Part I, Digital machines, is disappointingly brief (12 p.) and is divided into counters, adders, multipliers, and "the punch card machine." The discussion is largely devoted to mechanical parts used in desk calculators such as the Leibniz wheel and Napier's bones. There is a description of the Hollerith card sorter.

Part II, Continuous operators, sounds the keynote of the text: Numerical quantities can be represented by physical magnitudes. The magnitudes discussed range from linear displacements to the phase angle of alternating currents. There is a good treatment of linear networks. Among multiplying devices there is a description of "square" gears, variably wound potentiometers, and rectifiers. Integrators and differentiators, both mechanical and electrical, are treated in great detail. The rest of Part II is devoted to the theory of amplifiers, servomechanisms, selsyn units and other electrical devices and their uses in mathematical machines.

Part III, The solution of problems, is devoted to composite machines for solving systems of linear equations, ordinary and partial differential equations. These machines include the network analyzer, differential analyzer and two electronic computers for linear equations. One of the latter has been designed by the author and is fully described. The mathematical treatment here is particularly interesting. The reader will find this material under "Adjusters" (p. 84–94, unfortunately the book has no index).

Part IV, Mathematical Instruments, is concerned with planimeters, integrometers, harmonic analyzers and cinema-integraphs. There is a page and a half of bibliography arranged topically. This does not include a large number of references inserted in the text.

The reader, whether he be interested in mathematical machines from a technical or a purely mathematical point of view, will find something interesting on every page. It is to be hoped that a second volume dealing with the theories of the many other interesting devices developed during the war may be eventually forthcoming.

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NOTES

73. THE CHECKING OF FUNCTIONS TABULATED AT CERTAIN FRACTIONAL POINTS.—Many functions involving a parameter ν, in particular Bessel functions $J_\nu(x)$, $Y_\nu(x)$, $I_\nu(x)$, $K_\nu(x)$, etc., besides being tabulated for $\pm$ integral values of ν, as well as for $\nu = 0$, are often given for non-integral values of ν between -1 and 1, especially for $\nu = \pm \frac{3}{4}, \pm \frac{2}{3}, \pm \frac{1}{2}, \pm \frac{1}{3}$ and $\pm \frac{1}{4}$. When it is desired to perform the equivalent of a differencing check upon these or related functions (e.g. the zeros of these functions) considered as a function of ν for fixed x, due to the irregular interval in ν, it is necessary to take the divided differences. For any fixed set of n ν's, it is possible to obtain coefficients of f_ν for the last, i.e. $(n-1)$th, divided difference which should vanish if the function behaves as a polynomial of the $(n-2)$th degree in ν. Thus an error ϵ in any entry f_ν (this includes rounding errors) will usually show up by being multiplied by the coefficient of f_ν.

The coefficients which are given below are for three important cases likely to arise in practice, especially with Bessel functions:

(a) 7th divided difference for f_ν involving the 8 points $\nu = \pm \frac{3}{4}, \pm \frac{2}{3}, \pm \frac{1}{3}, \pm \frac{1}{4}$.
(b) 10th divided difference for f_ν involving the 11 points $\nu = \pm 1, \pm \frac{3}{4}, \pm \frac{2}{3}, \pm \frac{1}{3}, \pm \frac{1}{4}, 0$.
(c) 10th divided difference for f_ν involving the 11 points $\nu = \pm \frac{3}{4}, \pm \frac{2}{3}, \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{4}, 0$.

In case (c), omitting the $f_{\pm\frac{2}{3}}$ and $f_{\pm\frac{1}{3}}$ leaves 7 points at the uniform interval of $\frac{1}{4}$ in ν, which might be amenable to an ordinary differencing check. It is too cumbersome to work with more than 11 points f_ν. At any rate, if to (c) there were added f_ν for $\nu = \pm 1$, the set of ν's would include ± 1, $\pm \frac{3}{4}$, $\pm \frac{1}{2}$, $\pm \frac{1}{4}$ and 0, which are 9 points, again at the uniform interval of $\frac{1}{4}$, and to which an ordinary differencing check could be applied. The joint use of any two of (a), (b), or (c), when possible, lessens the likelihood of passing a double error which, by compensation, might yield a small divided difference in one case.

