Mathieu Functions of Integral Order and Their Tabulation

Introduction

Increasing attention is being given to scientific and technical problems which lead to differential equations of the Mathieu type. A considerable amount of analytical knowledge concerning the solutions of such equations exists. To a physicist or an engineer, however, a formal solution is seldom sufficient, but is merely a stepping-stone to quantitative results. There is thus an increasing need for tables giving numerical values of solutions which are of use in applications.

In a previous article[1] one of us gave an account of such tables as already exist. The aims of the present article are (a) to define a set of Mathieu functions appropriate to the solution of potential, wave, and analogous problems, (b) to make suggestions as to which should be tabulated, (c) to indicate which formulae are most convenient for computation.

Use of a common notation and agreement as to which functions are to be tabulated would greatly accelerate progress. The notation and definitions which we employ are a natural and systematic extension of those finally adopted[2] by INCE in his "Tables"[3] and embody the results of much thought and discussion. The '*e*' occurring throughout is of course '*e* for elliptic.'

Differential equations

If the two-dimensional wave equation[4]

(1) $$\nabla^2\phi + \kappa^2\phi = 0$$

is transformed from rectangular co-ordinates (x, y) to elliptic co-ordinates (ξ, η) by the formulae

(2) $$x = c\cosh\xi\cos\eta, \quad y = c\sinh\xi\sin\eta,$$

and a solution of the form

(3) $$\phi = X(\xi)\,Y(\eta)$$

is sought, it is found that $X(\xi)$ and $Y(\eta)$ must satisfy respectively the equations

(4) $$d^2Y/d\eta^2 + (a - 2q\cos 2\eta)Y = 0,$$

and

(5) $$d^2X/d\xi^2 - (a - 2q\cosh 2\xi)X = 0,$$

in which

(6) $$q = \tfrac{1}{4}\kappa^2c^2,$$

and a is the separation constant.

Equation (4) is known as Mathieu's equation. We adopt, as the canonical form,

(7) $$d^2y/dz^2 + (a - 2q\cos 2z)y = 0.$$

Corresponding to (5) we write

(8) $$d^2y/dz^2 - (a - 2q\cosh 2z)y = 0.$$

It is a very fortunate circumstance that (7) transforms into (8), and vice versa, if z is replaced by $\pm iz$.

In other problems the sign of κ^2 in (1) is changed, and we are then led to the pair of equations

(9) $$d^2y/dz^2 + (a + 2q\cos 2z)y = 0,$$

(10) $$d^2y/dz^2 - (a + 2q\cosh 2z)y = 0,$$

which, again, are interchanged by replacing z by $\pm iz$.

We note also that (7) is transformed into (9), and vice versa, by replacing z by $\pm(\frac{1}{2}\pi \pm z)$, and (8) into (10) and vice versa by replacing z by $\pm(\frac{1}{2}i\pi \pm z)$.

Physical considerations are usually such that solutions of (7) or (9) must admit the period 2π in z. Such solutions we shall term, for brevity, 'periodic solutions,' although solutions of period $2s\pi$, where s is any integer, exist. The present article deals mainly with periodic solutions of (7) or (9) and with the corresponding solutions of (8) and (10); usually q will be regarded as real and positive. We shall frequently write

(11) $$q = k^2,$$

and it is clear from (6) that k, rather than q, is likely to be physically significant.

Periodic solutions of (7)

In order that a solution of (7), with a prescribed value of q, may be periodic, the separation constant a must be one of an infinite sequence of characteristic numbers. The corresponding solutions fall into four classes, according to their symmetry or anti-symmetry, about $z = 0$ and $z = \frac{1}{2}\pi$. Following HEINE,[5] and using the ce, se notation[6] suggested by WHITTAKER, Ince[8] defines these as[7]

(12.1) $$ce_{2n}(z, q) = \sum_{r=0}^{\infty} A_{2r}^{(2n)} \cos 2rz, \qquad (a_{2n}),$$

(12.2) $$se_{2n+1}(z, q) = \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)} \sin(2r+1)z, \qquad (b_{2n+1}),$$

(12.3) $$ce_{2n+1}(z, q) = \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)} \cos(2r+1)z, \qquad (a_{2n+1}),$$

(12.4) $$se_{2n+2}(z, q) = \sum_{r=0}^{\infty} B_{2r+2}^{(2n+2)} \sin(2r+2)z, \qquad (b_{2n+2}).$$

The symbols in parentheses denote the corresponding characteristic numbers. As listed, for a given positive q, these are in order of increasing magnitude. In future formulae, the appropriate characteristic number can be inferred immediately from the coefficients and the order of the functions. In the above, n may be zero or a positive integer; the corresponding function has n real zeros in the open interval $0 < z < \frac{1}{2}\pi$.

