151	$\ln(1-9\cdot10^{-4})$	485	486	485 507
152	$\ln(1+8\cdot10^{-4})$	567	566	566 326
151	$\ln(1-7\cdot10^{-4})$	859	860	859 672
151	$\ln(1-5\cdot10^{-4})$	785	786	785 574
152	$\ln(1+5\cdot10^{-4})$	340	339	339 355
151	$\ln(1-2\cdot10^{-4})$	811	810	810 371
151	$\ln(1-1\cdot10^{-4})$	734	735	734 571
152	$\ln(1+1\cdot10^{-4})$	402	401	401 071
151	$\ln(1-8\cdot10^{-5})$	613	614	613 570
152	$\ln(1+8\cdot10^{-5})$	796	797	796 578
151	$\ln(1-6\cdot10^{-5})$	899	898	898 045
151	$\ln(1-5\cdot10^{-5})$	846	845	845 433
152	$\ln(1+5\cdot10^{-5})$	980	981	980 850
151	$\ln(1-4\cdot10^{-5})$	446	445	445 439
151	$\ln(1-3\cdot10^{-5})$	774	773	773 336
151	$\ln(1-1\cdot10^{-5})$	682	683	682 540
151	$\ln(1-9\cdot10^{-6})$	599	597	597 457
151	$\ln(1-8\cdot10^{-6})$	358	357	357 389
151	$\ln(1-7\cdot10^{-6})$	606	605	604 608
151	$\ln(1-5\cdot10^{-6})$	448	447	447 389
152	$\ln(1+5\cdot10^{-6})$	457	458	457 806
151	$\ln(1-1\cdot10^{-6})$	858	857	857 267
152	$\ln(1+1\cdot10^{-6})$	523	524	523 684

H. S. UHLER, 8 Jan. 1943

On 2 February 1943 Mr. Uhler drew my attention to the fact that five more last-figure errors in Grimpen's 84-place table on p. XXV are suggested by comparison with H. M. Parkhurst's 102-place table (see RMT **86,** p. 20); in the cases of log 23, log 41, log 61 and log 97 there should be unit increases, but in the case of log 83 there should be a unit decrease. I found that Parkhurst and Grimpen were in complete agreement in the cases of log 31, log 43 and log 59, referred to above; hence it is Sharp's terminal digits which seem then to be slightly erroneous. On 3 May 1943 Mr. Uhler reported that he had completely checked both of Grimpen's tables, p. XXIV–XXV, and that the only errors were those in the terminal figure indicated above.—EDITOR.

The correct value of π to 707D was calculated by William Shanks and may be found on p. 1 of the *Anhang* by P. & S., and in G. Peano, *Formulario Mathematico*, 5 ed., v. 5, Turin, 1908, p. 256. Shanks gave the value of π to 607D in his *Contributions of Mathematics, comprising chiefly the Rectification of the Circle . . .*, London, 1853, p. 86–87. That the last 8 digits were incorrect, was shown when he extended his value of π to 707D, giving at the same time arctan (1/5) and arctan (1/239), each to 709D, R. So. London, *Proc.*, v. 21, 1873, p. 319. But there were still errors in the 460–462nd, and in the 513–515th decimal places. These were corrected in the value Shanks gave, *idem*, v. 22, 1874, p. 45. Two new errors here introduced in the 326th and 680th decimal places were easily checked from the arctangent values referred to above, and used by Shanks in computing the value of π. See RMT **95.** A reference may be given to J. P. Ballantine "The best (?) formula for computing π to a thousand places," Amer. Math. Mo., v. 46, 1939, p. 499–501.

R. C. A.

UNPUBLISHED MATHEMATICAL TABLES

We have referred to unpublished mathematical tables (a) of COMRIE in RMT **82;** and (b) of RICHE DE PRONY, of SANG, of PETERS, and of Princeton University, in the first article of this issue.

2[D].—J. R. AIREY, *Sines and Cosines in Radian Arguments.* Ms. in Mr. Comrie's possession.

After the death of Airey in 1937, his calculations and manuscript tables came into my possession. Most of these had, of course, been published, although in many cases, e.g., the Fresnel integral, more decimals (usually within a unit of the last decimal) thus became available.

