Guide to Tables of Elliptic Functions

by

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CONTENTS

Note: In the various parts, differences are referred to as in FMR *Index*. In the description of a table, Δ means that first differences only are given; Δ^2 means first and second differences, δ^2 means second differences only, $\delta_m{}^2$ means modified second differences, Δ^4 means first to fourth differences, δ^4 means second and fourth differences, and so on.

EDITORIAL NOTE

Of material which HARRY BATEMAN had prepared for his Subcommittee L–Report on Higher Mathematical Functions we have already published the part dealing with BESSEL Functions (*MTAC*, no. 7, 1944). Since in varied research problems there seemed now to be an increasing demand for tabular information regarding Elliptic Functions, we invited Dr. FLETCHER, of the University of Liverpool, to prepare a Guide to this material as another part of our Section L–Report,—quite independent of anything H.B. had collected. The following comprehensive survey indicates not only all tables, but also all known errata, including many previously unpublished, discovered in Dr. FLETCHER's personal investigations. We are indeed grateful for this valuable aid in the promotion of scientific inquiry.

R. C. A.

PREFACE

In the title, "elliptic functions" is to be understood in a general sense, as including elliptic integrals and theta functions; in fact, existing tables, and therefore the present *Guide*, relate largely to elliptic integrals. "Tables" means numerical tables; extensive lists of formulae are given in several of the works quoted, but no account is given here of the enormous amounts of algebraical calculation which have been devoted to transformations of order higher than the third. The outlook is practical, and in sympathy with LUCIEN LÉVY's dictum that "the most elementary properties of the elliptic functions are the most useful."

It will be observed that the *Guide* is divided into the following three parts: Part I, a descriptive *Guide* to all tables considered, referring, with constant obvious abbreviations, to material listed in the detailed bibliography of Part II; and Part III, devoted to lists of errata in certain tables marked with an asterisk in Part II.

There is naturally in Part I an appreciable amount of duplication with FMR *Index* regarding material published before 1946; but in the present Part II much greater detail is offered, and Part III is almost entirely new.

Great care has been taken in Part III to indicate the nature of the verifications made and the status of the error lists given. A mere list of known errors, unaccompanied by any statement regarding its degree of completeness, is of limited use to the computer. Unless it can be asserted that the table in question contains no other gross errors, or at any rate no errors exceeding a definite size, he is still in the position of not having complete confidence in any entry. In several cases I have stated that a table is free from gross error (or free from any error); such a statement is more valuable than an imposing but uncharacterized list of errors. In a few cases where I have not examined a table, but merely communicate a few errors found by others, I have been careful to make clear the limited nature of the information given.

For information on the numerical calculation of elliptic integrals and functions, one may consult, among other references, L. V. KING, *On the Direct Numerical Calculation of Elliptic Functions and Integrals*, Cambridge, 1924, and a number of papers by S. C. VAN VEEN in Akad. Wetens., Amsterdam, *Proc.*, v. 44–45, 1941–42.

In performing about half the differencing which was done on the National accounting machine in the Mathematical Laboratory at Liverpool I had the invaluable assistance of Miss OLIVE E. VANES. I also wish to thank various library officials, friends and colleagues, and the University of Liverpool, for encouragement and arrangement of facilities for carrying on my investigations; and Professor R. C. ARCHIBALD for his encouragement and his care in editing the manuscript.

PART I

GUIDE TO TABLES

Section I: TABLES CONCERNING THE MODULUS

The description of tables of elliptic integrals and functions is complicated by the fact, which will become very noticeable, that various ways of specifying the modulus k have been used, so that the various tables employ several different argument-systems. The older use of the modular angle $\sin^{-1} k$ (which we shall denote by θ, except, for a special reason, in Section IX) and the more recent use of k^2 are by far the most important for general mathematical purposes, since they facilitate tabulation of functions of the complementary modulus k' defined by $k' = \sqrt{(1 - k^2)} = \cos\theta$ (k and k' will almost always be real and between 0 and 1); but other arguments (k, k', $\log k$, $\log k'^2$, $\tan^2\theta$, etc.) are convenient for special purposes.

Several tables for conversion of system (tables of θ against k^2, k^2 against θ, and k against k^2) are given in HAYASHI 1, 2. These tables contain a number of errors. In most cases a computer, presumed to be using a calculating machine, will do best to perform his own squarings and extractions of square root, when a reliable table, such as BARLOW-COMRIE, does not suffice; while the standard trigonometric and logarithmic tables are usually to be preferred, because of their great extent and accuracy, to any small table offered among elliptic tabulations.

Two tables for applying LANDEN's transformation will now be mentioned. The formulae for decreasing the modulus from k to k_1, so making $K_1'/K_1 = 2K'/K$, that is, doubling the period ratio, are

$$k_1 = (1 - k')/(1 + k') < k, \qquad k_1' = 2\sqrt{k'}/(1 + k') > k'.$$

$k_1'^2$ is tabulated to 7D with first differences for $k'^2 = 0(.001)1$ in NAGAOKA & SAKURAI 1 (p. 49). Successive applications of Landen's transformation provide a method of computing elliptic integrals, and 14D logarithms of the sequence of moduli and complementary moduli are given in LEGENDRE 3 (T.VI), 5 (T.VI) for $\theta = 0(0°.1)15°(0°.5)45°$.

Elliptic calculations are frequently made by methods involving the evaluation of arithmetic-geometric means, usually between 1 and k'. The process is a variant form of repeated Landen transformation. Usually only its end-result is of interest. But on account of their extent, it may be noticed that the first six sets of number triplets are given to 17D for $k^2 = 0(.01)1$ in the unpublished NBSCL 2. See also the description of BARTKY 1 in Section IX.

LEGENDRE 4 (p. 226) gives a table of modular angles in the cubic transformation. If the modular angles are θ and λ ($\theta < \lambda$), and the corresponding quarter-periods are K, K', Λ, Λ', then $K'/K = 3\Lambda'/\Lambda$, provided that

$$p = \sin(M - 30°)/\cos M, \quad q = \cos(M - 30°)/\sin M, \quad 30° \leqslant M \leqslant 60°,$$
$$\sin^2\theta = p^3 q, \qquad \sin^2\lambda = pq^3.$$

LEGENDRE 4 gives θ and λ to 5D of 1° for $M = 30°(1°)60°$. At $M = 45°$ we have the well-known case $\theta = 15°$, $\lambda = 75°$; moreover the second half of the table may be written down from the first, by changing M, θ, λ into $90° - M$, $90° - \lambda$, $90° - \theta$ respectively. For a related table, see F. W. NEWMAN 1, 2 in Section XI.

Section II: COMPLETE ELLIPTIC INTEGRALS OF THE FIRST AND SECOND KINDS

The incomplete elliptic integrals of the first and second kinds will be denoted by, respectively,

$$F(\phi) = \int_0^\phi (1 - k^2\sin^2\phi)^{-\frac{1}{2}}d\phi \qquad \text{and} \qquad E(\phi) = \int_0^\phi (1 - k^2\sin^2\phi)^{\frac{1}{2}}d\phi.$$

These are functions of two variables, namely ϕ and k (or $\sin^{-1} k$, k^2, etc.). In this section we are concerned only with the corresponding complete integrals in which the upper limit is $\frac{1}{2}\pi$, which we shall denote as usual by

$$K = \int_0^{\frac{1}{2}\pi} (1 - k^2\sin^2\phi)^{-\frac{1}{2}}d\phi \qquad \text{and} \qquad E = \int_0^{\frac{1}{2}\pi} (1 - k^2\sin^2\phi)^{\frac{1}{2}}d\phi.$$

These are functions of one variable, indicating the modulus, only, and are what is most often required in applications of elliptic integrals. Along with tables of K, E and their logarithms, we consider tables of $2K/\pi$, $2E/\pi$ and $\pi/2K$; the last of these is equal to M, the arithmetic-geometric mean of 1 and k'. Sections III–VII will be concerned with further expressions involving K, E and the corresponding functions K', E' of the complementary modulus.

As k tends to unity, there is apt to be difficulty in interpolating a value of K (which tends to infinity), and to a smaller extent one of E. Auxiliary tables for the calculation of K and E in such a case have been given by various authors. These are duly listed in **E** below. Attention is particularly drawn to the recent table of KAPLAN 1 (listed in $\mathbf{A}_3$ below), which tabulates K and E to 10D with argument $\log k'^2$. It is interesting to find that, with this argument, the second differences of K become less than those of E. Towards the end of KAPLAN's table, in the region where interpolation in ordinary tables is practically impossible, interpolation is linear to 9D in both K and E.

The lists at the end of this Section refer to tables in which k^2 is real. But diagrams for K and E when k^2 is complex are given in JAHNKE & EMDE 1_2–1_4.

The tables of CAMBI 1 tabulate complete integrals for moduli given by

$$k^2 = -e^{ix} = e^{-iy}, \qquad k'^2 = 1 + e^{ix} = 1 - e^{-iy},$$

where $y = \pi - x$. For $x = 0(\pi/20)\pi$, with additional arguments when necessary near

$x = \pi$ ($k^2 = 1$), the tables give, to 7–10D, values of

$$k',\ K,\ K',\ K/K',\ -\log|q'|,\ k'K \text{ and } e^{\frac{1}{2}i\psi}(1 - E'/K').$$

On the circle of tabulation, $q' = e^{-\pi K/K'}$ is purely imaginary. For fuller details, see RMT **565**.