Recurrence relations between the coefficients, derived by substituting these series in the differential equation (7), lead to a technique (developed by GOLDSTEIN[8] and Ince,[3] and now well-known), for the determination of characteristic numbers and the ratios of successive coefficients. The latter are made precise by the normalization rule

$$\int_0^{2\pi} y^2 dz = \pi, \tag{13}$$

so that

$$1 = 2A_0^2 + \sum_{r=1}^{\infty} A_{2r}^2 = \sum_{r=0}^{\infty} B_{2r+1}^2 = \sum_{r=0}^{\infty} A_{2r+1}^2 = \sum_{r=0}^{\infty} B_{2r+2}^2. \tag{14}$$

The functions so defined may be termed elliptic cylinder functions, or Mathieu functions, of (zero and) positive integral order. They have been tabulated by Ince[3] for $n = 0, 1, 2$, $q = 1(1)10$, $0 \leq z \leq \frac{1}{2}\pi$.

As $q \to 0$, $ce_0(z, q) \to 1/\sqrt{2}$,

$$ce_m(z, q) \to \cos mz, \quad se_m(z, q) \to \sin mz \qquad (m > 0),$$

and as regards symmetry or antisymmetry about $z = 0$ or $z = \frac{1}{2}\pi$, the functions behave as do these limiting forms; it is therefore sufficient to tabulate the functions over the first quadrant.

The series (12.1)–(12.4) are absolutely and uniformly convergent for all finite (real or complex) values of z. For small or moderate values of q and n, and for real values of z, convergence is satisfactorily rapid for computation.

Periodic solutions of (9)

It has already been mentioned that (9) may be obtained from (7) by replacing z by $\pm (\frac{1}{2}\pi \pm z)$. Hence if $f(z, q)$ satisfies (7), then $f(\frac{1}{2}\pi - z, q)$ satisfies (9).

Consequently we define (following Ince[3]) [9]

$$ce_{2n}(z, -q) = (-)^n ce_{2n}(\tfrac{1}{2}\pi - z, q) = (-)^n \sum (-)^r A_{2r} \cos 2rz, \tag{15.1}$$

$$ce_{2n+1}(z, -q) = (-)^n se_{2n+1}(\tfrac{1}{2}\pi - z, q) = (-)^n \sum (-)^r B_{2r+1} \cos (2r+1)z, \tag{15.2}$$

$$se_{2n+1}(z, -q) = (-)^n ce_{2n+1}(\tfrac{1}{2}\pi - z, q) = (-)^n \sum (-)^r A_{2r+1} \sin (2r+1)z, \tag{15.3}$$

$$se_{2n+2}(z, -q) = (-)^n se_{2n+2}(\tfrac{1}{2}\pi - z, q) = (-)^n \sum (-)^r B_{2r+2} \sin (2r+2)z. \tag{15.4}$$

The signs have been chosen so that the limiting forms as $q \to 0$ are $+1/\sqrt{2}$, $+\cos mz$, or $+\sin mz$, as the case may be.

It is clear that tables of the periodic solutions of (7) will also yield the values of the corresponding periodic solutions of (9).

Second solutions of (7) and (9)

It is known that two linearly independent solutions of (7) or (9), both admitting the period 2π, cannot coexist for the same values of a and q ($\neq 0$).

The second solution corresponding to a periodic Mathieu function of integral order is therefore not periodic.

Second solutions corresponding to $ce_m(z, q)$ and $se_m(z, q)$ will be denoted by $fe_m(z, q)$ and $ge_m(z, q)$, respectively.[10] Not being periodic, they will rarely be of importance in applications. Goldstein[11] has given (unnormalized) definitions, and an elaborate method for computing the coefficients. A much less laborious method of computation has been discovered, and it is hoped that an account of this will be published shortly.

