The tables of $J_n(x)$ and of $k_n(x)$ marked with asterisks are available in the form of bound photostats for loan for a limited period. It should be emphasized that these are not the Committee's final tables and that they are only made available, by courtesy of the Committee, in order not to hold up the work of establishments having a requirement for them.

Two other tables from which values can be supplied are those of the zeros of J_n and J'_n. j_{ns} is tabulated to 4D for $n = 0(1)19$, $s = 1(1)8$, and j'_{ns} to 4D for $n = 0(1)22$, $s = 1(1)9$.

MECHANICAL AIDS TO COMPUTATION

17[Z].—H. A. PETERSON & C. CONCORDIA, "Analyzers for use in engineering and scientific problems," *General Electric Review*, v. 48, September, 1945, p. 29–37. 20.7 × 28.5 cm.

This paper describes each of the following four analyzers in use by the General Electric Company:

(1) The direct current network analyzer,
(2) The alternating current network analyzer,
(3) The transient network analyzer,
(4) The differential analyzer.

A short account of the engineering applications of each analyzer is given, together with illustrations, and a bibliography of 63 titles, of which 14 of the 18 on Differential Analyzers were given in *MTAC*, v. 1, p. 452–454.

D. H. L.

NOTES

45. DAWSON'S OR POISSON'S INTEGRAL.—Various tables involving $\int_0^x e^{t^2}dt$ have been already noted, *MTAC*, v. 1, p. 322–323, 422–423. H. G. Dawson's table, appearing originally in London Math. So., *Proc.*, v. 29, 1898, p. 521–522, was recomputed and reprinted with corrections by H. M. TERRILL & LUCILE SWEENY, in Franklin Institute, *J.*, v. 238, Sept. 1944, p. 220f. An abridgement of Dawson's table is given in JAHNKE & EMDE, *Tables of Functions*, 1933 and later eds., $x = [0(.01)2; 4S]$. H. B. drew my attention to the fact that the integral here discussed occurs in Poisson's paper of 1815, "Sur la théorie des ondes," Acad. d. Sci., *Mémoires*, n.s., v. 1, 1818, p. 128–130. Two more references to tables involving the integral are as follows:

1. H. LAMB, "On water waves due to disturbance beneath the surface," London Math. So., *Proc.*, s. 2, v. 21, 1922, p. 367; table of $F(x) = x + (1 - 2x^2)e^{-x^2}\int_0^x e^{t^2}dt$, and of $F'(x)$, for $x = [0(.1)2; 3D]$. Also, p. 368, graph of $F(x)$, $0 < x \leqslant 2$.
2. Y. NOMURA, "On the waves of water of finite depth due to disturbance beneath the surface," Japan, Tôhoku teikoku daigaku, *Science Reports*, s. 1, *math. phys. chem.*, v. 25, 1936, p. 1082; table of $\psi_0(x) = 2\int_0^\infty t^2e^{-t^2}\sin(2xt)dt = x + (1 - 2x^2)e^{-x^2}\int_0^x e^{t^2}dt$, $x = [0(.1)2(1)6(2)10; 4D]$. Also, for the same range, tables of

$$\psi_s(x) = \sum_{r=0}^{s} \frac{x^{2r+1}}{(s-r)!r!} \frac{d^s}{d(x^2)^s}[x^{2s-2r-1}\psi_0(x)], \; s = 1, 2, 3.$$

R. C. A.

46. Four-point Lagrangean interpolation coefficients for unusual fractions of the interval.—In QR 20 it is suggested that $\tan^{-1}(m/n)$ could be obtained by interpolation in the NYMTP *Table of Arc Tan x*, 1942. A suitable formula is the four-point Lagrange formula, which may be written

$$f_p = A_{-1}f_{-1} + A_0f_0 + A_1f_1 + A_2f_2.$$

In this f_p is written for $f(a + pw)$, where w is the tabular interval. Since

$$A_{-1}(p) = A_2(1 - p) = A_2(q), \quad A_0(p) = A_1(1 - p) = A_1(q)$$

where $q = 1 - p$, we have also

$$f_q = A_2f_{-1} + A_1f_0 + A_0f_1 + A_{-1}f_2,$$

so that we need a table of coefficients only for $0 \leqslant p \leqslant 1/2$.