II–A₁. *K and E, argument θ*

15D, Spenceley 1, $\theta = 0(1°)90°$
12D, Legendre 3 (T.VIII), 5 (T.VIII), 7, $\theta = 0(1°)90°$, Δ^6 to 45°, then Δ
12D, Moseley 1, $\theta = 0(1°)90°$, from Legendre
12D, K only, F. W. Newman 2 (p. 29, 132), $\theta = 0(1°)90°$, from Legendre
9–10D, Legendre 3 (T.IX), 5 (T.IX), 6, 7, 8, $\theta = 0(1°)90°$ ($\phi = 90°$ in double-entry tables)
9–10D, Hippisley 1, $\theta = 0(1°)10°(5°)80°(1°)90°$
9D, S.A.E.W.E.B. 1 (p. 345), $\theta = 0(2°.5)90°$
7D, Boll 1 (p. 351), $\theta = 0(1°)70°(0°.5)80°(0°.2)89°(0°.1)90°$
7D, K only, Láska 1, $\theta = 0(1°)90°$
6D, Rosa & Cohen 1 and Rosa & Grover 1, $\theta = 0(1°)90°$, Δ^2
6D, Gray 1_2, $\theta = 0(1°)90°$, Δ
6D, Russell 1, $\theta = 0(1°)90°$
5D, Kiepert 1, $\theta = 0(0°.2)90°$
5D, Lévy 1, Hancock 1, Hayashi 2, $\theta = 0(1°)70°(0°.5)80°(0°.2)89°(0°.1)90°$
5D, Dale 1, Fricke 1, $\theta = 0(1°)90°$
5D, K only, Dwight 2, $\theta = 86°(1')90°$, Δ
4D, Silberstein 1, $\theta = 0(1°)45°(0°.5)80°(0°.2)89°(0°.1)90°$
4D, Peirce 2, Burington 1, Allen 1, $\theta = 0(1°)65°(0°.5)80°(0°.2)89°(0°.1)90°$
4D, Rosenbach, Whitman & Moskovitz 1, $\theta = 0(1°)70°(30')80°(12')89°(6')89°30'(3')90°$
4D, Jahnke & Emde 1, Glazenap 1, $\theta = 0(1°)70°(0°.5)80°(0°.2)89°(0°.1)90°$
4D, Ligowski 1, $\theta = 0(0°.5)89°.5(0°.1)90°$
4D, Hodgman 1, $\theta = 0(1°)90°$, also K to 3D for $\theta = 85°(6')89°(2')89°40'(1')90°$
4D, Fowle 1, Peirce 1, Hütte 1, $\theta = 0(1°)90°$
3D, K, Dwight 1, $\theta = 0(1°)50°(30')70°(12')82°(6')89°(2')89°40'(1')90°$, Δ
3D, E, Dwight 1, $\theta = 0(1°)90°$, Δ
3D, Montessus de Ballore 1, small tables

II–A₂. *K and E, argument k²*

16D, K only, NBSCL 2, $k^2 = 0(.01)1$
12D, K, Hayashi 1, $k^2 = .835(.001)1$
10D, K, Hayashi 1, $k^2 = 0(.001).834$, $k^2 = 0(10^{-7})10^{-6}(10^{-6}).00249$
8D, K, Hayashi 1, $k^2 = .00250(10^{-5}).003$, $1 - k^2 = 0(10^{-7})10^{-6}(10^{-6}).003$
10D, E, Hayashi 3, $k^2 = 0(.001)1$
9D, Milne-Thomson 2, $k^2 = 0(.01)1$, Δ
7D, Samoĭlova-Îakhontova 1, $k^2 = 0(.001)1$, Δ
7D, Milne-Thomson 3, $k^2 = 0(.01)1$
6D, Nagaoka & Sakurai 1, $k^2 = 0(.001)1$, Δ
5D, Hayashi 2, $k^2 = 0(.001)1$
5D, E, Dwight 3, $k^2 = 0(.001)1$, $I\Delta$
4D, K, Dwight 3, $k^2 = 0(.001).997(10^{-5}).99999(10^{-7})1$, $I\Delta$ in most of range
4D, Jahnke & Emde 1_3, 1_4, $k^2 = 0(.01)1$, $I\Delta$

II–A₃. *K and E, other arguments*

12D, K, Fletcher 3, $k = 0(.01).7(.005)1$, Δ^n
12D, E, Fletcher 3, $k = 0(.01).9(.005)1$, Δ^n
11D, E only, Schmidt 1 (p. 204), $k' = 0(.01)1$
10D, Kaplan 1, $\log k'^2 = -1(.005) - 2(.01) - 6$, $\mu\delta^2$ or $\frac{1}{2}\mu\delta^2$
10D, Fletcher 2, $k = 0(.01)1$

7D, Rosa & Cohen 1 and Rosa & Grover 1, $\tan\theta = 0(.1)1(.5)3(1)5(2.5)12.5$
5D, E only, Schlömilch 1, $k' = 0(.01)1$
4D, $4E$ only, Jelínek 1, $k' = 0(.01)1$
4D, $2E$ only, Moore 1, $k' = 0(.01)1$ [very inaccurate, see Part III]
4D, E, Cecconi 1, $k = 0(.01)1$ [see also Section VII-4]
3D, E only, Wayne 1, $k' = 0(.01)1$

II-B. *Log K and log E*

12D, Legendre 3 (T.I), 5 (T.I), 8, $\theta = 0(0°.1)90°$, Δ^3 to 70°, then Δ^4. (From 0 to 15° and from 75° to 90°, 14D values, not very accurate, are given for the functions, but the differences throughout are those of the 12D values.)
8D, Rosa & Cohen 1 and Rosa & Grover 1, $\theta = 45°(0°.1)90°$, Δ^2
8D, Russell 1, $\theta = 80°(0°.1)90°$, Δ^2
7D, Bertrand 1 (p. 714), $\theta = 0(0°.5)90°$, K and E called ω and ζ
7D, Greenhill 3, Potin 1, $\theta = 0(0°.5)90°$
7D, Legendre 1 (p. 118), $\theta = 0(1°)90°$
7D, Rosa & Cohen 1 and Rosa & Grover 1, $\tan\theta = 0(.1)1(.5)3(1)5(2.5)12.5$
6D, Hodgman 1, $\theta = 0(1°)90°$, also $\log K$ to 5D for $\theta = 85°(6')89°(2')89°40'(1')90°$
6D, Fowle 1, $\theta = 0(1°)90°$
5D, Bohlin 1, Witkowski 1, $\theta = 0(0°.1)90°$
4D, Hoüel 1, $\theta = 0(1^g)100^g$

II-C. $2K/\pi$, $2E/\pi$, $M = \frac{1}{2}\pi/K$ *and their logarithms*

12D, $2K/\pi$, $2(K-E)/\pi$, Kaplan 2, $\tan^2\theta = -.005(.005)+.160$, δ_m^2
6D, $2K/\pi$, $2E/\pi$, Heuman 1, $\theta = 0(0°.1)90°$, Δ
9D, $2K/\pi$, also $(2K/\pi)^2$, S.A.E.W.E.B. 1 (p. 345), $\theta = 0(2°.5)90°$
14D, $\log(2K/\pi)$, Legendre 3 (T.VI), 5 (T.VI), $\theta = 0(0°.1)15°(0°.5)45°$
4D, $\log(2K/\pi)$, Hoüel 1, $\theta = 0(1^g)100^g$
17D, M, NBSCL 2, $k^2 = 0(.01)1$
11-15D, M, Gauss 2 (p. 363), $k' = .2, .6, .8$
12D, M, Fletcher 3, $k = 0(.01).7(.005)1$, Δ^n
10D, M, Fletcher 2, $k = 0(.01)1$
10D, $M\ (=1/\vartheta_3^2)$, Milne-Thomson 2, $k^2 = 0(.01)1$, Δ
12D, M, Hayashi 1, $k^2 = .835(.001)1$
8D, M, Hayashi 1, $k^2 = .997(10^{-5}).99999(10^{-7})1$
7D, M, $\log M$, Gauss 2 (p. 403), $\theta = 0(0°.5)90°$
See also Section V.

II-D. K'/K, E/K

10D, K'/K, Hayashi 1, $k^2 = 0(.001)1$
8D, K'/K, K/K', Hayashi 1, $k^2 = 0(10^{-7})10^{-5}(10^{-5}).003$
7D, E/K, Milne-Thomson 4, $k^2 = 0(.1)1$

II-E. *Auxiliary tables, mainly for fairly large k*

13-12D, K_1, K_2, E_1, E_2, Airey 1, $k'^2 = 0(.00001).00010$, δ^2, $K = K_1\ln(4/k') - K_2$, $E = E_1\ln(4/k') + E_2$
12-11D, K_1, K_2, E_1, E_2, Airey 1, $k'^2 = 0(.0001).0010$, δ^2
10D, K_1, K_2, Airey 1, $k'^2 = 0(.001).170$, δ^2
10D, E_1, E_2, Airey 1, $k'^2 = 0(.001).100$, δ^2
8D, logarithms of C, C', C_1, C_1', C_2, Witt 1, $\theta = 13°(1°)36°(30')45°(20')50°(10')60°$, Δ, where $K = \sec\ \frac{1}{2}\theta\ (\sec\theta)^{\frac{1}{2}}C$, $E = \cos^3\frac{1}{2}\theta\ (\sec\theta)^{\frac{1}{2}}C_1$, $C' = 2C/\pi$, $C_1' = 2C_1/\pi$, $C_2 = C - C_1$. More complicated functions for $\theta \geqslant 60°$. See also Witt 2
8D, $\log M$, Idelson 1, $\theta = 60°(2°)64°(1°)75°$, where $2K/\pi = \sec^2\frac{1}{2}\psi\ (\sec\frac{1}{2}\theta)^{\frac{1}{2}}\ (\sec\theta)^{\frac{1}{2}}\ M$, $\cos\psi = (\cos\theta)^{\frac{1}{2}}$. More complicated function for $\theta > 75°$

8D, $\log (K/2\pi) + 2 \log (1 + \sqrt{k'})$, ROBBINS 2, $k = .4(.01).6(.001).849$
7D, $\log [(4K/\pi) \cos \frac{1}{2}\theta (\cos \theta)^{\frac{1}{2}}]$, NEWCOMB 1 (p. 69), $k = .45(.01).75$
7–11D, $K/\ln (4/k')$, DWIGHT 2, $\theta = 86°(1')90°$, Δ
7D, $K/\log (4/k')$, SAMOĬLOVA-ÎAKHONTOVA 1, $k^2 = .95(.001)1$, Δ
6D, $2K/\pi - a \log (90 - \theta)$, $a = (2/\pi) \ln 10 = 1.465871$, HEUMAN 1, $\theta = 65(0.1)90$, Δ
5D, first 8 coefficients in expansions of $K - \frac{1}{2}\pi + \ln k'$ and $K' + (2K/\pi) \ln k - \ln 4$ in powers of k^2, TANNERY & MOLK 1 (v. 3, p. 215)
4D, $K - \ln (4/k')$, $\ln (4/k')$, JAHNKE & EMDE 1_3, 1_4, $k^2 = .7(.01)1$, $I\Delta$
4D, $\beta = \log K - \log \ln (4/k')$, HOÜEL 1 (p. 57), $\theta = 50^g(1^g)100^g$
4D, $4E/\pi(1 + k')$, BOLL 1 (p. 348), $\epsilon = (1 - k')/(1 + k') = 0(.01)1$
4D, $4E/\pi(1 + k')$, BOLL 1 (p. 349), $k' = 0(.01)1$

Section III: JACOBI'S NOME q

q is defined as $e^{-\pi K'/K}$. Thus $\ln q = -\pi K'/K$, or $\log q = -\mu\pi K'/K$, where $\mu\pi = \pi \log e = 1.36437\ 63538\ 41841$, so that a table of $\log q$ is easily computed from a table of K, if θ or k^2 is the argument. Also

$$\log \log (1/q) = \log (\mu\pi) + \log K' - \log K,$$

where

$$\log (\mu\pi) = 0.13493\ 41839\ 94670\ 6,$$

so that $\log \log (1/q)$ is very easily computed from a table of $\log K$. It is not essential that a table of $\log q$ or of $\log \log (1/q)$ should extend beyond $\theta = 45°$ or $k^2 = \frac{1}{2}$, for if q' is the complementary nome $e^{-\pi K/K'}$, we have

$$\log q \log q' = \mu^2\pi^2 = 1.86152\ 28349\ 22757$$

and

$$\log \log (1/q) + \log \log (1/q') = 2 \log (\mu\pi) = 0.26986\ 83679\ 89341\ 3.$$

If $q = \Sigma a_n k^{2n}$, the exact values of a_n were calculated for $n = 1(1)12$ by F. TISSERAND, published in HERMITE 1_1, and reproduced in HERMITE 1_2 and in TANNERY & MOLK 1 (v. 4, p. 121).