First solution of (8)

Since (7) is transformed into (8) by writing iz for z, we can derive a set of solutions of (8) by applying this transformation to (12.1)–(12.4): of these

$$Ce_{2n}(z, q) = ce_{2n}(iz, q) = \sum A_{2r} \cosh 2rz$$

is typical. The rate of convergence of such series decreases rapidly with increasing (real) z. For large z they are consequently inconvenient for computation. Moreover, asymptotic properties cannot readily be derived from them. Fortunately, other developments are known, involving Bessel functions, which do not have these defects. The set of formulae, in which $k = +\sqrt{q}$ and primes denote derivatives, is:

$$(16.11) \qquad Ce_{2n}(z, q) = ce_{2n}(iz, q) = \sum A_{2r} \cosh 2rz$$

$$(16.12) \qquad = \frac{ce_{2n}(\frac{1}{2}\pi, q)}{A_0} \sum (-)^r A_{2r} J_{2r}(2k \cosh z)$$

$$(16.13) \qquad = \frac{ce_{2n}(0, q)}{A_0} \sum A_{2r} J_{2r}(2k \sinh z)$$

$$(16.14) \qquad = \frac{ce_{2n}(0, q) ce_{2n}(\frac{1}{2}\pi, q)}{A_0^2} \sum (-)^r A_{2r} J_r(ke^{-z}) J_r(ke^{z}).$$

$$(16.21) \qquad Se_{2n+1}(z, q) = i^{-1} se_{2n+1}(iz, q) = \sum B_{2r+1} \sinh (2r+1)z$$

$$(16.22) \qquad = \frac{se_{2n+1}(\frac{1}{2}\pi, q)}{kB_1} \tanh z \sum (-)^r (2r+1) B_{2r+1} J_{2r+1}(2k \cosh z)$$

$$(16.23) \qquad = \frac{se'_{2n+1}(0, q)}{kB_1} \sum B_{2r+1} J_{2r+1}(2k \sinh z)$$

$$(16.24) \qquad = \frac{se'_{2n+1}(0, q) se_{2n+1}(\frac{1}{2}\pi, q)}{kB_1^2} \sum (-)^r B_{2r+1} \times \{J_r(ke^{-z}) J_{r+1}(ke^{z}) - J_{r+1}(ke^{-z}) J_r(ke^{z})\}.$$

(16.31) $Ce_{2n+1}(z, q) = ce_{2n+1}(iz, q) = \sum A_{2r+1} \cosh (2r+1)z$

(16.32) $= -\dfrac{ce'_{2n+1}(\frac{1}{2}\pi, q)}{kA_1} \sum (-)^r A_{2r+1} J_{2r+1}(2k \cosh z)$

(16.33) $= \dfrac{ce_{2n+1}(0, q)}{kA_1} \coth z \sum (2r+1) A_{2r+1} J_{2r+1}(2k \sinh z)$

(16.34) $= -\dfrac{ce_{2n+1}(0,q) ce'_{2n+1}(\frac{1}{2}\pi,q)}{kA_1^2} \sum (-)^r A_{2r+1}$
$\times \{J_r(ke^{-z}) J_{r+1}(ke^z) + J_{r+1}(ke^{-z}) J_r(ke^z)\}.$

(16.41) $Se_{2n+2}(z, q) = i^{-1} se_{2n+2}(iz, q) = \sum B_{2r+2} \sinh (2r+2)z$

(16.42) $= -\dfrac{se'_{2n+2}(\frac{1}{2}\pi,q)}{k^2 B_2} \tanh z \sum (-)^r (2r+2) B_{2r+2} J_{2r+2}(2k \cosh z)$

(16.43) $= \dfrac{se'_{2n+2}(0, q)}{k^2 B_2} \coth z \sum (2r+2) B_{2r+2} J_{2r+2}(2k \sinh z)$

(16.44) $= -\dfrac{se'_{2n+2}(0,q) se'_{2n+2}(\frac{1}{2}\pi,q)}{k^2 B_2^2} \sum (-)^r B_{2r+2}$
$\times \{J_r(ke^{-z}) J_{r+2}(ke^z) - J_{r+2}(ke^{-z}) J_r(ke^z)\}.$

The series involving Bessel functions converge absolutely and uniformly for all finite values of z. The rate of convergence of those involving products of Bessel functions increases with increasing z.

For large *real* values of z these functions oscillate with increasing rapidity, while the amplitude tends exponentially to zero.

As $q \to 0$, $Ce_0(z, q) \to 1/\sqrt{2}$, and, if $m > 0$, $Ce_m(z, q) \to \cosh mz$, $Se_m(z, q) \to \sinh mz$.

