[B]$ and sends each x_j to $x_j^{p_e}$. Since K is algebraically closed, φ is a homeomorphism in the Zariski-topology, and since φ commutes with the projection $A \rightarrow B$, it suffices to show that this projection restricts to a closed embedding from $Z' := \varphi(Z)$ into B . Let J' be the ideal of Z' . By the previous paragraph, J' contains an element of the form $k_{0,j} + x_j$ with $k_{0,j} \in K[B]$ for each j . Hence the map $K[B] \rightarrow K[A]/J'$ is surjective, so $Z' \rightarrow B$ is a closed embedding, as desired. \square

Remark 22. In characteristic 0, the Zariski-homeomorphism from the lemma is in fact a closed embedding. In positive characteristic, it need not be.

2.8. Extending the field. Let $P : \mathbf{Vec} \rightarrow \mathbf{Vec}$ be a finite-degree polynomial functor over the infinite field K , let L be an extension field of K , and denote by \mathbf{Vec}_L the category of finite-dimensional vector spaces over L . We construct a polynomial functor $P_L : \mathbf{Vec}_L \rightarrow \mathbf{Vec}_L$ as follows. For every $U \in \mathbf{Vec}_L$ we fix a $V_U \in \mathbf{Vec}$ and an isomorphism $\psi_U : U \rightarrow L \otimes_K V_U$ of L -vector spaces.

At the level of objects, P_L is defined by $P_L(U) := L \otimes_K P(V_U)$. To define P_L on morphisms we proceed as follows. For $U, U' \in \mathbf{Vec}_L$ the polynomial map $P : \text{Hom}_K(V_U, V_{U'}) \rightarrow \text{Hom}_K(P(V_U), P(V_{U'}))$ extends uniquely to a polynomial map

$$P'_L : L \otimes_K \text{Hom}_K(V_U, V_{U'}) \rightarrow L \otimes_K \text{Hom}_K(P(V_U), P(V_{U'}))$$

of the same degree; here we use that K is infinite. The domain and codomain of P'_L are canonically $\text{Hom}_L(L \otimes_K V_U, L \otimes_K V_{U'})$ and $\text{Hom}_L(P_L(U), P_L(U'))$, respectively. Hence for $\varphi \in \text{Hom}_L(U, U')$ we may set

$$P_L(\varphi) = P'_L(\psi_{U'} \circ \varphi \circ \psi_U^{-1}).$$

A simple calculation shows that P_L is indeed a polynomial functor $\mathbf{Vec}_L \rightarrow \mathbf{Vec}_L$.

Furthermore, if X is a \mathbf{Vec} -closed subset of P , then we obtain a \mathbf{Vec} -closed subset X_L of P_L by letting $X_L(U)$ be the Zariski closure of $\{1 \otimes q \mid q \in X(V_U)\}$ in $P_L(U) = L \otimes_K P(V_U)$. The following lemma is straightforward.

Lemma 23. *The map $X \mapsto X_L$ from \mathbf{Vec} -closed subsets of P to \mathbf{Vec}_L -closed subsets of P_L is inclusion-preserving and injective. Consequently, Noetherianity of P_L implies that of P .*

2.9. Proof of Theorem 1. If $d = 0$, then $P(V)$ is a finite-dimensional space independent of V , and for every linear map $\varphi : V \rightarrow W$ the map $P(\varphi)$ is the identity, so the theorem is the topological corollary to Hilbert’s basis theorem. We therefore may and will assume that $d > 0$. Furthermore, by Lemma 23 we may assume that K is algebraically closed, so that we can use Lemma 21.

We proceed by induction, assuming that the theorem holds for all polynomial functors $P' \prec P$ in the well-founded order from §2.2; this is our *outer* induction hypothesis. From §2.5 we recall the definition of δ_X . We now order \mathbf{Vec} -varieties inside the fixed P by $X > Y$ if either $X_Q \supsetneq Y_Q$ or else $X_Q = Y_Q$ and $\delta_X > \delta_Y$.