There is one unpublished table of not inconsiderable value, especially to table-makers. It consists of four neatly written foolscap volumes, with 31 lines to a page, giving sines and cosines to 13 decimals at interval 0·0001 radians up to 0·8 radians, i.e., just past the first octant. From this, with the aid of multiples of $\frac{1}{2}\pi$, any sine or cosine can be found. The table can best be described by illustrating a typical opening with headings supplied.

Left hand page sin x	Δ'	sin x	Δ'	x	*Right hand page* cos x	Δ'	cos x	Δ'
·16721 08424 8	9 85912 8	827	798	·1680	·98592 11602 1	1 67260 1	132·	138
30 94337 6	896 1	625	067	1	90 44342 0	7358 7	994·	728
40 80233 7		692		2	88 76983 3		266	

The first column gives the sine to 11 decimals, and the second its difference. The third column gives the 11th, 12th and 13th decimals, which can replace the previous 11th decimal, with lowering by 1 of the tenth decimal when the 11th is 0 and this group is 950 or more, as in the case of cos $0^r\cdot1681$. The next column gives the 11th, 12th and 13th decimals of the first difference, which can replace the 11th decimal of the previous difference, as before. The arrangement on the right hand page is similar.

Actually the table was first made to 11D, the number used in Airey's table at interval $0^r\cdot001$ in the Br. Ass. Adv. Sci., *Report* for 1916, and in its *Mathematical Tables*, v. 1, 1931. The decision to extend it to 13D was made later. The 13th decimal has been worked to approximately half a unit, the half unit being indicated by a following central dot.

In 1931 the 11-figure values were compared under the present writer's direction with a 10-figure table (to 1·6 radians) at the same interval that he had made on a Burroughs machine. This enabled 12 errors in Airey's table to be corrected, as well as 10-end-figure errors between 0·8 and 1·2 units in the 10-figure table.

The published tables at interval $0^r\cdot001$ are the Br. Ass. (Airey's) table mentioned above, Van Orstrand's 23-place table in Nat. Acad. Sci., Washington, *Memoirs*, v. 14, part V, 1925, and Hayashi's 12-place (after $x=0\cdot1$) table in *Sieben- und mehrstellige Tafeln der Kreis- und Hyperbelfunktionen*, Berlin, 1926, which is notoriously inaccurate, and should not be used unless corrected. There is only one table at interval $0^r\cdot0001$, namely the New York W.P.A. *Tables of Circular and Hyperbolic Sines and Cosines for Radian Arguments*, 1939, to 9D. All these tables go to $1^r\cdot6$ or 2^r, and so have the slight advantage that multiples of $\frac{1}{2}\pi$ can always be *subtracted from* a large angle.

A table at interval $0^r\cdot001$ is virtually linear to seven decimals, and one at interval $0^r\cdot0001$ to nine. If we need twelve decimals in order to produce accurate 10-figure tables involving sines and cosines, an interval of $0^r\cdot0001$ is desirable in order that the effect of third differences may be negligible. Second differences are, of course, simply the first four decimals of the function, with a negative sign.

There is no doubt that Airey's manuscript could be used for an 80-page 12-decimal table that would be accurate to about 0·6 units of the last decimal—ample for all computing purposes. Where the 13th decimal is 4· (which might be 4 or 5), it would be rounded down, and 5· would be rounded up. A 5 (i.e., something between 4·75 and 5·25) would be treated in accordance with the rule that leaves the last figure even. That Airey had hoped to publish the table (perhaps privately) is shown by a printer's 12-line specimen, in two different types, in one of the volumes. I should be glad to have any expressions of opinion as to the advisability or otherwise of publication, bearing in mind the cost of printing, the existence of the W.P.A. table at the same interval to nine decimals, and the limited demand for such a table.

L. J. C.

3[L].—L. J. Comrie, *The Bessel Functions J_0 and J_1.* Ms. in Mr. Comrie's possession.