First solutions of (10)

A similar set of solutions of (10) exists, and can be obtained by writing $(\frac{1}{2}i\pi + z)$ for z in (16.11)–(16.44).

(17.11) $Ce_{2n}(z, -q) = (-)^n \sum (-)^r A_{2r} \cosh 2rz$

(17.12) $= (-)^n \dfrac{ce_{2n}(\frac{1}{2}\pi, q)}{A_0} \sum A_{2r} I_{2r}(2k \sinh z)$

(17.13) $= (-)^n \dfrac{ce_{2n}(0, q)}{A_0} \sum (-)^r A_{2r} I_{2r}(2k \cosh z)$

(17.14) $= (-)^n \dfrac{ce_{2n}(0, q) ce_{2n}(\frac{1}{2}\pi, q)}{A_0^2} \sum (-)^r A_{2r} I_r(ke^{-z}) I_r(ke^z).$

(17.21) $Ce_{2n+1}(z, -q) = (-)^n \sum (-)^r B_{2r+1} \cosh (2r+1)z$

(17.22) $= (-)^n \dfrac{se_{2n+1}(\frac{1}{2}\pi,q)}{kB_1} \coth z \sum (2r+1) B_{2r+1} I_{2r+1}(2k \sinh z)$

(17.23) $= (-)^n \dfrac{se'_{2n+1}(0, q)}{kB_1} \sum (-)^r B_{2r+1} I_{2r+1}(2k \cosh z)$

(17.24) $= (-)^n \dfrac{se'_{2n+1}(0, q) se_{2n+1}(\frac{1}{2}\pi, q)}{kB_1^2} \sum (-)^r B_{2r+1}$
$\times \{I_r(ke^{-z}) I_{r+1}(ke^z) + I_{r+1}(ke^{-z}) I_r(ke^z)\}.$

$$(17.31)\quad Se_{2n+1}(z,-q) = (-)^n\sum(-)^r A_{2r+1}\sinh(2r+1)z$$

$$(17.32)\quad = (-)^{n+1}\frac{ce'_{2n+1}(\frac{1}{2}\pi,q)}{kA_1}\sum A_{2r+1}I_{2r+1}(2k\sinh z)$$

$$(17.33)\quad = (-)^n\frac{ce_{2n+1}(0,q)}{kA_1}\tanh z \times \sum(-)^r(2r+1)A_{2r+1}I_{2r+1}(2k\cosh z)$$

$$(17.34)\quad = (-)^n\frac{ce_{2n+1}(0,q)ce'_{2n+1}(\frac{1}{2}\pi,q)}{kA_1^2}\sum(-)^r A_{2r+1} \times \{I_r(ke^{-z})I_{r+1}(ke^z) - I_{r+1}(ke^{-z})I_r(ke^z)\}.$$

$$(17.41)\quad Se_{2n+2}(z,-q) = (-)^n\sum(-)^r B_{2r+2}\sinh(2r+2)z$$

$$(17.42)\quad = (-)^{n+1}\frac{se'_{2n+2}(\frac{1}{2}\pi,q)}{k^2B_2}\coth z \times \sum(2r+2)B_{2r+2}I_{2r+2}(2k\sinh z)$$

$$(17.43)\quad = (-)^n\frac{se'_{2n+2}(0,q)}{k^2B_2}\tanh z \times \sum(-)^r(2r+2)B_{2r+2}I_{2r+2}(2k\cosh z)$$

$$(17.44)\quad = (-)^{n+1}\frac{se'_{2n+2}(0,q)se'_{2n+2}(\frac{1}{2}\pi,q)}{k^2B_2^2}\sum(-)^r B_{2r+2} \times \{I_r(ke^{-z})I_{r+2}(ke^z) - I_{r+2}(ke^{-z})I_r(ke^z)\}.$$

As regards convergence, these series behave as do the corresponding ones of (16.11)–(16.44). The sign has been so chosen that as $q \to 0$ they tend to $+1/\sqrt{2}$, $+\cosh mz$, or $+\sinh mz$, as the case may be.

For large (real) values of z they tend exponentially to infinity.