Tables of Lagrange coefficients, when $10\,p$ or $100\,p$ is an integer, are common; the NYMTP *Tables of Lagrangian Interpolation Coefficients*, 1944, gives the coefficients to 10D when $10\,000\,p$ is an integer. These tables also give the coefficients in fractional form when $12\,p$ is an integer.

The accompanying Tables give the coefficients when $14\,p$, $18\,p$, $22\,p$, $26\,p$ or $30\,p$ is an integer. These coefficients are rarely needed, and no published table is known to the writer. They have been checked by application to the functions $f(x) = 1$ and $f(x) = 2x - 1$.

p	DA_{-1}	DA_0	DA_1	DA_2	q	D
1/14	− 351	+ 15795	+ 1215	− 195	13/14	16464
1/7	− 39	+ 936	+ 156	− 24	6/7	1029
3/14	− 825	+ 14025	+ 3825	− 561	11/14	16464
2/7	− 60	+ 810	+ 324	− 45	5/7	1029
5/14	− 1035	+ 11799	+ 6555	− 855	9/14	16464
3/7	− 66	+ 660	+ 495	− 60	4/7	1029
1/2	− 1	+ 9	+ 9	− 1	1/2	16
1/18	− 595	+ 33915	+ 1995	− 323	17/18	34992
1/9	− 68	+ 2040	+ 255	− 40	8/9	2187
1/6	− 55	+ 1155	+ 231	− 35	5/6	1296
2/9	− 112	+ 1848	+ 528	− 77	7/9	2187
5/18	− 2015	+ 27807	+10695	−1495	13/18	34992
1/3	− 5	+ 60	+ 30	− 4	2/3	81
7/18	− 2233	+ 23925	+15225	−1925	11/18	34992
4/9	− 140	+ 1365	+ 1092	− 130	5/9	2187
1/2	− 1	+ 9	+ 9	− 1	1/2	16
1/22	− 903	+ 62307	+ 2967	− 483	21/22	63888
1/11	− 105	+ 3780	+ 378	− 60	10/11	3993
3/22	− 2337	+ 58425	+ 9225	−1425	19/22	63888
2/11	− 180	+ 3510	+ 780	− 117	9/11	3993
5/22	− 3315	+ 53703	+15795	−2295	17/22	63888
3/11	− 228	+ 3192	+ 1197	− 168	8/11	3993
7/22	− 3885	+ 48285	+22533	−3045	15/22	63888
4/11	− 252	+ 2835	+ 1620	− 210	7/11	3993
9/22	− 4095	+ 42315	+29295	−3627	13/22	63888
5/11	− 255	+ 2448	+ 2040	− 240	6/11	3993
1/2	− 1	+ 9	+ 9	− 1	1/2	16