The most usual way of computing q, other than by using its definition as given above, is to put

$$2\epsilon = (1 - \sqrt{k'})/(1 + \sqrt{k'}),$$

when we have

$$\epsilon = (q + q^9 + q^{25} + \ldots)/(1 + 2q^4 + 2q^{16} + \ldots),$$

which inverts (see WEIERSTRASS & SCHWARZ 1, p. 56) into

$$q = \epsilon + 2\epsilon^5 + 15\epsilon^9 + 150\epsilon^{13} + \ldots.$$

The first 14 terms of the series are given in LOWAN, BLANCH & HORENSTEIN 1.

It should be noted that q and ϵ are called h and $\frac{1}{2}l$ respectively in WEIERSTRASS & SCHWARZ 1 and elsewhere. The small difference $q - \epsilon$ (called $q - \frac{1}{2}l$) is tabulated to 8D for $q = 0(.01).14$ in NAGAOKA 1, and to 8D with Δ for $q = .02(.002).1(.001).15$ in NAGAOKA 2; the second table is reproduced in ROSA & GROVER 1. NAGAOKA & SAKURAI 1 (p. 56) tabulates $\log 2\epsilon$ (called $\log l$) to 7D with Δ for $k^2 = 0(.001).5$.

For complex k^2, see CAMBI 1 in Section II; also the diagrams in JAHNKE & EMDE 1_2 (p. 120), 1_3–1_4 (p. 46), giving k^2 as a function of τ, where $q = e^{i\pi\tau}$.

III–A. *q and its powers*

16S, q, SPENCELEY 1, $\theta = 0(1°)90°$
14–15D, q, HIPPISLEY 1, $\theta = 0(5°)80°(1°)90°$
8D, q, GLAISHER 1, $\theta = 0(1°)90°$
7–5D, q, LÁSKA 1, $\theta = 0(1°)90°$
17D, q^n, NBSCL 2, $n = \frac{1}{4}, \frac{1}{2}, 1(1)4(2)8, 9, 12, 16, 20, 25$, $k^2 = 0(.01)1$

10D, q, MILNE-THOMSON 1, $k^2 = 0(.01)1$, Δ
8D, q, MILNE-THOMSON 3, $k^2 = 0(.01)1$
8D, q, SAMOĬLOVA-ÎAKHONTOVA 1, $k^2 = 0(.001)1$, Δ
8D, q, HAYASHI 1, $k^2 = 0(.001).5$
5D, q, HAYASHI 2, $k^2 = 0(.001).5$
7D, q, GROVER 6 (p. 251), $k^2 = 0(.005).1$, Δ^2
11D, $q^{25/4}$, HAYASHI 1, $k^2 = .3(.001).5$
10D, q^4, $q^{9/4}$, HAYASHI 1, $k^2 = 0(.001).5$
8D, $q^{1/4}$, HAYASHI 1, $k^2 = 0(.001).5$

III–B. $\pm$ *log q and auxiliary functions*

10D, log $(1/q)$, PLANA 1, $\theta = 0(0^\circ.1)45^\circ(1^\circ)90^\circ$ [highly inaccurate, see Part III]
10D, log q, INNES 2, $\theta = 0(1^\circ)45^\circ$
10D, log $(q \cot^2 \frac{1}{2}\theta)$, INNES 2, $\theta = 0(1^\circ)45^\circ$, log v^4
10D, log (q/ϵ), INNES 2, $\theta = 0(1^\circ)31^\circ(30')40^\circ(10')45^\circ$
8D, log q, MEISSEL 1, $\theta = 0(1')90^\circ$
8D, log q, GLAISHER 1, $\theta = 0(1^\circ)90^\circ$
5D, log q, JACOBI 1, $\theta = 0(0^\circ.1)90^\circ$, Δ
5D, log q, BERTRAND 1, LÉVY 1, POTIN 1, $\theta = 0(5')90^\circ$
5D, log q, FRICKE 1, $\theta = 0(10')90^\circ$
5D, log q, SCHLÖMILCH 2, SILBERSTEIN 1, $\theta = 0(1^\circ)90^\circ$
5D, log $(1/q)$, LÁSKA 1, $\theta = 0(1^\circ)90^\circ$
4D, log q, JAHNKE & EMDE 1, $\theta = 0(5')90^\circ$
4D, log q, BOHLIN 1, GLAZENAP 1, $\theta = 0(0^\circ.1)90^\circ$
4D, log q, HOÜEL 1, $\theta = 0(1^g)100^g$
4D, $\alpha = \log(16q/k^2)$, HOÜEL 1 (p. 57), $\theta = 0(1^g)50^g$
3D, log q, MONTESSUS DE BALLORE 1, $\theta = 0(1^\circ)86^\circ(30')89^\circ(20')89^\circ40'$, $89^\circ55'$
15D, $-\log q$, NBSCL 2, $k^2 = 0(.01)1$
10D, log q, HAYASHI 1, $k^2 = 0(.001)1$
8D, log q, HAYASHI 1, $k^2 = 0(10^{-7})10^{-5}(10^{-5}).003$, $1 - k^2$ = same
7D, log q, NAGAOKA & SAKURAI 1, $k^2 = 0(.001)1$, Δ
7D, log (q/k^2) and ln (k^2/q), NAGAOKA & SAKURAI 1, $k^2 = 0(.001).05$, Δ
5D, log q, GROVER 6 (p. 251), $k^2 = 0(.005).1$, Δ^2

III–C. *log log (1/q) and log ln (1/q)*

12D, log log $(1/q)$, VERHULST 1, $\theta = 0(0^\circ.1)45^\circ$. Computed by LOXHAY. From 0 to 15°, values to 14 decimals are given, but they are inaccurate, as in LEGENDRE's table of log K, on which they are based. See Part III
10D, log log $(1/q)$, INNES 2, $\theta = 0(1^\circ)45^\circ$, from VERHULST 1
7D, log log $(1/q)$, BERTRAND 1, POTIN 1, $\theta = 0(0^\circ.5)90^\circ$
7D, log ln $(1/q)$, NAGAOKA & SAKURAI 1, $k^2 = 0(.001)1$, Δ
5D, log ln $(1/q)$, GROVER 6 (p. 251), $k^2 = 0(.005).1$, Δ^2

Section IV. THETA FUNCTIONS OF ZERO ARGUMENT

The general notation for theta functions which we shall use is

$$\begin{aligned}
\vartheta_1(x) &= 2q^{\frac14}(\sin x - q^2 \sin 3x + q^6 \sin 5x - q^{12} \sin 7x + \ldots),\\
\vartheta_2(x) &= 2q^{\frac14}(\cos x + q^2 \cos 3x + q^6 \cos 5x + q^{12} \cos 7x + \ldots),\\
\vartheta_3(x) &= 1 + 2q \cos 2x + 2q^4 \cos 4x + 2q^9 \cos 6x + \ldots,\\
\vartheta_4(x) &= 1 - 2q \cos 2x + 2q^4 \cos 4x - 2q^9 \cos 6x + \ldots,
\end{aligned}$$

where, as in Section III above, $q = e^{-\pi K'/K}$. This is the functional notation used, for example, in WHITTAKER & WATSON, and we shall adhere to it in spite of the fact that definitions in which x is replaced by πv have been adopted in important tabulations.

In this section we are concerned only with theta functions and their x-derivatives at $x = 0$. Since $\vartheta_1(0)$ vanishes, $\vartheta_1'(0)$ is considered in its place, and we have

$$\begin{aligned}
\vartheta_1' &= \vartheta_1'(0) = 2q^{\frac{1}{4}}(1 - 3q^2 + 5q^6 - 7q^{12} + \dots),\\
\vartheta_2 &= \vartheta_2(0) = 2q^{\frac{1}{4}}(1 + q^2 + q^6 + q^{12} + \dots),\\
\vartheta_3 &= \vartheta_3(0) = 1 + 2q + 2q^4 + 2q^9 + \dots,\\
\vartheta_4 &= \vartheta_4(0) = 1 - 2q + 2q^4 - 2q^9 + \dots.
\end{aligned}$$

On account of the difference of notation mentioned above, the $\vartheta_1'(0)$ of Nagaoka & Sakurai and Hayashi is our $\pi\vartheta_1'$, and will be so described.

The four functions just defined are connected by Jacobi's identity $\vartheta_1' = \vartheta_2\vartheta_3\vartheta_4$. Moreover, each is expressible in terms of K (and moduli), as follows:

$$\vartheta_1' = \sqrt{(8kk'K^3/\pi^3)}, \qquad \vartheta_2 = \sqrt{(2kK/\pi)},$$
$$\vartheta_3 = \sqrt{(2K/\pi)} = \sqrt{(1/M)}, \qquad \vartheta_4 = \sqrt{(2k'K/\pi)}.$$

If $M = 1/\vartheta_3^2 = 1 - 4g_1q + 4g_2q^2 - 4g_3q^3 + \dots$, the g's are all integral, and are given up to g_{27} in Glaisher 2.

Nagaoka & Sakurai 1 and Hayashi 1 also tabulate quotients which, in our notation, are

$$\pi^2\vartheta_1'''/\vartheta_1', \qquad \pi^2\vartheta_2''/\vartheta_2, \qquad \pi^2\vartheta_3''/\vartheta_3, \qquad \pi^2\vartheta_4''/\vartheta_4.$$

In their notation, the π^2 factors are absent. Mainly in consequence of Jacobi's identity $\vartheta_1' = \vartheta_2\vartheta_3\vartheta_4$, the first quotient is the sum of the other three. The expressions for ϑ_1''', ϑ_2'', ϑ_3'', ϑ_4'' are evidently

$$\begin{aligned}
\vartheta_1''' &= -2q^{\frac{1}{4}}(1 - 27q^2 + 125q^6 - \dots),\\
\vartheta_2'' &= -2q^{\frac{1}{4}}(1 + 9q^2 + 25q^6 + \dots),\\
\vartheta_3'' &= -8(q + 4q^4 + 9q^9 + \dots),\\
\vartheta_4'' &= 8(q - 4q^4 + 9q^9 - \dots).
\end{aligned}$$

It should not be overlooked that the quotients mentioned are expressible in terms of K and E (and moduli), as follows:

$$\pi^2\vartheta_1'''/\vartheta_1' = 4K\{(2 - k^2)K - 3E\}, \qquad \pi^2\vartheta_2''/\vartheta_2 = -4KE,$$
$$\pi^2\vartheta_3''/\vartheta_3 = -4K(E - k'^2K), \qquad \pi^2\vartheta_4''/\vartheta_4 = 4K(K - E).$$

Other notations for theta functions will be mentioned in Section X.