Second solutions of (8)

Since the J and Y Bessel functions satisfy the same differential equations and recurrence formulae, and since it is by virtue of these that the Bessel function series in (16.11)–(16.44) satisfy (8), it follows that solutions of (8), linearly independent of (16.11)–(16.44), can be obtained by replacing therein J by Y. Of the resulting series, those involving $Y_m(2k\cosh z)$ converge only when $|\cosh z| > 1$, those involving $Y_m(2k\sinh z)$ only when $|\sinh z| > 1$. Those containing $J_m(ke^{-z})Y_p(ke^z)$ are useful *only when the real part of z is positive*, the ratio of successive terms tending to $k^2e^{2z}/4r^2$; those containing $J_m(ke^z)Y_p(ke^{-z})$ *only when the real part of z is negative*, the ratio of successive terms tending to $k^2e^{2z}/4r^2$; and those containing $Y_m(ke^{-z})Y_p(ke^z)$ not at all, the ratio of successive terms tending to unity. There remain

$$(18.11)\quad Fey_{2n}(z,q) = \frac{ce_{2n}(\frac{1}{2}\pi,q)}{A_0}\sum(-)^r A_{2r}Y_{2r}(2k\cosh z) \qquad |\cosh z| > 1$$

$$(18.12)\quad = \frac{ce_{2n}(0,q)}{A_0}\sum A_{2r}Y_{2r}(2k\sinh z) \qquad |\sinh z| > 1,\ \mathrm{Re}(z) > 0$$

$$(18.13)\quad = \frac{ce_{2n}(0,q)ce_{2n}(\frac{1}{2}\pi,q)}{A_0^2}\sum(-)^r A_{2r}J_r(ke^{-z})Y_r(ke^z).$$

$$(18.21)\quad Gey_{2n+1}(z,q) = \frac{se_{2n+1}(\frac{1}{2}\pi, q)}{kB_1}\tanh z \times \sum (-)^r(2r+1)B_{2r+1}Y_{2r+1}(2k\cosh z) \qquad |\cosh z| > 1$$

$$(18.22)\quad = \frac{se'_{2n+1}(0,q)}{kB_1}\sum B_{2r+1}Y_{2r+1}(2k\sinh z) \qquad |\sinh z| > 1,\ \mathrm{Re}(z) > 0$$

$$(18.23)\quad = \frac{se'_{2n+1}(0,q)se_{2n+1}(\frac{1}{2}\pi,q)}{kB_1^2}\sum (-)^r B_{2r+1} \times \{J_r(ke^{-z})Y_{r+1}(ke^z) - J_{r+1}(ke^{-z})Y_r(ke^z)\}.$$

$$(18.31)\quad Fey_{2n+1}(z,q) = -\frac{ce'_{2n+1}(\frac{1}{2}\pi,q)}{kA_1}\sum (-)^r A_{2r+1}Y_{2r+1}(2k\cosh z) \qquad |\cosh z| > 1$$

$$(18.32)\quad = \frac{ce_{2n+1}(0,q)}{kA_1}\coth z\sum(2r+1)A_{2r+1}Y_{2r+1}(2k\sinh z) \qquad |\sinh z| > 1,\ \mathrm{Re}(z) > 0$$

$$(18.33)\quad = -\frac{ce_{2n+1}(0,q)ce'_{2n+1}(\frac{1}{2}\pi,q)}{kA_1^2}\sum (-)^r A_{2r+1} \times \{J_r(ke^{-z})Y_{r+1}(ke^z) + J_{r+1}(ke^{-z})Y_r(ke^z)\}.$$

$$(18.41)\quad Gey_{2n+2}(z,q) = -\frac{se'_{2n+2}(\frac{1}{2}\pi,q)}{k^2B_2}\tanh z \times \sum (-)^r(2r+2)B_{2r+2}Y_{2r+2}(2k\cosh z) \qquad |\cosh z| > 1$$

$$(18.42)\quad = \frac{se'_{2n+2}(0,q)}{k^2B_2}\coth z\sum(2r+2)B_{2r+2}Y_{2r+2}(2k\sinh z) \qquad |\sinh z| > 1,\ \mathrm{Re}(z) > 0$$

$$(18.43)\quad = -\frac{se'_{2n+2}(0,q)se'_{2n+2}(\frac{1}{2}\pi,q)}{k^2B_2^2}\sum (-)^r B_{2r+2} \times \{J_r(ke^{-z})Y_{r+2}(ke^z) - J_{r+2}(ke^{-z})Y_r(ke^z)\}.$$

The functions here defined and their derivations remain finite as $z \to 0$, but the series involving functions with arguments $\cosh z$ or $\sinh z$ converge non-uniformly as $|\cosh z| \to 1$ or $|\sinh z| \to 1$, so that term-by-term differentiation of these series is not legitimate at the limits of convergence.