p	DA_{-1}	DA_0	DA_1	DA_2	q	D
1/26	− 1275	+103275	+ 4131	− 675	25/26	105456
1/13	− 150	+ 6300	+ 525	− 84	12/13	6591
3/26	− 3381	+ 98049	+12789	−2001	23/26	105456
2/13	− 264	+ 5940	+ 1080	− 165	11/13	6591
5/26	− 4935	+ 91791	+21855	−3255	21/26	105456
3/13	− 345	+ 5520	+ 1656	− 240	10/13	6591
7/26	− 5985	+ 84645	+31185	−4389	19/26	105456
4/13	− 396	+ 5049	+ 2244	− 306	9/13	6591
9/26	− 6579	+ 76755	+40635	−5355	17/26	105456
5/13	− 420	+ 4536	+ 2835	− 360	8/13	6591
11/26	− 6765	+ 68265	+50061	−6105	15/26	105456
6/13	− 420	+ 3990	+ 3420	− 399	7/13	6591
1/2	− 1	+ 9	+ 9	− 1	1/2	16
1/30	− 1711	+159123	+ 5487	− 899	29/30	162000
1/15	− 203	+ 9744	+ 696	− 112	14/15	10125
1/10	− 57	+ 1881	+ 209	− 33	9/10	2000
2/15	− 364	+ 9282	+ 1428	− 221	13/15	10125
1/6	− 55	+ 1155	+ 231	− 35	5/6	1296
1/5	− 6	+ 108	+ 27	− 4	4/5	125
7/30	− 8533	+135309	+41181	−5957	23/30	162000
4/15	− 572	+ 8151	+ 2964	− 418	11/15	10125
3/10	− 119	+ 1547	+ 663	− 91	7/10	2000
1/3	− 5	+ 60	+ 30	− 4	2/3	81
11/30	−10241	+114513	+66297	−8569	19/30	162000
2/5	− 8	+ 84	+ 56	− 7	3/5	125
13/30	−10387	+103071	+78819	−9503	17/30	162000
7/15	− 644	+ 6072	+ 5313	− 616	8/15	10125
1/2	− 1	+ 9	+ 9	− 1	1/2	16

J. C. P. MILLER

47. THE GRAEFFE PROCESS.—Protagonists of the root-squaring method[1] for evaluating the roots of an algebraic polynomial equation usually claim that it renders possible the determination, in one process, of all the roots, both real and complex. This assertion is not true of the method as ordinarily expounded, but Lehmer's criticism,[2] that the process fails utterly in such simple cases as the equation

$$(1) \qquad x^4 + x^3 + x^2 + x + 1 = 0,$$

seems unduly harsh. Applying the Graeffe algorithm, (1) reproduces itself; hence the square of any root of (1) is also a root, all the roots are of modulus unity, and it is not a big step thence to deduce the roots in the forms

$$x = \cos 2\pi/5 \pm i \sin 2\pi/5,\ \cos 4\pi/5 \pm i \sin 4\pi/5.$$

Numerical values can be calculated by Runge's method,[3] applying the Graeffe algorithm to a modified equation obtained from (1) by adding a constant to the roots.

The simple Graeffe algorithm, applied to any algebraic polynomial equation whatever, separates the roots into groups having distinct[4] moduli, and determines the moduli, whatever their multiplicities. The real parts of all roots whose moduli are simple or double can be determined without ambiguity by using the additional algorithm of Brodetsky & Smeal.[5] These statements together imply that almost every practical equation can be completely solved by the combined algorithm; in those exceptional cases where moduli of higher multiplicity are found, however, the complete determination of the roots requires a further calculation with a different technique. The method of Brodetsky & Smeal uses an infinitesimal change of origin, and rests on the assumption that this change will not alter the way in which the equation breaks up under repeated application of the Graeffe algorithm. This assumption is not justified when roots exist having the same modulus but different real parts. Runge's method is still available, however, and can determine the roots without ambiguity. The factors corresponding to the completely determined roots should first be removed, and the change of origin then made: if there is more than one modulus of multiplicity higher than double, ambiguity is still avoidable by choosing the change of origin less than the least difference between the moduli concerned.

It remains to consider the case where roots exist having moduli so nearly equal that separation by the ordinary Graeffe process is too tedious to be practicable. Again it will be best to remove the factors corresponding to the other roots, leaving a residual equation to which we apply the algorithms of Graeffe, and Brodetsky & Smeal, after a change of origin.[6] In general, separation will be effected in this way, and the roots of the proposed equation can be inferred from those of the modified equation. If, on the other hand, separation of the modified equation is still impracticable, then the roots are clearly nearly equal (in conjugate pairs, if complex). Knowledge of the approximate moduli with or without the shift of origin enables us, as in the normal Runge process, to infer the approximate value of the roots. A high degree of accuracy is not to be expected, in view of the poorly determinate nature of nearly equal roots.[7]