Since δ_X takes values in a well-ordered set, for any strictly decreasing sequence $X_1 > X_2 > \dots$, $(X_i)_Q$ must become strictly smaller infinitely often; but this is impossible since Q is Noetherian by the outer induction hypothesis. Hence $>$ is a well-founded order on **Vec**-subvarieties of P .

We set out to prove, by induction along this well-founded order, that each **Vec**-variety $X \subseteq P$ is Noetherian as a **Vec**-topological space. Our *inner* induction hypothesis states that this holds for each **Vec**-variety $Y < X$ inside P .

First assume that $\delta_X = \infty$, which means that X is the pre-image of its projection X_Q , and let $Y \subsetneq X$ be any proper closed **Vec**-subset. Then either $Y_Q \subsetneq X_Q$ or else $Y_Q = X_Q$ and $\delta_Y < \delta_X$. Hence $Y < X$, so that Y is Noetherian by the inner induction hypothesis. Since any inclusion-wise strictly decreasing chain of closed **Vec**-subsets of X must contain such a Y as its first or second element, X is Noetherian, as well.

So we may assume that $\delta_X \in \mathbb{Z}_{\geq 1}$. Take $U \in \mathbf{Vec}$ of minimal dimension for which $\mathcal{I}_X(U)$ contains a nonzero homogeneous element of degree δ_X , and let $f \in K[P(U)]$ be a homogeneous polynomial representing this element. Regarding f as a polynomial with coefficients from $K[Q(U)]$ in coordinates that restrict to a basis of $R(U)^*$, we may remove from f all terms with coefficients that vanish identically on $X_Q(U)$, and then at least one nonconstant term survives.

By Lemma 18 applied to f with $W' = P(U)$ and $W = R(U)$, this implies that there exists an $r_0 \in R(U)$ such that the directional derivative $h := \partial_{r_0} f \in K[P(U)]$ in the sense of §2.6 also has at least some coefficient in $K[Q(U)]$ that does not vanish on $X_Q(U)$. Let $e_0 \in \mathbb{Z}_{\geq 0}$ be the exponent e in the Lemma 18; so the map $r \mapsto (\partial_r f)(q)$ lies in $\text{Add}(R(U))_{e_0}$ for each $q \in P(U)$. Since coordinate functions on $R_d(U)$ were assigned degree d (§2.3), we have $\deg(h) = \deg(f) - dp^{e_0}$ and in particular $\deg(h) < \deg(f)$. By minimality of the degree of f , the polynomial h does not vanish identically on X .

Let Y be the largest closed **Vec**-subset of X such that h does vanish identically on $Y(U)$, i.e., $Y(V) = \{p \in X(V) \mid h(P(\varphi)p) = 0 \text{ for all } \varphi \in \text{Hom}(V, U)\}$. Then either $Y_Q \subsetneq X_Q$ or else $Y_Q = X_Q$ and $\delta_Y \leq \deg(h) < \delta_X$. Hence $Y < X$, so Y is Noetherian by the inner induction hypothesis.

Define Z by $Z(V) := X(V) \setminus Y(V)$. This is typically *not* a **Vec**-subset of X , since for $\varphi \in \text{Hom}_{\mathbf{Vec}}(V, V')$ the map $P(\varphi)$ might map points of $Z(V)$ into $Y(V')$, i.e., outside $Z(V')$. Indeed, since we chose U of minimal dimension, when we pull back f to a $P(V)$ for $\dim V < \dim U$, we obtain a polynomial that is zero modulo the ideal of $X_Q(V)$. This implies that the pull-back of h is identically zero on $X(V)$, so that $Z(V) = \emptyset$. To remedy this, we now construct a **Vec**-variety Z' closely related to Z , by shifting our polynomial functor over U as in §2.4.