For computational purposes the series involving Bessel function products behave admirably for all finite real positive values of z (compare the numerical example below).

For large (real) values of z the functions oscillate in the same manner as the corresponding $Ce_m(z, q)$ or $Se_m(z, q)$, with a phase difference of $\frac{1}{2}\pi$.

Solutions of (8) exist, derived from $fe_m(z, q)$ and $ge_m(z, q)$ by replacing z by iz; we shall denote them by $Fe_m(z, q)$ and $Ge_m(z, q)$. In applications, however, the functions $Fey_m(z, q)$ and $Gey_m(z, q)$ are much more convenient—and this is true quite apart from the slow convergence of the series defining $Fe_m(z, q)$ and $Ge_m(z, q)$ for large (real) values of z.

Second solutions of (10)

These may be obtained from the Bessel function series in (17.12)–(17.44) by replacing I_m by $(-)^m K_m$. As regards convergence, the resulting series behave as do the corresponding series in (18.11)–(18.43). We introduce a factor $1/\pi$ to systematize the asymptotic relations, and define:

(19.11) $$Fek_{2n}(z, -q) = (-)^n \frac{ce_{2n}(\frac{1}{2}\pi, q)}{\pi A_0} \sum A_{2r} K_{2r}(2k \sinh z) \qquad |\sinh z| > 1,\ \mathrm{Re}(z) > 0$$

(19.12) $$= (-)^n \frac{ce_{2n}(0, q)}{\pi A_0} \sum (-)^r A_{2r} K_{2r}(2k \cosh z) \qquad |\cosh z| > 1$$

(19.13) $$= (-)^n \frac{ce_{2n}(0, q) ce_{2n}(\frac{1}{2}\pi, q)}{\pi A_0^2} \sum A_{2r} I_r(ke^{-z}) K_r(ke^z).$$

(19.21) $$Fek_{2n+1}(z, -q) = \frac{se_{2n+1}(\frac{1}{2}\pi, q)}{\pi k B_1} \coth z \times \sum (2r+1) B_{2r+1} K_{2r+1}(2k \sinh z) \qquad |\sinh z| > 1,\ \mathrm{Re}(z) > 0$$

(19.22) $$= (-)^n \frac{se'_{2n+1}(0, q)}{\pi k B_1} \sum (-)^r B_{2r+1} K_{2r+1}(2k \cosh z) \qquad |\cosh z| > 1$$

(19.23) $$= (-)^n \frac{se'_{2n+1}(0, q) se_{2n+1}(\frac{1}{2}\pi, q)}{\pi k B_1^2} \sum B_{2r+1} \times \{I_r(ke^{-z}) K_{r+1}(ke^z) - I_{r+1}(ke^{-z}) K_r(ke^z)\}.$$

(19.31) $$Gek_{2n+1}(z, -q) = (-)^{n+1} \frac{ce'_{2n+1}(\frac{1}{2}\pi, q)}{\pi k A_1} \sum A_{2r+1} K_{2r+1}(2k \sinh z) \qquad |\sinh z| > 1,\ \mathrm{Re}(z) > 0$$

(19.32) $$= (-)^n \frac{ce_{2n+1}(0, q)}{\pi k A_1} \tanh z \times \sum (-)^r (2r+1) A_{2r+1} K_{2r+1}(2k \cosh z) \qquad |\cosh z| > 1$$

(19.33) $$= (-)^{n+1} \frac{ce_{2n+1}(0, q) ce'_{2n+1}(\frac{1}{2}\pi, q)}{\pi k A_1^2} \sum A_{2r+1} \times \{I_r(ke^{-z}) K_{r+1}(ke^z) + I_{r+1}(ke^{-z}) K_r(ke^z)\}.$$

(19.41) $$Gek_{2n+2}(z, -q) = (-)^{n+1} \frac{se'_{2n+2}(\frac{1}{2}\pi, q)}{\pi k^2 B_2} \coth z \times \sum (2r+2) B_{2r+2} K_{2r+2}(2k \sinh z) \qquad |\sinh z| > 1,\ \mathrm{Re}(z) > 0$$