This brief discussion shows that Graeffe's method, with the auxiliary processes which have been devised to determine the real parts of the complex roots, is competent to solve any algebraic polynomial equation whatever. If objections are to be raised against the method, it must not be on account of its lack of generality, but mainly on account of the fact that it is neither self-correcting nor self-checking. If one makes an error in the numerical work, then essentially the wrong equation is solved. This is a weakness in any computing process, but when the process is of a nature that seems to invite mistakes (and the wanderings of the decimal point under the Graeffe algorithm are most error-provoking), it constitutes a serious defect. A further defect is the difficulty of assessing the accuracy of the roots obtained: an independent checking and correcting process is desirable. For these and other reasons, the author has found that iterative methods of evaluating the roots are much to be preferred, save where root location is a matter of difficulty. A paper on some new and powerful iteration methods is being prepared; in the interim the methods given by SHIH-NGE LIN[8] are often all that is needed.

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[1] See E. T. WHITTAKER & G. ROBINSON, *The Calculus of Observations*, third ed., London, 1940.

[2] D. H. LEHMER, "The Graeffe process as applied to power series," *MTAC*, v. 1, p. 377f.

[3] C. RUNGE & H. KÖNIG, *Vorlesungen über numerisches Rechnen*, Berlin, 1924. The method was given earlier in C. RUNGE, *Praxis der Gleichungen* (*Sammlung Schubert*), Leipzig, 1900. [EDITORIAL NOTE: There was a second improved edition of this work (*Göschen's Lehrbücherei*, v. 2), Berlin, 1921.]

[4] There is a practical limitation here, to which we return below. The moduli must be sufficiently distinct to separate in a reasonable number of steps: regarding 8 as the reasonable limit, moduli in the ratios 1.0072 or 1.0036 to 1 will be separated, to 1 part in 10^4 or 10^8 respectively.

[5] S. BRODETSKY & G. SMEAL, "On Graeffe's method for complex roots of algebraic equations," Camb. Phil. So., *Proc.*, v. 22, 1924, p. 83f.

[6] A shift of origin through a distance about equal to the approximate value of the modulus is recommended as certain to separate the roots, if not actually nearly equal.

[7] A. OSTROWSKI, "Sur la continuité relative des racines d'équations algébriques," Académie d. Sci., Paris, *Comptes Rendus*, v. 209, 1939, p. 777f, has illustrated this very forcibly by comparison of $z^4 - 4z^3 + 6z^2 - 4z + 1 = 0$, roots 1, 1, 1, 1; with $z^4 - 4z^3 + 5.999951z^2 - 4z + 1 = 0$, roots 1.0872, .9198, .9965 $\pm$ .0836i.

[8] SHIH-NGE LIN, "A method for finding roots of algebraic equations," *J. Math. Phys.*, v. 22, 1943, p. 60f.

48. GUIDE (*MTAC*, no. 7), SUPPL. 2 (for Suppl. 1, see *MTAC*, v. 1, p. 403–404).—In L. SILBERSTEIN, *Bell's Mathematical Tables*, London, 1922, or *Synopsis of Applicable Mathematics with Tables*, New York, 1923, there are tables of $J_0(x)$, $J_1(x)$, $x = [0(.01)15.5;\ 5D]$, p. 143–150, and $K_0(x)$, $x = [0(.01)1(.1)9.9;\ 4D]$, p. 152.

G. GIORGI, "Sugli integrali dell'equazione di propagazione in una dimensione," Circolo Matem. d. Palermo, *Rendiconti*, v. 52, 1928, p. 311–312; tables of $I_1(x)$, $x = [0(.1)6(1)11;\ 4\text{–}6S]$, $Bn(x) = I_1(x)e^{-x}/x$, $x = [0(.1)6(.5)$-$9(1)20(5)30(10)100(100)500,\ 1000;\ 4\text{–}6D]$.

In R. C. KNIGHT, "The potential of a sphere inside an infinite circular cylinder," *Q. J. Math.*, v. 7, 1936, p. 130, there is a table of $I_{2n} = \int_0^\infty K_0(m)m^{2n}dm/I_0(m)$, $n = [0(1)6;\ 5D$ or $S]$. There are also tables for the same integral, but with the limits 0 to 1, 1 to 5, and 5 to ∞.