Divided Difference Formulae:

(a) $\nu = \pm \frac{3}{4}, \pm \frac{2}{3}, \pm \frac{1}{3}, \pm \frac{1}{4}$; 7th divided difference

$$= \frac{1}{85085}[21\,28896\,(f_{\frac{3}{4}} - f_{-\frac{3}{4}}) - 42\,45696\,(f_{\frac{2}{3}} - f_{-\frac{2}{3}})$$
$$+ 174\ 49344\,(f_{\frac{1}{3}} - f_{-\frac{1}{3}}) - 183\ 30624\,(f_{\frac{1}{4}} - f_{-\frac{1}{4}})].$$

(b) $\nu = \pm 1, \pm \frac{3}{4}, \pm \frac{2}{3}, \pm \frac{1}{3}, \pm \frac{1}{4}, 0$; 10th divided difference

$$= \frac{1}{4\ 25425}[10\,50192\,(f_{-1} + f_1) - 324\,40320\,(f_{-\frac{3}{4}} + f_{\frac{3}{4}}) + 573\,16896$$
$$\times (f_{-\frac{2}{3}} + f_{\frac{2}{3}}) - 2944\ 57680\,(f_{-\frac{1}{3}} + f_{\frac{1}{3}})$$
$$+ 3910\ 53312\,(f_{-\frac{1}{4}} + f_{\frac{1}{4}}) - 2450\ 44800 f_0].$$

(c) $\nu = \pm \frac{3}{4}, \pm \frac{2}{3}, \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{4}, 0$; 10th divided difference

$$= \frac{1}{4\ 25425}[454\ 16448\,(f_{-\frac{3}{4}} + f_{\frac{3}{4}}) - 1637\ 62560\,(f_{-\frac{2}{3}} + f_{\frac{2}{3}})$$
$$+ 5376\ 98304\,(f_{-\frac{1}{2}} + f_{\frac{1}{2}}) - 18845\ 29152\,(f_{-\frac{1}{3}} + f_{\frac{1}{3}})$$
$$+ 19552\ 66560\,(f_{-\frac{1}{4}} + f_{\frac{1}{4}}) - 9801\ 79200 f_0].$$

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74. POISSON'S OR DAWSON'S INTEGRAL AND ANOTHER INTEGRAL.—In W. O. SCHUMANN, *Elektrische Durchbruchfeldstärke von Gasen*, Berlin, 1923, p. 243 and 241, are tables of $f(x) = \int_0^x e^z dt$, 2–5S, $z = t^2$, of $\log f(x)$, 4–5S, and of $e^{-w}f(x)$, 3–6D, $w = x^2$, each for $x = [.1(.1)2.6(.2)7.4]$. The reference to Schumann in FMR *Index*, p. 219, cannot be verified. See *MTAC*, v. 2, p. 55, N**45**; and RMT **378** and **424**, ARLEY. On p. 242 are graphs of $f(x)$, and on p. 234–235 is a small 4D table of $MS(x) = \int_0^x e^{-u}dt$, $u = t^4$, for $x = 0(.1)1, \infty$; graphs of $S(x)$, and of $MS(x)$, are given on p. 237. $M = \Gamma(1.25)$ is the value of the integral when $x = \infty$, and is approximately .9064. More generally BIERENS DE HAAN, *Nouvelles Tables d'Intégrales Définies*. Leyden, 1867, and New York, 1939, Table 26(4), gives $\int_0^\infty e^{-v}dt = n^{-1}\Gamma(n^{-1})$, if $v = t^n$. But $\int_0^x e^{-v}dt = n^{-1}\Gamma(n^{-1}, x^n) = xM(n^{-1},\ 1 + n^{-1},\ -x^n)$ where $\Gamma(p + 1, x) = \int_0^x e^{-x}x^p dx$, or as in K. PEARSON, *Tables of the Incomplete* Γ*-function*, London, 1922, $y_n = n[\Gamma(1/n)]^{-1} \int_0^x e^{-v}dt = I(u, p) = I[\sqrt{n}\, x^n, (1 - n)/n]$. For $n = 4$, there is a table, p. 118–126, of y_4 for $u = 2x^4 = [.1(.1)27; 7D]$, $p = -.75$; also, p. 164, $u = [0(.1)6; 5D]$. See also the little tables for this particular case by F. EMDE, *Z. f. angew. Math. u. Mech.*, v. 14, 1934, p. 336–339, $S(x)$, $x = [.8, .9(.01)1, 1.1; 6D]$.

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