Set $P' := P \circ \text{Sh}_U$ and $X' := X \circ \text{Sh}_U$ and consider the open **Vec**-subset Z' of X' defined by $Z'(V) := \{q \in X(U \oplus V) \mid h(q) \neq 0\}$. Here we regard h as a polynomial on $P(U \oplus V)$ via the map $P(U \oplus V) \rightarrow P(U)$ coming from the projection $\pi_U : U \oplus V \rightarrow U$ along V . As the maps $\pi_U \circ g$ with $g \in \text{GL}(U \oplus V)$ are Zariski-dense in $\text{Hom}(U \oplus V, U)$, Z' and $Z \circ \text{Sh}_U$ are related by

$$\begin{aligned}
 (*) \quad Z(U \oplus V) &= \{q \in X(U \oplus V) \mid \exists \psi \in \text{Hom}(U \oplus V, U) : h(P(\psi)q) \neq 0\} \\
 &= \{q \in X'(V) \mid h(g(q)) \neq 0 \text{ for some } g \in \text{GL}(U \oplus V)\} = \bigcup_{g \in \text{GL}(U \oplus V)} gZ'(V).
 \end{aligned}$$

Recall that R is an irreducible subfunctor of P_d . Write $R' := R \circ \text{Sh}_U = R'_0 \oplus \dots \oplus R'_d \subseteq P'$ where R'_e is homogeneous of degree e and $R'_d = R$ by Lemma 14. Define $Q' := P'/R'_d$ and note that $Q' \prec P$ since Q' has degree at most d and the degree- d part of Q' is equal to P_d/R by Lemma 14. In particular, Q' is a Noetherian **Vec**-topological space by the outer induction hypothesis.

Remark 24. Note that for $e < d$, Q'_e typically will have dimension larger than P_e ! Also, Q' is not equal to $Q \circ \text{Sh}_U$: there is a surjection $Q' \rightarrow Q \circ \text{Sh}_U$ with kernel R'/R'_d .

The map $P(\pi_U) : P(U \oplus V) \rightarrow P(U)$ has $R'_d(V)$ in its kernel, so we can regard h as a polynomial on $Q'(V)$, as well. Consider the basic open **Vec**-subset of Q' defined by $B(V) := \{q \in Q'(V) \mid h(q) \neq 0\}$. As Q' is Noetherian, so is B with its induced topology.

Lemma 25. *For every K -vector space V the projection $Z'(V) \rightarrow B(V)$ is a Zariski-homeomorphism with a closed subset of $B(V)$.*

Before proving this lemma, we use it to complete the proof of Theorem 1. First, since B is Noetherian, Lemma 25 implies that so is Z' .

Then suppose that $X = X_0 \supseteq X_1 \supseteq \dots$ is a sequence of closed **Vec**-subsets of X . By Noetherianity of Y there exists an n_0 such that for all $V \in \mathbf{Vec}$ the sequence $(X_n(V) \cap Y(V))_n$ is constant for $n \geq n_0$. By Noetherianity of Z' there exists an n_1 such that for all $V \in \mathbf{Vec}$ the sequence $(X_n(U \oplus V) \cap Z'(V))_n$ is constant for $n \geq n_1$. Using (*) we find

$$\begin{aligned} X_n(U \oplus V) \cap Z(U \oplus V) &= \bigcup_{g \in \text{GL}(U \oplus V)} X_n(U \oplus V) \cap gZ'(V) \\ &= \bigcup_{g \in \text{GL}(U \oplus V)} g(X_n(U \oplus V) \cap Z'(V)), \end{aligned}$$

where in the last step we used that $X_n(U \oplus V)$ is $\text{GL}(U \oplus V)$ -stable. We find that, for each V of dimension at least $\dim U$, the sequence $(X_n(V) \cap Z(V))_n$ is constant for $n \geq n_1$. Since $Z(V) = \emptyset$ for V of dimension less than $\dim U$, we find that X_n is constant for $n \geq \max\{n_0, n_1\}$. This proves Noetherianity of X and concludes the proof of the inner induction step. □

Example 26. We pause and see what Lemma 25 says in the running Examples 11,13,16. Here $U = K^2$ and we take $V = K^n$. Then $Z'(V)$ is the variety of $(n + 2) \times (n + 2)$ -matrices

$$M = \begin{bmatrix} C & D \\ E & F \end{bmatrix}$$

of rank 1 such that the upper-left 2×2 -submatrix C has a nonzero coordinate z_{12} (recall that $h = 2z_{12}$).