R. ZURMÜHL, "Zur numerischen Integration gewöhnlicher Differentialgleichungen zweiter und höherer Ordnung," *Z. angew. Math. Mech.*, v. 20, 1940, p. 116; tables of $y = \int_0^{\frac{1}{2}\pi} e^{-x\sin t}dt = \frac{1}{2}\pi[I_0(x) - \mathbf{L}_0(x)]$, and $-y' = 1 - \frac{1}{2}\pi[I_1(x) - \mathbf{L}_1(x)]$, $x = [0(.1)2(.2)10;\ 7\text{–}8D]$. Errors up to 2 units in the last decimal possible. $\mathbf{L}_0(x) = -i\mathbf{H}_0(ix)$ (Watson, p. 329). See also R. Müller, *Z. angew. Math. Mech.*, v. 19, 1939, p. 54, where there are tables of y and $-y'$, $x = [0(.1)1(.2)13.6(.4)16;\ 3\text{–}5D]$. On p. 53 Müller gives also a table of $\bar{K}_0(x)$, $x = [0, .02, .04, .1(.1)2.6(.2)13.6(.4)16;\ 6\text{–}9S]$, δ^2.

R. C. A.

49. A VOLUME OF TABLES BY KULIK.—Brown University has recently acquired a copy of tables by JAKOB PHILIPP KULIK (1793–1863), entitled *Handbuch mathematischer Tafeln*, and published by Christoph Penz at Graz, in 1824. liv, 149 p. + 1 p. Druckfehler. 13.8 × 20 cm. The book came from the fine library of CHARLES N. HASKINS of Dartmouth College. This volume is not listed by DE MORGAN (1861), GLAISHER (1873), HENDERSON (1926), LEHMER (1941), Wölffing (1903), or in catalogues of Bibliothèque Nationale (1925), British Museum (1890), Crawford Library of the Edinburgh Univ. (1890), Hamburg Math. So. Lib. (1890, 1894, 1906, 1913), Library of Congress (1944), Pulkowa Observatory (1860, 1880), R. Astron.

So. (1886, 1900), Stadtbibliothek of Frankfort a. M., math. Abt. (1909). In only two sources did I find the work listed, namely: in POGGENDORFF, *Biog.-liter. Handwörterbuch*, v. 1, Leipzig, 1863, a contribution from Kulik; and in D. BIERENS DE HAAN, Akad. v. Wetenschappen, Amsterdam, *Verhandelingen*, v. 15, 1875, "Tweede Ontwerp eener Naamlijst van Logarithmentafels."

In the "Vorerinnerung," dated November, 1823, Kulik states that his Tafeln are an extract from a larger work, to be published in the year 1824, and entitled *Collectio tabularum mathematico-physicarum locupletissima, vollständige Sammlung mathematisch-physicalischer Tafeln.* Among various volumes of tables which Kulik wrote I do not find that this one is ever mentioned, not even in the bibliography sent by Kulik to Poggendorff (1863) nearly 40 years later. But he did publish at Graz, in 1825, the following volume of 266 p.: *Divisores numerorum decies centena millia non excedentium. Accedunt tabulae auxiliares ad calculandos numeri cujuscunque divisores destinatae. Tafeln der einfachern Factoren aller Zahlen unter einer Million nebst Hülfstafeln zur Bestimmung der Factoren jeder grösseren Zahl.* This v. is mentioned in none of the bibliographies listed above, except Poggendorff (1863), but it is surveyed in BAASMTC, *volume V, Factor Table*, London, 1935, p. xiii.

About a decade ago L. J. C. drew my attention to *Astron. Nachrichten*, v. 3, 1825, col. 192, where Kulik describes a work with the following title: *Canon logarithmorum naturalium in 48 notis decimalibus pro omnibus numeris inter 1 et 11000 denuo in computum vocatus ab Jac. Phil. Kulik.* He stated (v. 4, 1826, col. 47) that 192 of the projected 288 pages had already been printed but that the printing of further pages depended upon securing subscribers. The book seems never to have been completed. Is there any library which has these 192 pages? See *Scripta Mathematica*, v. 4, 1936, p. 340.