(19.42) $$= (-)^n \frac{se'_{2n+2}(0, q)}{\pi k^2 B_2} \tanh z \times \sum (-)^r (2r+2) B_{2r+2} K_{2r+2}(2k \cosh z) \qquad |\cosh z| > 1$$

(19.43) $$= (-)^{n+1} \frac{se'_{2n+2}(0, q) se'_{2n+2}(\frac{1}{2}\pi, q)}{\pi k^2 B_2^2} \sum B_{2r+2} \times \{I_r(ke^{-z}) K_{r+2}(ke^z) - I_{r+2}(ke^{-z}) K_r(ke^z)\}.$$

These functions behave, for small values of z, in a manner similar to the corresponding *Fey* or *Gey* function. For large (real) values of z they tend exponentially to zero.

The numerical behavior of these series, and, in particular, the superiority of the Bessel function product series for computation will be demonstrated later.

Since

$$K_\nu(x) = \tfrac{1}{2}\pi e^{\frac{1}{2}(\nu+1)\pi i}\{J_\nu(ix) + iY_\nu(ix)\} \tag{20}$$

there are relations of the type

$$Fey_{2n}(z, -q) = iCe_{2n}(z, -q) - 2Fek_{2n}(z, -q). \tag{21}$$

For real values of q and z or, $Fek_{2n}(z, -q)$ is real, so that $Fey_{2n}(z, -q)$ is complex, and will rarely be of importance in the applications. If and when, however, tabulation of functions for imaginary or complex values of z is considered, these relations may be useful.

Functions to be tabulated

Having defined solutions of the several differential equations, we now suggest a set of functions which, in view of technical applications, should be tabulated.

FUNCTIONS

Characteristic Number	Trigonometric	Hyperbolic			
a_{2n}	$ce_{2n}(z, q)$	$Ce_{2n}(z, q)$	$Fey_{2n}(z, q)$	$Ce_{2n}(z, -q)$	$Fek_{2n}(z, -q)$
b_{2n+1}	$se_{2n+1}(z, q)$	$Se_{2n+1}(z, q)$	$Gey_{2n+1}(z, q)$	$Ce_{2n+1}(z, -q)$	$Fek_{2n+1}(z, -q)$
a_{2n+1}	$ce_{2n+1}(z, q)$	$Ce_{2n+1}(z, q)$	$Fey_{2n+1}(z, q)$	$Se_{2n+1}(z, -q)$	$Gek_{2n+1}(z, -q)$
b_{2n+2}	$se_{2n+2}(z, q)$	$Se_{2n+2}(z, q)$	$Gey_{2n+2}(z, q)$	$Se_{2n+2}(z, -q)$	$Gek_{2n+2}(z, -q)$

There are some reasons (based upon its physical interpretations) for preferring k to q as a tabular parameter. Apart from the range $0 \leqslant z \leqslant \frac{1}{2}\pi$ for the periodic functions, no *natural* upper limits to the ranges of z, $q(0^+ k)$, n exist, and present knowledge of technical applications is too meagre to permit authoritative suggestions as to the useful extent of these ranges. Developments asymptotic in one or more of these variables are known, and tabulation might cease when one or two terms of such a development provide adequate accuracy.

In the first instance, tables of low accuracy, covering wide ranges of z and q, and for the lower values of n, are desirable: these would constitute a numerical exploration of the field, which is essential. It will, however, be justifiable, and in the long run economical, to compute fundamental quantities (especially characteristic numbers) to high accuracy. The ultimate goal is a set of tables interpolable in q (or k) as well as in z, to an accuracy of at least five significant figures, and for sufficient of the lower values of n, say from 0 to 6. The experience of Ince[8] and HIDAKA[12] indicates that interpolability in q necessitates an interval of the order 0.1 in this parameter. Consequently, such a set of tables will be laborious to produce, and voluminous to publish. They form no one-man project, so many contributors will be needed. One important purpose of this article is to suggest a suitable scheme into which all contributions could fit, so that both economy of labor and co-ordination of results may be achieved. In any such scheme Ince's Tables[3] are foundation-stones, well and truly laid.

Computation: numerical illustrations

Methods for calculating characteristic numbers and the coefficients A, B are well established, and the rate of convergence of the series (12.1)–(12.4) is satisfactory—at least for values of n and q likely to be required.