IV–A. ϑ_1', ϑ_2, ϑ_3, ϑ_4

15D, ϑ_3/ϑ_4[$=$ D(90)], ϑ_4/ϑ_3, Spenceley 1, $\theta = 0(1°)90°$
10–12D, ϑ_2[$=$ H(K)], ϑ_4[$= \Theta(0)$], ϑ_3/ϑ_4[$= C(0) = 1/\sqrt{k'}$], Hippisley 1, $\theta = 0(5°)80°(1°)90°$
9D, log ϑ_3 [column log $\Theta(\omega)$], Bertrand 1 (p. 714), $\theta = 0(0°.5)90°$
9D, log ϑ_3 [column log $\vartheta_2(0)$], Potin 1 (p. 832), $\theta = 0(0°.5)90°$
8D, $\pi\vartheta_1'$ [col. $\vartheta_1'(0)$], ϑ_2, ϑ_3, ϑ_4 [col. $\vartheta_0(0)$], Hayashi 1, $k^2 = 0(.001).5$
5D, $\pi\vartheta_1'$ [col. $\vartheta_1'(0)$], ϑ_2, ϑ_3, ϑ_4 [col. $\vartheta_0(0)$], Hayashi 2, $k^2 = 0(.001).5$
7D, logs of these 4 functions, Nagaoka & Sakurai 1, $k^2 = 0(.001).5$, Δ
7D, log $(\pi\vartheta_1'/\sqrt{k})$, Nagaoka & Sakurai 1, $k^2 = 0(.001).05$, Δ

Use may also be made of the values for zero argument in the small double-entry tables of Hoüel 1 and Jahnke & Emde 1 (see Section X). It will be noticed that ϑ_3/ϑ_4 (see Spenceley 1 and Hippisley 1 above) is merely an algebraic function of the modulus.

IV–B. ϑ_1''', ϑ_2'', ϑ_3'', ϑ_4''

8D, $\pi^2\vartheta_1'''/\vartheta_1'$, $\pi^2\vartheta_2''/\vartheta_2$, $\pi^2\vartheta_3''/\vartheta_3$, $\pi^2\vartheta_4''/\vartheta_4$, Hayashi 1, $k^2 = 0(.001).5$
7D, logs of these 4 quotients, Nagaoka & Sakurai 1, $k^2 = 0(.001).5$, Δ
7D, log $(-\pi^2\vartheta_3''/k^2\vartheta_3)$, log $(\pi^2\vartheta_4''/k^2\vartheta_4)$, Nagaoka & Sakurai 1, $k^2 = 0(.001).05$, Δ

Section V. Laplace Coefficients and Related Functions

The Laplace coefficients $b_s^{(i)}$ occur in the expansion

$$(1 + \alpha^2 - 2\alpha \cos x)^{-s} = \tfrac{1}{2}b_s^{(0)} + \sum_{i=1}^{\infty} b_s^{(i)} \cos ix,$$

where $s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots$. The expansion is used in dynamical astronomy, where α is most often equal to the ratio of the mean distances of two planets from the Sun. The coefficients, and their derivatives with respect to α, are all linear functions of the complete elliptic integrals of the first and second kinds with modulus $k = \alpha$; in particular,

$$b_{\frac{1}{2}}^{(0)} = 4K/\pi, \qquad b_{\frac{1}{2}}^{(1)} = 4(K - E)/\pi k.$$

The principal tables multiply the Laplace coefficients by various integral powers of α and $\sqrt{(1 - \alpha^2)}$ before tabulation, or use variant forms of the functions. The argument is usually α, $\log \alpha$ or $\alpha^2/(1 - \alpha^2)$, that is, in the notation of our other sections, k, $\log k$ or $\tan^2 \theta$.

A full description of the various tables entails some algebraical complication; for details one may consult p. 333–335 of FMR *Index*. When all derivatives of all Laplace coefficients are considered, the functions may be regarded as somewhat general hypergeometric functions rather than merely as standard forms of elliptic integrals. Consequently only a few brief indications are given here.

Tables with argument α. The most extensive tables are in Runkle 1 (on errors, see Witt 2). Brown & Brouwer 1 has minor tables with argument α. See also Robbins 2 (and Innes 5), Fletcher 1, Witt 2, Norén & Raab 1, Andoyer 1, and Legendre 2 (p. 289), 5 (p. 551).

Tables with argument $\log \alpha$. Masal 1 and Gyldén 1 are extensive. The integrals tabulated in Masal 1, namely

$$\beta_n^{(s)} = (2/\pi) \int_0^{\frac{1}{2}\pi} (1 - \alpha^2 \sin^2 \phi)^{-\frac{1}{2}s} \sin^{2n} \phi d\phi,$$

are perhaps more germane to this *Guide* than most functions discussed in the present section.

Tables with argument $\alpha^2/(1 - \alpha^2)$. This is the argument of the major tables in the important Brown & Brouwer 1.

Coefficients used in calculating Laplace coefficients. See Runkle 1, Leverrier 1, Newcomb 1, Wellmann 1, Brown 1, 2, Williams 1, Witt 2, Brown & Brouwer 1, and Brown & Shook 1.

Section VI. Combinations of K and E occurring in connection with Currents in Circles and Coils

Below are listed a number of standard tables, many of which may be regarded as source tables, of four main kinds designed for electrical use. Some of them have doubtless been copied into various electrical text-books and hand-books, but no attempt has been made to extend the lists below to include all these.

The function f_1[$= M/4\pi\sqrt{(Aa)}$, in a common notation] occurs in connection with the mutual inductance of two circles with a common axis; f_2 in connection with the attraction between currents in such circles. The function f_3 (often called K) occurs in connection with the self-inductance of cylindrical solenoids, as Nagaoka's correction factor for finite length; in this case $\tan \theta$ is the ratio of diameter to length of the solenoid, and f_3 tends to unity for long coils (small θ). As the signs will be found to be wrong or misleading in a number of places in the literature, it may be added that f_1, f_2 and f_3 are all positive when defined as below.

Spielrein's function $f(\alpha) = f_4$ occurs in connection with the self-inductance of flat coils. If the windings lie in a plane between two concentric circles of radii a, A, and the number of turns, N, is large, each turn being approximately circular, the self-inductance in absolute electromagnetic units is $N^2Af(\alpha)$, where $\alpha = a/A < 1$. It will be observed that $f(\alpha)$ involves integrals of K and E with respect to the modulus k, so that it is essentially more complicated than the other three functions, but it is convenient to include it here on account of the electrical relatedness. It may be noted that

$$\int_0^1 Kdk = 2G, \qquad \int_0^1 Edk = G + \tfrac{1}{2},$$

where G is CATALAN's constant,

$$G = \tfrac{1}{2}\int_0^{\frac{1}{2}\pi} \phi \operatorname{cosec} \phi d\phi = 1^{-2} - 3^{-2} + 5^{-2} - 7^{-2} + \cdots = 0.91596\ 55941\ 77219,$$

so that $f(0) = \frac{8}{3}\pi(2G - 1) = 6.96957\ 04257$.

GROVER 1, 6 (p. 252) give tables relating to the maximum attraction between currents in two circles with a common axis, and the distance between the planes of the circles which produces this maximum.

Further references will be found in ROSA & GROVER 1, in GROVER 6, and in the extensive bibliography of HAK 1 (687 items, p. 232–246).

VI–A1. $f_1 = (2/k - k)K - 2E/k$

7D, $\log f_1$, MAXWELL 1, $\theta = 60°(0°.1)90°$, reprinted in MASCART & JOUBERT 1, A. GRAY 1_1, and FOWLE 1_1, and (partly corrected by comparison with ROSA *et al.*) in A. GRAY 1_2
7D, $\log f_1$, ROSA & COHEN 1 and ROSA & GROVER 1, $\theta = 60°(0°.1)90°$, Δ, recomputation of MAXWELL 1
5D, f_1, JAHNKE & EMDE 1_1 (p. 77), $\theta = 60°(0°.1)90°$, antilogs of MAXWELL 1
4–6D, $4\pi f_1$, PIDDUCK 1, $\theta = 5°(6')89°(1')89°54'$
About 6S, $4\pi f_1$, $\log(4\pi f_1)$, NAGAOKA & SAKURAI 2 (p. 140), $k^2 = 0(.001)1$, Δ^2
About 6S, $4\pi f_1/1000$, $\log(4\pi f_1/1000)$, GROVER 5, $k^2 = 0(.005)1$, Δ^n, also auxiliary tables
4S, $2\pi k f_1$, CURTIS & SPARKS 1, $k^2 = 0(.01)1$, Δ, also auxiliary tables
4–5S, $4\pi f_1/1000$, $\log(4\pi f_1/1000)$, GROVER 6 (p. 79), $k'^2 = 0(.01)1$, Δ, also auxiliary tables
About 4S, $4\pi f_1/1000$, BUREAU OF STANDARDS 1, TERMAN 1 (p. 69), $k' = .01(.001).016(.002).05(.01).95(.002)1$
8D, ϵ and $\log(1+\epsilon)$, NAGAOKA 2, $q = .02(.002).1(.001).15$, Δ for $\log(1+\epsilon)$, where $f_1 = 4\pi q^{\frac{1}{2}}(1+\epsilon)$. Also two further auxiliary functions with q' as argument. All reproduced in ROSA & GROVER 1. Brief tables of the same kind in NAGAOKA 1. NAGAOKA 5 tabulates $\log(1+\epsilon)$ to 7D with Δ for $h = 0(.001).2$, where $h = q_1 = q^2$, q and q_1 corresponding to k and LANDEN's modulus k_1, see Section I. Brief table of the same kind in NAGAOKA 3. (ϵ is not to be confused with the ϵ of Section III, which is denoted by $\frac{1}{2}l$ in NAGAOKA.)