In the *Handbuch* are 30 tables including the following:

1–2: All factors of numbers up to 21500 and the smallest factors up to 67100.
3–4: Squares and cubes of numbers up to 1000 and higher powers of numbers up to 100.
5: Square roots and cube roots of numbers up to 1000.
15, 19: Natural and logarithmic sin and tan.
16: Natural secants.
28: 11-place log of prime numbers up to 1811.

R. C. A.

50. ZEROS OF $z + \sin z$.—The zeros of $z + \sin z$ are situated symmetrically in the four quadrants. The first four zeros in the first quadrant were found as follows. A first approximation to the imaginary part, y, of the n^{th} zero is

$$y = \ln (4n - 1)\pi.$$

The imaginary part is a solution of

$$(\sinh^2 y - y^2)^{\frac{1}{2}} \coth y - \operatorname{arccos} (\sinh y) = 0$$

and was obtained by inverse interpolation. The real part of the zero is then given by

$$x = \arccos(-y/\sinh y).$$

Zeros of $z + \sin z$ where $z = x + iy$

n	$\pm x$	$\pm y$
1	4.2123922	2.2507286
2	10.7125374	3.1031487
3	17.0733649	3.5510873
4	23.3983552	3.8588090.

If one considers the roots of $\sin z = z$ as functions of n and interpolates for the roots corresponding to $n = 4\frac{1}{2}, 5\frac{1}{2}, \cdots, 9\frac{1}{2}$, one obtains the zeros of $z + \sin z$ for $n = 5, 6, \cdots, 10$. Using the first ten roots of $\sin z = z$ as given in an earlier article by HILLMAN & SALZER,[1] the above mentioned roots of $z + \sin z$ can be obtained to at least four decimal places.

NYMPT

BEATRICE S. MITTELMAN & A. P. HILLMAN

[1] A. P. HILLMAN & H. E. SALZER, "Roots of $\sin z = z$," *Phil. Mag.*, s. 7, v. 34, 1943, p. 575. See *MTAC*, v. 1, p. 141.

EDITORIAL NOTE.—In *Ingenieur-Archiv*, v. 11, 1940, p. 129, J. FADLE gave the first five zeros of $\sin z \pm z$, to 5D. Comparing with the seven-place values listed above, it appears that six last-figure endings of Fadle should be increased by unity, namely in the real parts of the second and fourth zeros, and in the imaginary parts of the first four zeros. Comparing Fadle's zeros of $\sin z - z$ to 5D with those found by Hillman & Salzer[1] to 6D, we find that all of Fadle's end-figures in the first three zeros, as well as the end-figure in the real part of the fourth zero, should be increased by unity.

QUERIES

16. TABLES OF $\sin nx/\sin x$.—Has any table been calculated for the function $\sin nx/\sin x$, for large integral values of n, say up to 100, and for values of x in radians?

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EDITORIAL NOTE: In NYMTP, *Tables of Sines and Cosines for Radian Arguments*, 1940, are values of $\sin x$, $x = [0(1)100; 8D]$.

QUERIES—REPLIES

19. CUBE ROOTS (Q 11, v. 1, p. 372; QR 15, v. 1, p. 432). The answer to Q 11 seems to be a definite 'No.' The table required is equivalent to one giving 5 or 6D for $N = 1000(0.1)2000$, and a table at 10 times this interval is already interpolable linearly, so that a printed table at interval 0.1, although it might be very convenient, cannot be considered an urgent need. For a table at interval 1, the 1930 and 1941 editions of *Barlow's Tables* seem most convenient.

Use of linear interpolation when the second difference is about 2 units means, of course, that the last figure is subject to a maximum error of about $1\frac{1}{2}$ units, whereas tabular values are usually kept within half a unit.