The lemma says that forgetting the skew-symmetric part of F is a closed embedding of $Z'(V)$ into the open subset of $U \otimes U + V \otimes U + U \otimes V + Q(V)$ where h is nonzero. We prove this by showing that, on Z' , each entry of the skew-symmetric part of F can be expressed as a rational function in the entries of C, D, E and the entries of the symmetric part of F , with a denominator equal to h .

We assume $n \geq 2$ and consider a surjective linear map $\varphi : V \rightarrow U$. Thinking of φ as a $2 \times n$ -matrix, we have

$$P(1_U \oplus t\varphi)M = C + t(\varphi E + D\varphi^T) + t^2\varphi F\varphi^T.$$

Since Z' is a **Vec**-closed subset, for all $M \in Z'(V)$, the 2×2 -determinant f vanishes on the latter 2×2 -matrix for all choices of t and of φ . Hence expanding $f(P(1_U \oplus t\varphi)M)$ as a polynomial in t , the coefficient k of t^2 also vanishes for all M, φ . Take $i < j$ in $\{1, \dots, n\}$ and specialise φ to the linear map sending the i th standard basis vector to e_1 , the j th standard basis vector to e_2 , and all other standard basis vectors to zero. Then the matrix above reads

$$\begin{bmatrix} C_{11} + tE_{i1} + tD_{1i} + t^2F_{ii} & C_{12} + tE_{i2} + tD_{1j} + t^2F_{ij} \\ C_{21} + tE_{j1} + tD_{2i} + t^2F_{ji} & C_{22} + tE_{j2} + tD_{2j} + t^2F_{jj} \end{bmatrix}.$$

The coefficient of t^2 in the determinant of this matrix is

$$k(C, D, E, F) = F_{ii}C_{22} + F_{jj}C_{11} - F_{ij}C_{21} - F_{ji}C_{12} + \dots$$

where the remaining terms do not involve F . Using, as in Example 11, the variables y, z for the symmetric and skew-symmetric parts of C , and using the variables y', z' for the symmetric and skew-symmetric parts of F , this reads

$$\begin{aligned} & y'_{ii}y_{22} + y'_{jj}y_{11} - (y'_{ij} + z'_{ij})(y_{12} - z_{12}) - (y'_{ij} - z'_{ij})(y_{12} + z_{12}) + \dots \\ & = 2z'_{ij}z_{12} + \dots = z'_{ij}h(C) + \dots \end{aligned}$$

where the dots in the last two expressions contain only variables that we do not discard in the projection $P'(V) \rightarrow Q'(V)$. This shows that, on Z' , the coordinate z'_{ij} can be expressed in the entries of C, D, E and the coordinates y'_{ij} on the symmetric part of F , as desired. ♣

Proof of Lemma 25. First assume that $\dim V \geq \max\{\dim U, d\}$. We want to apply Lemma 21 with W' equal to $P'(V)$, W equal to $R'_d(V) = R(V)$, B equal to $B(V)$, and Z equal to $Z'(V)$. Let J be the ideal of $Z'(V)$ in the coordinate ring of the pre-image of $B(V)$ inside $P'(V)$, and let J_{e_0} be as in the text preceding Lemma 21.

Fix any surjective linear map $\varphi : V \rightarrow U$. For $t \in K$ and $e \in \{0, \dots, d\}$ consider the linear map $\Phi_e(t) := P_e(1_U \oplus t\varphi) : P_e(U \oplus V) \rightarrow P_e(U)$. This map depends on t as a polynomial of degree at most e , hence decomposes as $\Phi_e(t) = t^0\Phi_{e0} + \dots + t^e\Phi_{ee}$ for unique linear maps $\Phi_{ei} : P_e(U \oplus V) \rightarrow P_e(U)$. Note that $\Phi_{ee} = P_e(0 \oplus \varphi)$ and $\Phi_{e0} = P_e(1_U \oplus 0) = P_e(\pi_U)$. On the other hand, decompose $P'_e := P_e \circ \text{Sh}_u = P'_{e0} \oplus \dots \oplus P'_{ee}$ where P'_{ei} is a homogeneous polynomial functor of degree i . Then Φ_{ei} is zero on $P'_{ej}(V)$ except when $i = j$. To see this, take a $q \in P'_{ej}(V)$, compute

$$\begin{aligned} \Phi_e(t)q &= P_e(1_U \oplus t\varphi)q = P_e(1_U \oplus \varphi)P_e(1_U \oplus t1_V)q \\ &= P_e(1_U \oplus \varphi)P'_e(t1_V)q = t^j P_e(1_U \oplus \varphi)q, \end{aligned}$$

and observe that the right-hand side is homogeneous of degree j in t .

