the tables described in No. 10 are in question. Because the reviewer in *MTAC*, v. 2, p. 186, followed the Harvard *Annals* v. 1, he listed "SCHEUTZ (1834)", rather than SCHEUTZ (1853).]

R. C. A.

A New Approximation to π

A. EDITORIAL NOTES: In *MTAC*, v. 2, p. 143–145 we noted various formulae which had been used for calculating π to many places of decimals. These included that of MACHIN (1706)

$$\frac{\pi}{4} = 4 \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{239}, \tag{1}$$

which was used by WILLIAM SHANKS (1812–1882) to compute π to 707D. The accuracy of this computation to 500D was verified by an independent calculation completed and published in 1854. No one appears to have checked the later figures until 1945, when Mr. D. F. FERGUSON, now connected with the Department of Mathematics of the University of Manchester, undertook the task. As we have already noted he used the formula

$$\frac{\pi}{4} = 3 \tan^{-1} \frac{1}{4} + \tan^{-1} \frac{1}{20} + \tan^{-1} \frac{1}{1985}, \tag{2}$$

given in LONEY's *Plane Trigonometry* (1893). We published his new computation for π from the 526th through the 620th decimal place (p. 145). Mr. Ferguson gave an account of his work in *Mathematical Gazette*, v. 30, May 1946, p. 89–90, and recorded there his figures for π from the 521st to the 540th decimal place. Mr. Ferguson found that Shanks' approximation to π was incorrect beyond 527D. By November 1946 he had carried on his calculations of the value of π to 700D, and by January 1947 to 710D.

In December 1945 we suggested to Dr. JOHN W. WRENCH, JR. that he might take up the wholly independent computation of π by means of Machin's formula (1). In April 1946 he reported that he was in communication with Mr. LEVI B. SMITH of Talbotton, Georgia, who began his work on computing $\tan^{-1} \frac{1}{239}$ in November 1940 and had by February 1944 completed the work to 820D, through the term $[173 \cdot 239^{173}]^{-1}$. Then Dr. W. took up actively the computation of $\tan^{-1} \frac{1}{5}$ so that his results might be combined with those of Mr. S. as in Machin's formula. He found the errors in work of Shanks, earlier pointed out by Mr. F., and others described below.

Early in January 1947 Dr. W. sent to us his new approximation to π to 808D given below, as a companion to the value of e to 808D (*MTAC*, v. 2, April 1946, p. 69). The value found by Mr. F. to 710D agrees with this.

π = 3.14159	26535	89793	23846	26433	83279	50288	41971	69399	37510
58209	74944	59230	78164	06286	20899	86280	34825	34211	70679
82148	08651	32823	06647	09384	46095	50582	23172	53594	08128
48111	74502	84102	70193	85211	05559	64462	29489	54930	38196
44288	10975	66593	34461	28475	64823	37867	83165	27120	19091
45648	56692	34603	48610	45432	66482	13393	60726	02491	41273
72458	70066	06315	58817	48815	20920	96282	92540	91715	36436
78925	90360	01133	05305	48820	46652	13841	46951	94151	16094
33057	27036	57595	91953	09218	61173	81932	61179	31051	18548
07446	23799	62749	56735	18857	52724	89122	79381	83011	94912
98336	73362	44065	66430	86021	39494	63952	24737	19070	21798
60943	70277	05392	17176	29317	67523	84674	81846	76694	05132
00056	81271	45263	56082	77857	71342	75778	96091	73637	17872
14684	40901	22495	34301	46549	58537	10507	92279	68925	89235
42019	95611	21290	21960	86355	44191	19716	02977	46113	09960
51870	72113	49999	99837	29780	49951	05973	17328	16096	31867
50244	594(55)								

B. Report of Mr. Smith & Dr. Wrench, January 1947

As stated in the Editorial Notes formula (1) was employed by the present writers in their joint calculation of π. The calculation of $\tan^{-1}\frac{1}{5}$ consisted essentially of checking and extending to 850D the values of the individual terms of the corresponding series as published to 530D by Shanks.[1] On the other hand, the computation of $\tan^{-1}\frac{1}{239}$ was carried out to 820D entirely independently of earlier calculations of that number, and the resulting values of the terms of the series were checked against the corresponding data of Shanks only when the investigation was nearly completed.

An important preliminary step in the new evaluation of $\tan^{-1}\frac{1}{5}$ consisted of the formation of a definitive table of powers of 2. This original table contains the exact values of 2^n for $n = 1(2)1207$, and has been collated with an unpublished table of non-consecutive powers to 2^{671} computed by Professor H. S. Uhler[2] and also with a table of 2^n, $n = 13(12)721$, given by Shanks.[1] No discrepancies were found. In addition to this comparison with previous tables of high powers of 2, every entry beyond 2^{531} in the new table was checked by the Fermat-Euler theorem.