I have two bound Burroughs-script tables, one of J_0 and the other of J_1, both being to 12D, for $x=0(0\cdot001)16(0\cdot01)25$. The argument is printed in small red figures, and the function (which has the decimal printed in the correct position) has been ruled into groups of 5, 5 and 2 figures. First and second differences are given, the former being in complementary form when they have the opposite sign to that of the function; the latter are always in direct form. The tables are printed on foolscap (one side only), with the usual machine spacing of one-sixth of an inch.

These tables were prepared about 1933–34, while I was engaged on producing the Br. Ass. Adv. Sci., *Mathematical Tables*, v. 6, *Bessel Function*, part I, 1936, in which these values appear (with their second differences) to 10D. The part up to $x=15\cdot 5$ was formed by subtabulating to tenths the 12-figure table of Ernst Meissel, as given originally in the *Berliner Abhandlungen* for 1888, and reprinted in A. Gray and G. B. Mathews, *A Treatise on Bessel Functions*, London, Macmillan, 1895; 2nd ed. by Gray and T. M. Macrobert, 1922. For the range $x=15\cdot 5$ to $x=25$, the sources were a manuscript table lent by H. T. Davis, original calculations based on Meissel's table of $J_n(x)$ for integral values of n and x (in Gray and Mathews), and comparison with Hayashi's *Tafeln der Besselschen, Theta, Kugel- und anderer Funktionen*, Berlin, Springer, 1930. The latter contains 22 errors in this range.

The tables were a stepping stone to the published 10-figure tables mentioned. They are, however, being retained in case these values should ever be required to more decimals.

L. J. C.

MECHANICAL AIDS TO COMPUTATION

2[Z].—S. Lilley, "Mathematical Machines," *Nature*, v. 149, 25 Apr. 1942, p. 462–465.

This is a pleasantly written survey article, beginning with the "rise of modern arithmetic" (based on material in Stevin's decimal arithmetic of 1585, and Napier's logarithms of 1614), then on to discussion of "future trends." The whole concludes with an excellent selected bibliography. Its a good article for the uninformed reader, desiring to get a general idea of achievements up to the present, especially if he reads also about half of the score of sources (1914–40) which have been brought to his attention.

The earliest calculating machines of Pascal, Morland, and Leibniz, discussed in MAC **1**, are merely mentioned, as well as several others dating from the eighteenth century. Taken together such machines specializing in the operations of addition, subtraction, multiplication, and division. But each of these machines was a failure, because "industrial revolution" had not caused the development of techniques for producing gears of adequate exactness and durability. Such a revolution took place in Great Britain in the first half of the nineteenth century when problems of power and its transmission, arising from the steam engine, were effectively handled. In this period Charles Babbage[1] (1792–1871), mathematician and scientific mechanician, was the only one appreciating economic trends and attempting to employ mathematics to assist in this work. His remarkable Difference Engine (invented in 1812), and, the incomplete "Analytical Engine" (1833+), of which the Hollerith punch-card machine is the modern counterpart, were products of his genius.

We are told that with the possible exception of the arithmometer (1820) designed and introduced by Charles X. Thomas of Colmar, no successful machine was produced until the 1880's, when the continually growing demand, coupled with the accurate machine tools which were then available caused an extremely rapid development of many efficient machines. The Comptometer (1877) invented by D. E. Felt of Chicago, was the first successful Key-operated machine; and many other successful machines were made about the same time. The German Brunsviga Calculating Machine,[2] based on the invention of a Russian engineer, first appeared in 1892, and the completion of the 20000th machine was celebrated in 1912. During the past fifty years no fundamental change has occurred in the ordinary calculating machine, although a multitude of detailed improvements have increased its speed and efficiency. Especially in the late nineteenth and early twentieth centuries did many new specialized machines appear for rapidly solving particular types of problems. Reference may be made to two of these. The National Accounting Machine[3] is very similar to the one planned by Babbage, and has been extensively used in computing the British *Nautical Almanac*. W. J. Eckert tells us[4] that the first extensive use of the early Hollerith in astronomy was made by L. J. Comrie. He used it for building a table from successive differences, and for adding large numbers of harmonic terms.[5] Modern Hollerith machines seem to be capable