VI–A2. $f_2 = k\{(2 - k^2)E/(1 - k^2) - 2K\} = \sin\theta\{(1 + \sec^2\theta)E - 2K\}$

8D, $\log f_2$, ROSA, DORSEY & MILLER 1, $\theta = 55°(0°.1)70°$, Δ^2, recomputation of RAYLEIGH & SIDGWICK 1
7D, $\log f_2$, RAYLEIGH & SIDGWICK 1, $\theta = 55°(0°.1)69°.9$, reprinted in MASCART & JOUBERT 1_2
About 5S, f_2, JAHNKE & EMDE 1_1 (p. 79), $\theta = 55°(0°.1)69°.9$, antilogs of RAYLEIGH & SIDGWICK 1
About 6S, πf_2, $\log(\pi f_2)$, NAGAOKA & SAKURAI 2 (p. 161), $k^2 = 0(.001)1$, Δ^2
5D, $\log(\pi f_2)$, NAGAOKA 4, $q = .02(.002).1(.001).15$, Δ
About 4S, πf_2, GROVER 6 (p. 250), $k^2 = 0(.01)1$, Δ^2, abridged from NAGAOKA & SAKURAI 2

VI–A3. $f_3 = (4/3\pi)[(K\cos^2\theta - E\cos 2\theta)\sin^{-2}\theta\cos^{-1}\theta - \tan\theta]$

6D, f_3, $\log(\pi^2 f_3)$, NAGAOKA & SAKURAI 2, $k^2 = 0(.001)1$, $\tan\theta = 0(.001)1$, $\cot\theta = 0(.001)1$, Δ^2
6D, f_3, HAK 1 (p. 16), $\tan\theta = 0(.001).002(.002).01(.01)1$, $\cot\theta = 0(.001).002(.002).01(.01)1$
6D, f_3, $\log f_3$, NAGAOKA 2, $\theta = 0(1°)90°$, Δ^2
6D, f_3, ROSA & GROVER 1 (T.XX), $\theta = 0(1°)90°$, Δ^2, from NAGAOKA 2
6D, f_3, NAGAOKA 2, $\tan\theta = 0(.01)1(.05)2(.1)5(.5)10$, Δ

6D, f_3, ROSA & GROVER 1 (T.XXI), $\tan\theta = 0(.01)1(.05)2(.1)5(.5)10$, Δ^2 to 5, then Δ^3, from NAGAOKA 2 with correction
6D, f_3, GROVER 6 (p. 144), $\cot\theta = 0(.01)1$, $\tan\theta = 0(.01)1$, Δ^2
5D, f_3, GROVER 3, $\tan\theta = 0(.01)1(.05)2(.1)5(.5)10(1)20(\text{var.})400$, Δ
4D, f_3, BUREAU OF STANDARDS 1, $\tan\theta = 0(.05)2(.1)5(.2)8(.5)10(1)20(\text{var.})100$, Δ
5D, $2\pi^2 f_3 \tan\theta$, ROSA & COHEN 1 (T.IV), ROSA & GROVER 1 (T.IV), $\tan\theta = .2(.1)1(.2)4$
4D, $2\pi^2 f_3 \tan\theta$, RUSSELL 1, $\tan\theta = .2(.1)1(.2)4$
4D, $\frac{1}{2}\pi f_3 \tan\theta$, EMDE 1, $\tan\theta = 0(.01)1$, $\cot\theta = 0(.01)1$
4–5S, $\pi^2 f_3 \tan\theta$, $\pi^2 f_3 \tan^2\theta$, JAHNKE & EMDE 1_2–1_4, BOLL 1 (p. 344), $\tan\theta = 0(.01)1$, $\cot\theta = 0(.01)1$
2–3D, $\pi^2 f_3$ (col. k), $\pi^2 f_3 \tan\theta$ (col. F), GROVER 4, 6 (p. 152), $\tan\theta = 0(.01)1$, $\cot\theta = 0(.01)1$

VI–A$_4$. $f_4 = 16\pi/[3(1-\alpha)^2]\int_\alpha^1 (1-\alpha^3 k^{-3})(K-E)dk$

6–7S, $f(\alpha) = f_4$, SPIELREIN 1, GROVER 2 (p. 570), $\alpha = 0(.05).9, 1$
4S, $f(\alpha) = f_4$, SPIELREIN 1, $\alpha = 0, .05(.01).95, .99, 1$
4S, $Q = (1-\alpha)^2 f_4$, BUTTERWORTH 1, $r_1/r_2 = \alpha = 0(.05)1$
5S, $\Phi = f_4/(1+\alpha)$, HAK 1 (p. 18), $\gamma = \alpha = 0(.05).9$
4–5S, $\Phi = f_4/(1+\alpha)$, HAK 1 (p. 18), $\rho = (1-\alpha)/(1+\alpha) = 0(.05)1$
5S, $P = 2f_4/(1+\alpha)$, GROVER 6 (p. 113), $c/2a = (1-\alpha)/(1+\alpha) = 0(.01)1$, Δ

Section VII. MISCELLANEOUS TABLES INVOLVING COMPLETE ELLIPTIC INTEGRALS AND THETA FUNCTIONS OF ZERO ARGUMENT

1. LEGENDRE functions $P_n(x)$, $Q_n(x)$ of half-odd-integral order n may be expressed in terms of complete elliptic integrals of the first and second kinds. BATEMAN[1] gives a treatment of the case $x \geqslant 1$, which arises in solving LAPLACE's equation in toroidal coordinates. For references to early work, see BATEMAN[1] and AIREY 1. Some authors (including AIREY 1 and LEVY & FORSDYKE 1) use a variant notation for toroidal functions, in that they write $Q_0, Q_1, Q_2, \cdots$ in place of $Q_{-\frac{1}{2}}, Q_{\frac{1}{2}}, Q_{\frac{3}{2}}, \cdots$, in order to avoid fractional suffixes; but we shall adhere to the usual Legendre function notation. Some formulae for low n have recently been stated explicitly by MILLER[2], for both cases, $-1 \leqslant x \leqslant 1$ and $x \geqslant 1$.

When $|x| \leqslant 1$, put $x = \cos\theta$. Then the elliptic modulus being given by

$$k^2 = \tfrac{1}{2}(1-x) = \sin^2\tfrac{1}{2}\theta, \qquad k'^2 = \tfrac{1}{2}(1+x) = \cos^2\tfrac{1}{2}\theta,$$

we have, for instance,

$$P_{-\frac{1}{2}}(x) = 2K/\pi, \qquad P_{\frac{1}{2}}(x) = 2(2E-K)/\pi,$$

and (see MILLER[2])

$$Q_{-\frac{1}{2}}(x) = K' \mp iK, \qquad Q_{\frac{1}{2}}(x) = 2E' - K' \pm i(2E-K).$$

Further P's and Q's may be obtained from the usual recurrence relation. No tables, except of course for $P_{-\frac{1}{2}}$ and $Q_{-\frac{1}{2}}$, are known, but they could evidently be very easily formed, with argument θ, from the tables of LEGENDRE or HEUMAN 1.

When $x \geqslant 1$, put $x = \cosh\sigma$. Then the elliptic modulus being given by

$$k^2 = 2/(x+1) = \operatorname{sech}^2\tfrac{1}{2}\sigma, \qquad k'^2 = (x-1)/(x+1) = \tanh^2\tfrac{1}{2}\sigma,$$

we have, for instance,

$$P_{-\frac{1}{2}}(x) = 2kK'/\pi, \qquad P_{\frac{1}{2}}(x) = 2(2E' - k^2K')/k\pi,$$
$$Q_{-\frac{1}{2}}(x) = kK, \qquad Q_{\frac{1}{2}}(x) = \{(2-k^2)K - 2E\}/k.$$

The values of $Q_n(x)$, $Q_n'(x)$, $Q_n''(x)$ are given to 6D or 7S for $n = -\frac{1}{2}(1)+2\frac{1}{2}$, $\sin^{-1}k = 83°$, $80°(5°)65°(10°)25°$ in LEVY & FORSDYKE 1. The values of $P_n(x)$, $P_n'(x)$, $Q_n(x)$, $Q_n'(x)$, as well as corresponding associated Legendre functions and their first derivatives, are given to about 6S for $n = -\frac{1}{2}(1)+4\frac{1}{2}$, $x = 1(.1)10$ in NBSCL 1. $P_n(\cosh\sigma)$, $P_n'(\cosh\sigma)$, $Q_n(\cosh\sigma)$, $Q_n'(\cosh\sigma)$ are tabulated to 4D for $n = -\frac{1}{2}(1)+\frac{3}{2}$, $\sigma = 0(.1)3$ in FOUQUET 1.

2. F. W. NEWMAN 1 has a number of tables in which the argument is $\rho = \pi K'/2K$, so that $q = e^{-2\rho}$, $\rho = \frac{1}{2}\ln(1/q)$. For the first of these tables, see Section XI. The remainder are as follows:

p. 129, 16D, $\frac{1}{4}\ln(1/k')$, $\rho = 1(.1)6.3$
p. 130, 16D, $\frac{1}{2}\ln(2K/\pi) = \ln\vartheta_3$, $\rho = 1(.1)6.3$
p. 131, 16D, $\frac{1}{2}(2K/\pi - 1)$, $\rho = 1(.1)6.1$
p. 132, 16D, $\ln Q$, $\rho = 1(.1)4.6$, where $Q^{-1} = (1-q^2)(1-q^4)(1-q^6)\cdots = (\vartheta_1'/2q^{\frac{1}{4}})^{\frac{1}{3}}$
p. 133, 16D, $\frac{1}{2}(1 - 2k'K/\pi)$, $\rho = 1(.1)6.3$
p. 134, 16D, kK/π, $\rho = 1(.1)6$
p. 135, 18D, $e^{-\rho}$, $\rho = .1(.1)37$

The last table barely falls within the scope of this *Guide*, but is included to complete the description of NEWMAN'S ρ-tables, and is relevant to the remainder of these. It is taken from NEWMAN'S original publication.[3] About 16D are correct.

3. It is well known that, on putting $q = e^{-4t}$, each term of the defining expansions of the four theta functions satisfies $\partial^2 f/\partial x^2 = \partial f/\partial t$, the diffusion or heat conductivity equation. The theta functions themselves are not usually the appropriate combinations of such terms in physical problems. But integrals of theta functions of zero argument have been tabulated in connection with a number of diffusion problems. For example,

$$\frac{4}{\pi^2}\int_0^t \vartheta_2 dt = 1 - \frac{8}{\pi^2}\left(e^{-t} + \frac{1}{9}e^{-9t} + \frac{1}{25}e^{-25t} + \cdots\right),$$

and the expression on the right is tabulated as the "simple diffusion function" in MCKAY 1 to 4D for $t = 0(.01)3.59$. Similarly a table in OLSON & SCHULTZ 1 involves $\int_0^t \vartheta_1' dt$, some in SHERWOOD 1 involve $\int_0^t \vartheta_2 dt$ and $\int_0^t(\int_0^t \vartheta_2 dt)dt$, some in A. B. NEWMAN 1 involve first integrals (with respect to t) of ϑ_2 and ϑ_3, some in A. B. NEWMAN 2 involve first and second integrals of ϑ_2 and ϑ_3, and some in A. B. NEWMAN 3 involve first and second integrals of ϑ_3 and ϑ_4. These tables have various arguments (t/π^2, $4t/\pi^2$, $4t$), and particulars will be omitted here (some will be found in FMR *Index*, p. 323). There appears to be need for a consolidated table of first and second integrals of theta functions of zero argument with respect to t or some simple multiple of t. It may be noted that F. W. NEWMAN'S ρ is $2t$.