The quotient arising from the application of this congruential check on the accuracy of the tabular value of 2^{2n-1} comprised, together with the appropriate number of antecedent zeros, the sequence of digits occupying the first $2n - 1$ decimal places of the approximation to the nth term of the series for $\tan^{-1}\frac{1}{5}$. If r denotes the residue determined by the preceding check, then the decimal evaluation of $r/(2n - 1)$ is also required, corresponding to all integral n between 1 and 425. Thus it was found desirable to calculate *de novo* a table of the complete periods of the reciprocals of all prime-powers (p^k, $k \geqq 1$, $p \neq 2, 5$) less than 800. In each of the many cases where the period consisted of an even number of digits an effective check involved the juxtaposition and subsequent addition of the two halves of the period so as to yield an unbroken sequence of 9's.[3] All remaining cases of evaluation of $r/(2n - 1)$ were checked by duplicate machine calculation, as was the summation of the terms of the series.

The calculation of the successive terms of the series for $\tan^{-1}\frac{1}{239}$ was performed by the recurrence formula

$$U_{n+1} = (2n - 1)U_n/(2n + 1)239^2,$$

where U_n and U_{n+1} denote respectively the nth term and its successor. The computations were carried to at least 820D and were checked modulo $10^{10} + 1$ every hundred decimal places. The same checking procedure was applied to the respective sums of the positive and negative terms of the series.

The final values of $\tan^{-1}\frac{1}{5}$ and $\tan^{-1}\frac{1}{239}$ were compared with the corresponding data to 709D of Shanks, and several errata in the latter were discovered. In addition to the two errors (described in **C**) which were independently discovered by Mr. Ferguson in Shanks' value of $\tan^{-1}\frac{1}{5}$, there exist in the same number a unit error in the 533rd place and a residual error of approximately $5.2193762669 \times 10^{-602}$. Shanks' approximation to $\tan^{-1}\frac{1}{239}$ is also erroneous, for it exceeds the present estimate of that number by nearly $4.77447473 \times 10^{-592}$.

Appended to this report are values of (**a**) $\tan^{-1}\frac{1}{5}$, and (**b**) $\tan^{-1}\frac{1}{239}$, both curtailed to 811D from more extended approximations appearing on the work sheets.

(a)

0.19739	55598	49880	75837	00497	65194	79029	34475	85103	78785
21015	17688	94024	10339	69978	24378	57326	97828	03728	80441
12628	11807	36913	60104	45647	98867	94239	35574	75654	95216
30327	00522	10747	00156	45015	56006	12861	85526	63325	73186
92806	64389	68061	89528	40582	59311	24251	61329	73139	93397
11323	35378	21796	08417	66483	10525	47303	96657	25650	48887
81553	09384	29057	93116	95934	19285	18063	64919	69751	94017
08560	94952	73686	73738	50840	08123	67856	15800	93298	22514
02324	66755	49211	02670	45743	78815	47483	90799	78985	02007
52236	96837	96139	22783	54193	25572	23284	13846	47744	13529
09705	46512	24383	02697	56051	83775	74220	87783	58531	52464
74933	09145	87633	82311	24903	32030	12680	51006	70223	31257
50509	42448	46026	71622	54894	07922	61404	67995	06236	59692
82873	05828	78720	53603	03457	07660	66681	37431	25662	67431
40899	26057	41703	54539	40465	13623	01101	58081	00262	13759
92595	89071	66648	51452	55706	79954	88100	43132	95466	83892
79036	88309	3							

(b)

0.00418	40760	02074	72386	45382	14959	28545	27410	48065	30763
19508	27019	61288	71817	78341	42289	32737	82605	81362	29094
54975	45066	64448	63756	05245	83947	89311	86505	89221	28833
09280	08462	71962	33077	33759	47634	60331	84734	14570	33198
60154	54814	80599	24498	30211	46039	12539	49527	60779	68815
58881	27339	78533	46518	04574	25481	35867	46447	51979	10232
83097	70020	64652	82763	46532	96910	48183	86543	56078	91959
14512	32220	94463	68627	66155	20831	67964	26465	74655	11032
51034	35262	82445	12693	55670	49968	44452	47904	33177	28393
07086	31401	93869	51950	37058	64107	70855	85540	45223	55388
14237	67708	36515	69182	52702	00229	30895	44950	04358	54409
34496	44014	24187	24950	92283	86239	54553	33565	11719	73747
02023	49475	97790	97469	50111	88854	76673	97957	31537	09303
27821	13089	84258	30836	77190	91008	39098	51655	10419	22416
78092	05326	86491	62667	40271	68444	24477	31579	64520	27549
57415	88258	29094	05850	90382	07331	75908	43199	77843	27604
28586	38373	5							

[1] W. Shanks, *Contributions to Mathematics comprising chiefly the Rectification of the Circle to 607 Places of Decimals*, London, 1853.
[2] *MTAC*, v. 2, p. 224, N66.
[3] H. Rademacher & O. Toeplitz, *Von Zahlen und Figuren*, second ed. Berlin, 1933, p. 133–135.