On the other hand, as has been remarked, the convergence of the various series for the 'hyperbolic' types of function is not everywhere satisfactory. The series involving Bessel function products converge everywhere in the contemplated ranges, and they converge at least as rapidly as the series for the corresponding periodic function. More important still is the fact that they converge with increasing rapidity as z increases, and so behave computationally in a manner similar to asymptotic expansions. We illustrate the behavior of the various developments by giving numerical values of the terms in the series (17.11)–(17.14) for $Ce_0(z, -4)$ and (19.12), (19.13) for $Fek_0(z, -4)$, for $z = 0$ and for $e^z = 2$ ($z \simeq 0.7$). The values of A_{2r} are taken from Ince's Tables,[3] and appear (actually as $(-)^r A_{2r}$) as the terms of (17.11) with $z = 0$.

$Ce_0(z, -4)$

$z = 0$

r	(17.11)	(17.12)	(17.13)	(17.14)
0	0.55971 72	1.29197 0	0.79002 0	0.83846 0
1	0.59897 00		0.48040 3	0.43686 6
2	0.12051 12		2131 6	1648 9
3	1203 74		23 2	15 6
4	70 68		1	1
5	2 71			
6	7			
	1.29197 04	1.29197 0	1.29197 2	1.29197 2

$Ce_0(z, -4)$

$e^z = 2$ ($z \simeq 0.7$)

r	(17.11)	(17.12)	(17.13)	(17.14)
0	0.55971 7	+6.30584 0	1.90410 5	2.30875 8
1	1.27281 1	−3.10418 6	1.30948 2	0.95236 4
2	0.96559 6	+ 9060 2	7688 0	3028 6
3	0.38529 .	− 60 2	119 1	25 7
4	9047 .	+ 1 3	7	2
5	139. .			
6	14. .			
	3.289.. .	3.29166 7	3.29166 5	3.29166 7

Already, at $z \backsimeq 0.7$, (17.11) is practically useless, and the superiority of (17.14) is becoming evident.

$\mathrm{Fek}_0(z, -4)$

	$z = 0$		$e^z = 2$	
r	(19.12)	(19.13)	(19.12)	(19.13)
0	0.00078 0079	+0.04189 15	0.00025 8013	+0.00227 970
1	130 1687	− 3841 39	39 7057	− 121 818
2	93 6566	+ 607 35	22 9653	+ 8 215
3	67 359.	− 47 80	12 1274	− 230
4	50 25..	+ 2 27	6 306.	
5	38 9...	− 7	3 30..	
			1 7...	
			est. rem. 2 1	
	0.0046.	0.00909 51	0.00114 0...	0.00114 137

At $z = 0$ (19.12) is computationally useless, and even at $e^z = 2$ it serves to determine three significant figures only, as against 6 from (19.13). For neither value of z does (19.11) converge.

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[1] W. G. BICKLEY, "The tabulation of Mathieu functions," *MTAC*, v. 1, July 1945, p. 409–419.

[2] We do, however, replace Ince's θ by q (compare *MTAC*, v. 1, p. 418), thereby avoiding a mixture of English and Greek symbols in the differential equation.

[3] E. L. INCE, "Tables of the elliptic-cylinder functions," R. So. Edinburgh, *Proc.*, v. 52, 1932, p. 355–423.

[4] But these are not the only problems which lead to equations of the group (7)–(10).

[5] H. E. HEINE, *Handbuch der Kugelfunktionen*, second ed., Berlin, 2 v., 1878–1881.

[6] *ce* from the initials of 'cosine elliptic,' *se* from those of 'sine elliptic.'

[7] The superscripts to A or B, denoting the order of the function, will be omitted where this can be done without ambiguity.

[8] S. GOLDSTEIN, "Mathieu functions," Cambridge Phil. So., *Trans.*, v. 23, 1927, p. 303–336.

[9] Here, and throughout the article, unless otherwise specified, $\sum$ implies $\sum_{r=0}^{\infty}$.

[10] The letters f, g are convenient and not likely to cause confusion with other mathematical functions.

[11] S. GOLDSTEIN, "The second solution of Mathieu's differential equation," Cambridge Phil. So., *Trans.*, v. 24, 1928, p. 223–230.

[12] K. HIDAKA, "Tables for computing the Mathieu functions of odd order . . . ," Imp. Marine Observatory, Kobe, Japan, *Memoirs*, v. 6, 1936, p. 137–157.