4. EMDE 2 proposes new normal forms of complete elliptic integrals, namely, quantities D, B, C defined by

$$D = (K - E)/k^2,$$
$$B = K - D = (E - k'^2 K)/k^2,$$
$$C = (D - B)/k^2 = [(2 - k^2)K - 2E]/k^4.$$

In the notation of Section VI, $C = f_1/k^3$. EMDE 2 tabulates D, B, C to 4–5D with $I\Delta$ for $k^2 = 0(.01)1$. For D and C the auxiliary functions

$$d = D + 1 - \ln(4/k'),$$
$$c = C + 2 - \ln(4/k')$$

are tabulated to 3–4D with $I\Delta$ for $k^2 = .7(.01)1$. These five tables are reproduced in JAHNKE & EMDE 1_3, 1_4. CECCONI 1 gives $(E - k'^2 K)/k^2$, which he calls G, to 4D for $k = 0(.01)1$; see also Section II–$\mathbf{A}_3$.

5. HILL 1 tabulates to 8D with Δ^2 for $\theta = 0(0°.1)50°$ the logarithms of three quantities $\mathfrak{K}$, $\mathfrak{L}'$ and $\mathfrak{N}$ which may be identified as follows:

$$\pi k'^2 \mathfrak{K} = 2E,$$
$$k^4 k'^2 E \mathfrak{L}' = 2(1 - k^2 k'^2)E - k'^2(1 + k'^2)K,$$
$$k^2 k'^2 E \mathfrak{N} = (1 + k^2)E - k'^2 K.$$

6. SMEKAL 1 has a table of quantities K, $Q(K)$, $Q(1/K)$, $1/K$ with argument α. This notation is so inconvenient for present purposes that the tabulated functions will be denoted instead by k, Q, Q^*, $1/k$. The primary argument of the table is

$$\alpha = 45°(3')44°30'(6')40°(15')35°(30')5°.$$

k is tabulated to 5D, the other functions to about 4D, all but $1/k$ having first differences. The definitions in terms of α are

$$k = \tan^2 \alpha, \qquad Q = \pi^{-1} \sin^2 \alpha (K - E \sec 2\alpha), \qquad Q^* = \pi^{-1} \cos^2 \alpha (K + E \sec 2\alpha),$$

where K and E have modular angle 2α. But k is the reduced modulus derived from $\sin 2\alpha$ by Landen's transformation (Section I), and in terms of this subsidiary argument we have

$$\begin{aligned} Q &= -\,2k^2dK/(\pi dk) = -\,2k(E - k'^2K)/(\pi k'^2), \\ Q^* &= 2d(kK)/(\pi dk) = 2E/(\pi k'^2), \end{aligned}$$

where K and E now have modulus k. Thus Q is related to EMDE's B and Q^* is identical with HILL's $\mathfrak{K}$ (see 4 and 5 above). Replacing k now by the symbol α usual in planetary theory (not SMEKAL's α), the identifications are

$$Q = -\,\tfrac{1}{2}\alpha^2 b_0', \qquad Q^* = \tfrac{1}{2}(b_0 + \alpha b_0'),$$

where $\frac{1}{2}b_0$ is the constant term in the Fourier expansion of $(1 + \alpha^2 - 2\alpha \cos x)^{-\frac{1}{2}}$, and $b_0' = db_0/d\alpha$. The symbol α just introduced in place of SMEKAL's K is appropriate, because the latter is the ratio of the radii of two circular orbits (compare Section V).

7. Coefficients in the Fourier expansions of two Jacobian functions (see Sections X and XI) are tabulated in GLAISHER 1. If

$$\left.\begin{aligned} 2 \operatorname{sn} u \operatorname{cn} u &= \Sigma A_n \sin 2nx \\ 2 \operatorname{am} u - 2x &= \Sigma B_n \sin 2nx \end{aligned}\right\} x = \pi u/2K,$$

we have

$$A_n = n\left(\frac{\pi}{kK}\right)^2 \frac{4q^n}{1 + q^{2n}}, \qquad B_n = \frac{1}{n}\frac{4q^n}{1 + q^{2n}}.$$

GLAISHER 1 gives A_1, B_1, about 7D, $\theta = 0(1°)89°$; $\log A_1$, $\log B_1$, about 8D, $\theta = 0(1°)89°$; A_2, B_2, A_3, B_3, 7D, $\theta = 0(5°)85°$.

8. SHOOK 1 tabulates the arithmetic-geometric mean of 1 and x to 4S for $x = 1(.1)20(1)100$.

9. DEVISON 1 tabulates (p. 334) $\frac{1}{2}\gamma(t) = \tan^{-1}(K'/K)$ to 3D of a radian for $t = k^2 = 0(.00001).0001(.0001).0005(.0002).0015, .002(.001).010, .013, .016, .020(.005).05(.01).10(.02).2(.1).5$.

This small table is then used for the evaluation by numerical quadrature of integrals such as $\int_0^1 \frac{\gamma(t)dt}{t - \lambda}$. See *MTAC*, v. 2, p. 268, 1947.

10. See also GREAT BRITAIN, ADMIRALTY COMPUTING SERVICE 1.

[1] H. BATEMAN, *Partial Differential Equations of Mathematical Physics*, Cambridge, 1932 and New York, 1944, p. 461.
[2] J. C. P. MILLER, *Math. Gazette*, v. 30, 1946, p. 240.
[3] F. W. NEWMAN, Camb. Phil. Soc., *Trans.*, v. 13, pt. 3, 1883, p. 145–241.

Section VIII. INCOMPLETE ELLIPTIC INTEGRALS OF THE FIRST AND SECOND KINDS

Almost all tables of $F(\phi)$ and $E(\phi)$ are derived ultimately from the great Table IX of LEGENDRE (it is to be noted that SAMOĬLOVA-ĬAKHONTOVA states that her tables were constructed by interpolation in LEGENDRE). The chief exception is the independently-computed table of $E(\phi)$ by SCHMIDT, which gives more values to more decimals than LEGENDRE's table of $E(\phi)$, but has the unorthodox arguments k' and $\sin \phi$; the values are arcs of ellipses with semi-axes $a = 1$, $b = k'$ between the points $(0, k')$ and $(\sin \phi, k' \cos \phi)$. HOÜEL's second table under B below is also probably independent of LEGENDRE. No attempt has been made to seek out all abbreviations of LEGENDRE's table, which is now easily accessible through the three facsimile reproductions of POTIN, EMDE and PEARSON.

For the second integral with argument of Jacobian type, see Section X.

KAPLAN 2 has recently given important auxiliary tables for the computation of $F(\phi)$ and $E(\phi)$ when both θ and ϕ are near 90°, so that interpolation in LEGENDRE is difficult or impossible. KAPLAN puts

$$r = k'/k = \cot\theta, \qquad x = \cos\phi,$$
$$K - F(\phi) = (2/\pi)K' \sinh^{-1}(x/r) + x(r^2 + x^2)^{\frac{1}{2}} f,$$
$$E - E(\phi) = (2/\pi)(K' - E') \sinh^{-1}(x/r) + x(r^2 + x^2)^{\frac{1}{2}} e,$$

and tabulates the auxiliary functions f and e to 10D for $x^2 = -.005(.005) + .160$ and $r^2 = -.005(.005) + .160$, the negative arguments $-.005$ being included to facilitate interpolation.

As an example of one of the more extended applications of the incomplete integrals to the formation of tables of physical interest, see OSBORN 1.

VIII–A. $F(\phi)$ *and* $E(\phi)$

12D, LEGENDRE 3 (T.II), 5 (T.II), $\theta = 45°$, $\phi = 0(0°.5)90°$, Δ^4 or Δ^5
12D, LEGENDRE 3 (T.VIII), 5 (T.VIII), 7, $\theta = 0(1°)90°$, $\phi = 45°$, Δ^5 or Δ^6
10D, LEGENDRE 3 (p. 84), 5 (p. 77), $\theta = 89°$, $\phi = 0(0°.5)90°$, Δ
9–10D, LEGENDRE 3 (T.IX), 5 (T.IX), 6, 7, 8, $\theta = 0(1°)90°$, $\phi = 0(1°)90°$
10D, KAPLAN 2, $\theta = 70°(5°)85°$, $\phi = 70°(5°)85°$
5D, LÉVY 1, BOHLIN 1, DALE 1, HANCOCK 1, HAYASHI 2, $\theta = 0(5°)90°$, $\phi = 0(1°)90°$
5D, $F(\phi)$, FRICKE 1, $\theta = 0(5°)90°$, $\phi = 0(1°)90°$
5D, $E(\phi)$, FRICKE 1, $\theta = 0(10°)90°$, $\phi = 0(2°)90°$
5D, BOLL 1 (p. 352), $\theta = 0(3°)90°$, $\phi = 0(3°)90°$
5D, KIEPERT 1, $\theta = 0(5°)90°$, $\phi = 0(5°)90°$
5D, BERTRAND 1, $\theta = 0(15°)90°$, also 10°, 80°, 89°, $\phi = 0$(var.)65°(1°)90°
5D, LÁSKA 1, $\theta = 0(15°)90°$, also 10°, 80°, $\phi = 0(1°)5°(5°)85°(1°)90°$
5D, $F(\phi)$ only, GREENHILL 3, $\theta = 45°$, $\phi = 0(0°.5)90°$
4D, GLAZENAP 1, $\theta = 0(5°)90°$, $\phi = 0(1°)90°$; also $F(\phi)$, $\theta = 87°, 89°$, $\phi = 0(1°)90°$
4D, JAHNKE & EMDE 1, $\theta = 0(5°)90°$, $\phi = 0(1°)90°$, also $F(\phi)$, $\theta = 89°$, $\phi = 0$(var.) 65°(1°)90°; $F(\phi)$ has 5D for $\phi = 0(1°)5°$
4D, SILBERSTEIN 1, $\theta = 0(5°)90°$, $\phi = 0(1°)90°$; $F(\phi)$ has 5D for $\phi = 0(1°)5°$
4D, ROSENBACH, WHITMAN & MOSKOVITZ 1, $\theta = 5°(5°)90°$, $\phi = 0(1°)90°$
4D, PEIRCE 1_2–1_5, $\theta = 0(15°)90°$, also 10°, 80°, $\phi = 0(1°)5°(5°)85°(1°)90°$
4D, LIGOWSKI 1, HÜTTE 1, $\theta = 0(10°)90°$, $\phi = 0(10°)90°$
3D, MONTESSUS DE BALLORE 1, $\theta = 0(15°)90°$, $\phi = 0(5°)90°$
3D, BYERLY 1, CAMPBELL 1, $\theta = 0(6°)30°$, 37°, 45°, 53°, 64°, 90°, $\phi = 0(5°)90°$
4D, HOÜEL 1, $\theta = 0(10^g)100^g$, $\phi = 0(10^g)100^g$
5D, SAMOĬLOVA-ÎAKHONTOVA 1, $k^2 = 0(.01)1$, $\phi = 0(1°)90°$, $I\Delta$ in k^2, Δ in ϕ
4D, $F(\phi)$ only, RENO 1, $k^2 = 0(.01)1$, $\phi = 0(1°)90°$
3D, GRIFFIN 1, $k = 0(.1).9, .95, .99, 1$, $\phi = 0(10°)90°$
11D, $E(\phi)$ only, SCHMIDT 1, $k' = 0(.01)1$, $\sin\phi = 0(.01)1$, Δ^2; also 10D, $\sin\phi = .990(.001).999$, $k' = .80(.01).99$, Δ^5 in k'