The $\Phi_e(t)$ together form the map $\Phi(t) := P(1_U \oplus t\varphi) : P(U \oplus V) \rightarrow P(U)$. Since X is a **Vec**-subset of P and f vanishes on $X(U)$, $f(\Phi(t)q) = 0$ for all $q \in X(U \oplus V)$ and $t \in K$. This implies that the coefficient $k(q)$ of $t^{d_p^{e_0}}$ in $f(\Phi(t)q)$ also vanishes identically on $X(U \oplus V)$.

To determine how $k(q)$ depends on $R'_d(V)$, consider $r \in R'_d(V)$ and $q = q_0 + \dots + q_d \in P(U \oplus V)$ with $q_e \in P_e(U \oplus V)$, and for variables t, s compute

$$\begin{aligned} & f(\Phi(t)(q_0 + \dots + q_{d-1} + (q_d + sr))) \\ &= f(\Phi_0(t)q_0 + \dots + \Phi_{d-1}(t)q_{d-1} + \Phi_d(t)(q_d + sr)) \\ &= f(\Phi_0(t)q_0 + \dots + \Phi_{d-1}(t)q_{d-1} + \Phi_d(t)q_d + t^d s \Phi_{dd}r) \\ &= f(\Phi_0(t)q_0 + \dots + \Phi_{d-1}(t)q_{d-1} + \Phi_d(t)q_d + t^d s P_d(0 \oplus \varphi)r) \\ &\equiv f(\Phi_0(t)q_0 + \dots + \Phi_{d-1}(t)q_{d-1} + \Phi_d(t)q_d) \\ &+ (t^d s)^{p^{e_0}} (\partial_{P_d(0 \oplus \varphi)r} f)(\Phi_0(t)q_0 + \dots + \Phi_{d-1}(t)q_{d-1} + \Phi_d(t)q_d) \pmod{((t^d s)^{p^{e_0}+1})} \end{aligned}$$

where in the second equality we have used that $r \in P'_{dd}(V)$ and in the last step we have used Lemma 18. We see that $t^{dp^{e_0}}$ is the lowest power of t to whose coefficient r contributes, and that this contribution is additive in r . More specifically, for each $s \in K$ we have $k(q + sr) = k(q) + s^{p^{e_0}} (\partial_r k)(q)$, and $(\partial_r k)(q)$ equals the value of the directional derivative $\partial_{R(\varphi)r} f$ at the point $\sum_{e=0}^d \Phi_e q_e = P(\pi_U)(q)$. In particular, $k \in J_{e_0}$.

From now on, assume that the image of q in $Q'(V)$ lies in $B(V)$. Since φ is surjective, so is $R(\varphi) : R(V) \rightarrow R(U)$. In particular, there exists an $r \in R(V)$ such that $R(\varphi)r = r_0$. For such an r we have $\ell(r) := (\partial_r k)(q) = h(q) \neq 0$, so $\ell \in \text{Add}(R(V))_{e_0}$ is not zero.

Keeping q fixed but replacing φ by $\varphi \circ g$ for $g \in \text{GL}(V)$, ℓ transforms into the additive function $r \mapsto \ell(R(g)r)$. Hence by varying g we find that the image of J_e in $\text{Add}(R(V))_e$ under the map $\tilde{k} \mapsto (r \mapsto (\partial_r \tilde{k})(q))$ from Lemma 21 contains a nonzero $\text{GL}(V)$ -submodule L of $\text{Add}(R(V))_e$. Since R is irreducible and $\dim V \geq d$, $R(V)$ is an irreducible $\text{GL}(V)$ -module [18, Lemma 3.4], and this implies the irreducibility of $(R(V))^*$ and of $\text{Add}(R(V))_{e_0}$ —indeed, raising to the power p^{e_0} gives a bijection from $\text{GL}(V)$ -submodules to $\text{GL}(V)$ -submodules of the latter.