C. Report of Mr. Ferguson, January 1947

In calculating my value of π to 710D I made use of formula (2) and found the following results for (a) $\tan^{-1}\frac{1}{4}$, (b) $\tan^{-1}\frac{1}{20}$, and (c) $\tan^{-1}\frac{1}{1985}$:

(a)

0.24497	86631	26864	15417	20824	81211	27581	09141	44098	38118
40671	27375	91466	73551	19587	64209	65745	34157	66870	19913
63834	80449	00371	18374	29548	54209	95059	97695	89869	60614
20373	52012	77087	38758	16557	21586	71598	26385	50632	05220
87873	06750	14341	56233	63482	63956	36978	08521	59107	32458
35238	13507	62999	55568	90112	58302	66262	33025	99157	53281
02760	62335	60275	36107	52021	78574	13846	85151	60692	64028
13351	40849	79441	00602	37988	46394	26115	26552	60206	13960
37795	44423	80726	04741	94391	78861	19777	28818	69907	30107
50590	15074	22498	58439	41420	59686	03414	25179	53473	22290
10855	05491	71108	69928	36288	80499	01695	01187	90130	24132
23900	98621	56045	96838	55524	50318	46160	29411	55558	05517
39341	51460	87605	90907	76086	95391	27283	64199	28418	54722
42486	09165	39440	71147	52423	62989	80883	85199	09565	57220
81885	50930	(59)							

(b)

0.04995	83957	21942	76141	00062	87034	84488	14912	77080	42350
71744	10853	45482	99835	95476	71033	50612	64888	70485	01265
49675	88718	56799	74803	45043	78235	17343	64195	86075	35558
34705	50031	66812	64425	55070	35889	99864	21844	62020	22011
35398	44491	94479	55125	91884	70605	15358	82203	57911	15507
66709	70265	20884	04697	53559	08904	34425	33211	75071	00898
99983	99369	89611	53196	70717	40134	40774	24235	31335	37603
73612	47259	31779	72222	52596	59464	82850	02739	09656	29682
61838	28530	42311	66214	89812	84597	81323	80425	73403	65277
35640	82643	15372	91283	73850	56089	49548	56557	20164	84879
81610	72192	83012	94406	89240	40051	11637	64820	56557	65999
47240	24101	35373	55511	99718	11544	33853	54021	44594	33781
36222	03768	16540	61055	38956	20032	50668	29159	05403	66710
93525	58744	35937	48968	67734	87127	37233	28015	36651	95672
97735	23477	(67)							

(c)

0.00050	37782	94913	08568	94071	15151	20340	68155	82974	27671
70794	50754	94924	49051	47331	91562	48721	41344	62457	31663
54356	72097	25292	46735	43656	70658	85121	98509	57714	34631
16201	87555	72950	86848	41560	75739	76456	56371	24816	21875
12054	38001	54144	09788	49786	28731	58173	88022	96546	91890
13988	03384	98768	15748	37461	32808	20136	07891	78079	24576
34848	81139	81141	28185	10421	04373	41755	93445	09515	54421
06064	77781	30180	52296	70643	13015	42264	54341	08263	07459
83039	69957	29909	17547	54316	44112	04827	48412	99637	24037
14450	28084	07818	72581	81602	03033	62489	37749	65168	46978
10408	29672	64677	37415	73398	53325	49265	37800	02819	17053
46292	72603	22836	58266	41037	79381	23834	28205	57905	00949
45767	62167	06957	55242	02247	36629	36425	52266	02749	98589
82687	23984	76364	21164	11631	63537	47742	14457	26882	79973
42113	22633	(35)							

$(b) + (c) = \tan^{-1}(5/99)$, which was independently computed and furnished complete agreement to 710D+.

My procedure for calculating $\tan^{-1} \frac{1}{4}$ was as follows:

(i). calculated $(1/4)^{2n+1}$ by dividing $(1/4)^{2n-1}$ by 16;
(ii). multiplied the result by 4096 and compared with $(1/4)^{2n-5}$;
(iii). divided $(1/4)^{2n+1}$ by $(2n + 1)$;
(iv). multiplied this by $16(2n + 1)$ and compared with $(1/4)^{2n-1}$;
(v). after copying $(1/4)^{2n+1}/(2n + 1)$ for the purpose of the series, checked by multiplying the copied figures by $2n + 1$.

At all steps of the work ample margins of overlap were allowed.

In the course of my work I discovered two errors in the results of Shanks. The explanation of the first of these which vitiates his final result beyond 527D was noted in January 1946 and is as follows: It was a question of an omission in the evaluation of the term $[497 \cdot 5^{497}]^{-1}$ in the 531st decimal place. I found the value to be (through 547D)

00804 82897 38430 58350,

while Shanks, carelessly omitting a zero, used

00848 28973 84305 83501

The second Shanks' error was the omission from 569D+ of the term $5^{-29}/29$ which comes in the series for $\tan^{-1} \frac{1}{5}$.