VIII–B. *Log* $F(\phi)$ *and log* $E(\phi)$

4D, HOÜEL 1, $\theta = 0(10^g)100^g$, $\phi = 0(10^g)100^g$
4D, log $F(\phi)$ only, HOÜEL 1, $\theta = 90^g(1^g)100^g$, $\phi = 90^g(1^g)100^g$

VIII–C. $F(\phi)/K$

4D, HOÜEL 1, $\theta = 0(10^g)100^g$, $\phi = 0(10^g)100^g$

VIII–D. *Other combinations*

6D, $[K - F(\phi)]/\sqrt{2}$, MACGREGOR 1, $\theta = 45°$ (lemniscate case), $\phi = 0(0°.5)90°$
3D, $\int_x^\infty (1 + y^4)^{-\frac{1}{2}} dy$, BORN 1 (p. 432), $x = 0(.1)1$, $1/x = 0(.1)1$; the integral equals $K - \frac{1}{2}F(2\tan^{-1}x)$ with $\theta = 45°$

2D, $P(\alpha) = \int_0^\alpha \sin^{\frac{1}{2}} t dt = 2\int_{\frac{1}{4}\pi-\frac{1}{2}\alpha}^{\frac{1}{4}\pi} (1 - 2\sin^2 u)^{\frac{1}{2}} du$, FREEMAN 1, $\alpha = 0(5°)180°$. Also functions involving $P(\alpha)$. To reduce integrand to standard form with $k^2 = \frac{1}{2}$, put $\sin u = 2^{-\frac{1}{2}} \sin \phi$.

Section IX. Elliptic Integrals of the Third Kind

The incomplete elliptic integral of the third kind, say

$$\int_0^\phi \frac{1}{1 - p\sin^2\phi} \frac{d\phi}{\sqrt{(1 - k^2\sin^2\phi)}},$$

is apparently a function of three variables, k, p and ϕ.

But in the "hyperbolic" cases, in which $0 < p < k^2$ or $p > 1$, the integral may be expressed in terms of theta and Jacobian zeta functions of real arguments, so that it can be evaluated by means of double-entry tables.

In the "circular" cases, which tend to be more important in practice (the spherical pendulum provides an elementary dynamical example), we have $k^2 < p < 1$ or $p < 0$, and the arguments of the theta and zeta functions become complex. The circular integral has apparently not been tabulated as a function of three variables.

But the complete integral with upper limit of integration $\frac{1}{2}\pi$ can in all cases be expressed in terms of complete and incomplete integrals of the first and second kinds. In the circular case $k^2 < p < 1$ (to which the case $p < 0$ may be reduced), the complete integral has been tabulated in HEUMAN 1. The actual function tabulated is

$$\Lambda_0(\alpha, \beta) = \frac{2}{\pi}\int_0^{\pi/2} \frac{\cos^2\alpha \sin\beta \cos\beta\sqrt{(1 - \cos^2\alpha\sin^2\beta)}}{\cos^2\alpha\cos^2\beta + \sin^2\alpha\cos^2\phi} \frac{d\phi}{\sqrt{(1 - \sin^2\alpha\sin^2\phi)}},$$

so that α is the modular angle and $p = \sin^2\alpha/(\sin^2\alpha + \cos^2\alpha\cos^2\beta)$. The expression in terms of first and second integrals is

$$\tfrac{1}{2}\pi\Lambda_0(\alpha, \beta) = KE'(\beta) - (K - E)F'(\beta),$$

where K and E have modulus $\sin\alpha$, and primes are used to denote that $F'(\beta)$ and $E'(\beta)$ have modulus $\cos\alpha$. HEUMAN 1 gives $\Lambda_0(\alpha, \beta)$ to 6 decimals without differences for $\alpha = 0(1°)90°$, $\beta = 0(1°)90°$, with a supplementary 6-decimal table for $\alpha = 0(0°.1)5°.9$, $\beta = 80°(1°)89°$.

I know of no other table of a standard form of third elliptic integral. A rather special inversion of HEUMAN's table is given in connection with the quantum mechanics of a freely rotating rigid body in G. W. KING 1, 2, 3; for a detailed account, see RMT **467**, **544**.

BARTKY 1 discusses the numerical calculation of a general form of complete elliptic integral by use of arithmetic-geometric means. Defining these by

$$m_i = \tfrac{1}{2}(m_{i-1} + n_{i-1}), \qquad n_i = (m_{i-1}n_{i-1})^{\frac{1}{2}}, \qquad m_0 = m, \qquad n_0 = n,$$

he shows that, if the difference between m_3 and n_3 may be neglected, that is, if m_3 gives a sufficiently accurate approximation to the mean $M = m_\infty = n_\infty$, then

$$\int_0^{\pi/2} \frac{F(R)d\phi}{R} = \frac{\pi}{8m_3}\{\tfrac{1}{2}F(m) + \tfrac{1}{2}F(n) + F(n_1) + F(m') + F(n')\},$$

where

$$R^2 = m^2\cos^2\phi + n^2\sin^2\phi$$

and

$$m' = n_2 + (n_2^2 - n_1^2)^{\frac{1}{2}}, \qquad n' = n_2 - (n_2^2 - n_1^2)^{\frac{1}{2}}.$$

Taking $m = 1$, $n = k'$, which gives R the Legendre normal form $(1 - k^2\sin^2\phi)^{\frac{1}{2}}$, he tabulates m_3, n_1, m', n' to $4D$ for $k' = .10(.01).49$ and to 5D for $k' = .50(.01)1$. As a special case, to calculate

$$\int_0^{\pi/2} \frac{1}{1 - p\sin^2\phi} \frac{d\phi}{(1 - k^2\sin^2\phi)^{\frac{1}{2}}},$$

take

$$F(R) = k^2(k^2 - p + pR^2)^{-1}.$$

NYSTRÖM 1 discusses the practical calculation, by graphical and planimetric methods, of the normal form

$$\Pi(k, \lambda, \phi) = \int_0^\phi \frac{\sqrt{(1 - k^2 \sin^2 \psi)}}{1 + \lambda^2 \sin^2 \psi} d\psi = \int_0^\phi \frac{\sqrt{(\cos^2 \psi + k'^2 \sin^2 \psi)}}{\cos^2 \psi + \lambda'^2 \sin^2 \psi} d\psi,$$

where $\lambda'^2 = \lambda^2 + 1 > 0$. This is a circular case. NYSTRÖM 1 gives no elliptic table, though he provides an approximate nomogram relating k', λ' and the complete integral $\Pi(k, \lambda, \frac{1}{2}\pi)$ in the ranges $0 \leqslant k' \leqslant 1$, $0 \leqslant \lambda' \leqslant 10$. But he gives double-entry tables and diagrams of the elementary functions

$$x(\lambda, \psi) = 3.6\pi^{-1} \tan^{-1} (\lambda' \tan \psi),$$
$$y(k, \psi) = \sqrt{(\cos^2 \psi + k'^2 \sin^2 \psi)},$$
$$z(k, \psi) = k' + (1 - k') \cos \psi - y(k, \psi),$$

the tables being for $\lambda' = 0(.1)1(1)10$, $k' = 0(.1).9$, $\psi = 0(5°)90°$. These enable one to evaluate $\Pi(k, \lambda, \phi)$, since it is proportional to the Stieltjes integral

$$\int_0^\phi y(k, \psi) dx(\lambda, \psi).$$

Use of the auxiliary function z in place of y gives increased accuracy, since z is in general much smaller than y, and the remaining contribution to Π is an elementary integral.

TALLQVIST 1, 2 tabulate respectively magnetic potential (solid angle) due to a circular current, and gravitational potential due to a uniform circular disc. Both may be expressed either as infinite series involving Legendre functions, or in terms of elliptic integrals, including those of the third kind.

A very complicated integral expressible in terms of the third elliptic integral is tabulated for seismological purposes in CONFORTO & VIOLA 1.

Section X. THETA FUNCTIONS, JACOBIAN AMPLITUDE, SECOND INTEGRAL AND ZETA FUNCTION

We shall use the four theta functions already defined in Section IV, in which tables of theta functions of zero argument have been listed. The connection with Jacobi's earlier notation is, if $M = \pi/2K$, and $x = Mu = \pi u/2K$,

$$\vartheta_1(x) = \vartheta_1(Mu) = \mathrm{H}(u), \qquad \vartheta_2(x) = \vartheta_1(\tfrac{1}{2}\pi - x) = \mathrm{H}(K - u),$$
$$\vartheta_4(x) = \vartheta_4(Mu) = \Theta(u), \qquad \vartheta_3(x) = \vartheta_4(\tfrac{1}{2}\pi - x) = \Theta(K - u).$$

All tables of theta functions employ, substantially, the argument x; that is, they divide the quadrant (for x) or the quarter-period K (for u) into a number (say 90 or 100) of equal parts. But MILNE-THOMSON's tables of Jacobian elliptic and zeta functions have argument u at an interval (.01) which is incommensurable with K for the tabular moduli.

The few tables of the amplitude $\phi = \mathrm{am}\, u$, defined by $u = F(\phi)$, also have x or u/K as argument.

To avoid confusion, we follow GUDERMANN and other writers in defining

$$\mathrm{el}\, u = \int_0^u \mathrm{dn}^2 u du = \int_0^\phi \sqrt{(1 - k^2 \sin^2 \phi)} d\phi = E(\phi),$$

el u being denoted more often by $E(u)$ in the literature. A quantity often tabulated instead of el u is the Jacobian zeta function, or periodic part of el u, defined by

$$\mathrm{Z}(u) = \mathrm{zn}\, u = \mathrm{el}\, u - Eu/K.$$

The GLAISHER and GREENHILL notations will be explained in connection with the descriptions of the corresponding tables.