We conclude that $L = \text{Add}(R(V))_{e_0}$, and since q was arbitrary in the pre-image of $B(V)$, the conditions of Lemma 21 are fulfilled. Hence the projection $Z'(V) \rightarrow B(V)$ is a Zariski-homeomorphism with a closed subset of $B(V)$, as desired.

Finally, if $\dim V < \max\{\dim U, d\}$, then take any embedding $\iota : V \rightarrow V'$ where V' does have sufficiently high dimension. Then we have a commuting diagram

$$\begin{array}{ccc} Z'(V) & \xrightarrow{P'(\iota)} & Z'(V') \\ \downarrow & & \downarrow \\ B(V) & \xrightarrow{P'(\iota)} & B(V') \end{array}$$

where all arrows except, a priori, the left-most one are homeomorphisms with closed subsets of the target space. But then so is the left-most one. □

2.10. Comments on the proof. The idea to do induction on P is not completely new: it is also used, in special cases, in [4, 9, 11, 15]. In these papers, more information than just Noetherianity is extracted from the proof, e.g., that the *tuple rank* of a tuple of matrices is bounded in a proper closed subvariety of a polynomial functor capturing matrices [11, 15], or that the *q-rank* of a cubic is bounded [9], or that the *strength* of a homogeneous form is bounded [4]. Our proof above does not directly yield such qualitative information. However, in [6] we repair this defect

for symmetric, alternating, and ordinary tensors and characteristic 0 or sufficiently large.

If K has characteristic 0, then symmetric powers S^dV are irreducible $\mathrm{GL}(V)$ -modules, and one can prove Noetherianity for direct sums of these without the need for more general polynomial functors—though also without the proof becoming any easier. But in general characteristic, symmetric powers need not be irreducible, and polynomial functors need not be completely reducible into irreducible summands, so reducing modulo an irreducible subfunctor is the only natural thing to do.

The idea further to do induction on δ_X and to use directional derivatives is new, but inspired by techniques used earlier in [10–12], where a determinant is regarded as an affine-linear polynomial in one matrix entry, whose coefficient is a determinant of lower order, and induction is done over that order.

2.11. An open problem. The most tantalising open problem in this area is the following.

Question 27. Let P be a finite-degree polynomial functor over an infinite field K . Does any sequence $I_1 \subseteq I_2 \subseteq \cdots$ of ideals in $K[P]$ eventually become constant?

For P of degree at most 1, the answer is yes, and it follows from the stronger statement that the ring $R[y_{ij} | i = 1, \dots, k, j \in \mathbb{N}]$, acted upon by $\mathrm{Sym}(\mathbb{N})$ via $\pi y_{ij} = y_{i\pi(j)}$, is $\mathrm{Sym}(\mathbb{N})$ -Noetherian for any Noetherian ground ring R [5, 8, 19]. For $P = S^2$ and $P = \bigwedge^2$ in characteristic 0, the answer is also yes, since we know all $\mathrm{GL}(V)$ -stable ideals from [2] and [1], respectively. In [22] a much stronger result than this is established for S^2 and \bigwedge^2 , namely, that finitely generated modules over $K[P]$ are also Noetherian. These questions were first raised, in the setting of twisted commutative algebras, in [30]. They remain widely open even for more general degree-two functors, and also for, say, \bigwedge^2 in positive characteristic.

ACKNOWLEDGMENTS

The author thanks Arthur Bik, Michał Lasoń, Florian Oosterhof, and Andrew Snowden for useful discussions and comments on an earlier version of this paper. The author also thanks the organisers of the April 2016 Banff workshop on *Free Resolutions, Representations, and Asymptotic Algebra* for bringing together participants with a wide variety of backgrounds—they have strongly influenced my understanding of these infinite algebraic structures. Finally, the author thanks the referees for several valuable suggestions, including the running Example 11.

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