HOÜEL 1 tabulates to 4D the common logarithms of $\vartheta_n(x)$ and $\vartheta_n'(x)/\vartheta_n(x)$, $n = 1(1)4$, for $\theta = 0(10^g)90^g$, $x = 0(10^g)100^g$. Natural values deduced (with correction) from these logarithms are given to 4S in JAHNKE & EMDE 1. (In both tables, our ϑ_4 is denoted simply by ϑ.)

ROSENHEAD 1 tabulates, for $\theta = 9°$, $n = 1(1)4$,

$$\vartheta_n(x), \quad \pi d\vartheta_n(x)/dx, \quad \text{to 3–4D for } x/\pi = 0(.05).5,$$
$$\vartheta_n(iy), \quad \pi d\vartheta_n(iy)/d(iy), \quad \text{to 4D for } y/\pi = 0(.05)1.$$

The GLAISHER manuscript (GLAISHER 3, now in the possession of the BAASMTC) relates to the theta functions

$$\Theta_1(x) = \vartheta_3\vartheta_1(x)/\vartheta_2 = \vartheta_1(x)/\sqrt{k}, \qquad \Theta_2(x) = \vartheta_4\vartheta_2(x)/\vartheta_2 = \vartheta_2(x)\sqrt{(k'/k)},$$
$$\Theta_3(x) = \vartheta_4\vartheta_3(x)/\vartheta_3 = \vartheta_3(x)\sqrt{k'}, \qquad \Theta_4(x) = \vartheta_4(x).$$

The four functions were computed in 1872–75 for the BAAS under GLAISHER's superintendence. They were tabulated to 8D, both naturally and logarithmically, for $\theta = 0(1°)89°$, $x = 0(1°)90°$. The tables were completely set up in type (356 pages), but were not published, and no printed copy appears to have survived. See BAAS *Reports* of 1873 (p. 171) and 1930 (p. 250), *Mess. Math.*, v. 6, p. 111–112, 1876, and *MTAC*, v. 3, p. 92, 1948.

In 1911 GREENHILL brought forward (BAASMTC 1) a scheme for the rearrangement of the elliptic tables. He suggested that tables be made of four theta functions defined by

$$A(r) = \vartheta_1(x)/\vartheta_2, \qquad B(r) = \vartheta_2(x)/\vartheta_2, \qquad C(r) = \vartheta_3(x)/\vartheta_4, \qquad D(r) = \vartheta_4(x)/\vartheta_4,$$

where $x = r°$. These have the property, which both the ordinary and the GLAISHER forms lack, that the division-values arising at bisection, trisection, etc. of the quadrant or quarter-period are algebraic functions of the modulus; values for $x = 0(3°)90°$ may, as in the case of the trigonometric functions, be expressed by surd formulae. They also have the advantage over GLAISHER's forms of retaining simple relationships between functions of complementary arguments, since

$$B(r) = A(90 - r), \qquad C(r) = D(90 - r).$$

GREENHILL suggested a semi-quadrantal tabular arrangement, giving, at one opening for each modulus,

$$u = rK/90 = F(\phi), \qquad \phi = \text{am}\, u, \qquad E(r) = \text{zn}\, u, \qquad D(r), \qquad A(r)$$

for $r = 0(1)90$. (The notation $E(r)$ for the zeta function $Z(u) = \text{zn}\, u$ is unfortunate.) The experimental tables of this kind for $\theta = 15°, 45°, 75°$ in BAASMTC 2, 4 are practically superseded by the corresponding definitive tables in HIPPISLEY 1 or SPENCELEY 1, described below. Only in the improved table, calculated by HIPPISLEY, for $\theta = 45°$ (BAASMTC 4) is appreciably more information given (ϕ to 0°.001 instead of to 1′) than in HIPPISLEY's Smithsonian tables. There are, however, tables of the same kind in BAASMTC 2, 3, 4 for the "singular" moduli for which

$$K/K' = 1/\sqrt{2}, \sqrt{2}, 2, 3/\sqrt{2}, 2\sqrt{2}, 3, 2\sqrt{3}, 4, 3\sqrt{2}, 5, 3\sqrt{3}.$$

Details of the transformations employed are given. These tables are rather rough, but not without interest, since some of the modular angles exceed 89°, and such cases contribute appreciably to three noteworthy graphs of $D(r)$, $A(r)$ and $E(r)$ in BAASMTC 3. BAASMTC 4 also gives u, zn u, $D(r)$, $A(r)$ to 15D for the 7 and 17 sections in the lemniscate case ($\theta = 45°$), the values of u/K or $r/90$ being 0(1/7)1 and 0(1/17)1 respectively. A few numerical values relating to lemniscate sections are contained in WILTON 1.

Various lemniscate constants are given in GAUSS 1_1 (p. 31), 1_2 (p. 150), 2 (p. 364, 413–421), and various rational powers of $e^{-\pi}$ to many figures in GAUSS 2 (p. 418–432) and BAASMTC 4.

The final result of the GREENHILL-HIPPISLEY work for integral degrees of modular angle was published in the Smithsonian tables of HIPPISLEY 1. These give, in columns headed

$$F(\phi), \phi, E(r), D(r), A(r),$$

the quantities $u = rK/90$, ϕ, zn u, $D(r)$, $A(r)$ for $\theta = 5°(5°)80°(1°)89°$, $r = 0(1)90$; all values are to 10D, except those of ϕ, which are to the nearest minute. The heading for each modular angle provides values of K, K', E, E', q, $\Theta(0) = \vartheta_4$, $H(K) = \vartheta_2$; these have been listed in Sections II, III, IV.

The important GREENHILL-HIPPISLEY work has been greatly extended in the recent monumental tables of the SPENCELEYS, the most important elliptic tables ever published. These give (i) tables for each degree of θ, compared with $\theta = 5°(5°)80°(1°)89°$ in HIPPISLEY 1, (ii) Jacobian functions sn, cn, dn (see Section XI), omitted in HIPPISLEY 1, as well as theta functions, etc. as described in the next paragraph, (iii) more decimals than in HIPPISLEY 1.

Apart from three columns giving sn u, cn u, dn u, the SPENCELEY 1 tables give, in columns headed

$$u = (r/90)K = F(\phi, k), \phi, E(\phi, k), A(r), D(r),$$

values of these quantities to 12D for $\theta = 1°(1°)89°$, $r = 0(1)90$. The tabulated functions are as in the HIPPISLEY 1 tables, except that ϕ is expressed in radians and that $E(\phi, k)$ is the second integral, or el u, while HIPPISLEY's $E(r)$ is the zeta function, or zn u. The table of ϕ is by far the most extensive in existence, and will make it comparatively easy to check LEGENDRE's great double-entry tables, by inverse interpolation in SPENCELEY 1. The arrangement of all tables is quadrantal. The headings for each modular angle give values of K, K', E, E', q, q', $D(90) = \vartheta_3/\vartheta_4$, $1/D(90) = \vartheta_4/\vartheta_3$; these have been listed in Sections II, III, IV.

On a few theta functions tabulated in NBSCL 2, see *MTAC*, v. 1, p. 126, 1943.

LEGENDRE 3 (p. 96), 5 (p. 88) gives ϕ to 6D of 1″ with Δ^3 and el $u = E(\phi)$ to 11D with Δ^4 for $\theta = 45°$, $200\, u/K = 0(1)20$. LEGENDRE 3 (T.VII), 5 (T.VII) gives ϕ to 7D of 1″ with Δ^3 for $\theta = 0(0°.1)45°$, $u = K/10$.

MILNE-THOMSON 4 gives $Z(u) =$ zn u to 7D, with second differences in u, for $k^2 = .1(.1)1$ and for u at interval .01 from 0 to a limit 2, 2.5 or 3, depending on k^2, but always greater than K (except of course at $k^2 = 1$, where K is infinite).

FOX & MCNAMEE 1 have recently tabulated the Jacobian zeta function of complex argument. They write

$$f_1(\psi_1, \phi, \alpha) + if_2(\psi_1, \phi, \alpha) = \text{zn}\,(K\psi_1 + iK'\phi, k) + i\pi\phi/(2K),$$

the notation ψ_1 and ϕ for the fractions of the quarter-periods arising from their potential problem, and α denoting the modular angle $\sin^{-1} k$. It is unnecessary to tabulate f_1 and f_2 separately, as the authors show that $f_2(\psi_1, \phi, \alpha) = f_1(1 - \phi, 1 - \psi_1, \frac{1}{2}\pi - \alpha)$. They tabulate f_1 to 3S for $\alpha = 1°(1°)5°(5°)85°(1°)89°$, $\psi_1 = 0(.1)1$, $\phi = 0(.1)1$.

On coefficients in an expansion for am u, see Section VII–7 (GLAISHER 1).

Section XI. JACOBIAN ELLIPTIC FUNCTIONS

If $x = Mu = \pi u/2K$, we have

$$\text{sn}\, u = \frac{1}{\sqrt{k}}\frac{\vartheta_1(x)}{\vartheta_4(x)}, \qquad \text{cn}\, u = \sqrt{\frac{k'}{k}}\frac{\vartheta_2(x)}{\vartheta_4(x)}, \qquad \text{dn}\, u = \sqrt{k'}\frac{\vartheta_3(x)}{\vartheta_4(x)}.$$

The connection with the theta functions of GLAISHER 3 is

$$\text{sn}\, u = \Theta_1(x)/\Theta_4(x), \qquad \text{cn}\, u = \Theta_2(x)/\Theta_4(x), \qquad \text{dn}\, u = \Theta_3(x)/\Theta_4(x).$$

Elliptic functions may be computed from HIPPISLEY's tables by the formulae

$$\text{sn}\, u = \frac{1}{\sqrt{k'}}\frac{A(r)}{D(r)}, \qquad \text{cn}\, u = \frac{B(r)}{D(r)}, \qquad \text{dn}\, u = \sqrt{k'}\frac{C(r)}{D(r)},$$

where $r = 90u/K$.

The great SPENCELEY 1 tables give sn u, cn u, dn u to 12D for $\theta = 1°(1°)89°$, $r = 90u/K = 0(1)90$, that is, $x = 0(1°)90°$.

The NBSCL 2 manuscript gives sn u, cn u, dn u to 15D for $k^2 = 0(.01)1$, $u/K = .01$, $.1(.1)1$, that is, $x = 1^g, 10^g(10^g)100^g$. The RENO 1 manuscript tabulates sn u for $k^2 = 0(.01)1$, $u/K = 0(.01)1$. BYERLY 1 gives sn u, cn u, dn u to 3D for $k^2 = \frac{1}{2}$, $u = .05(.1)1.85$.

MILNE-THOMSON 3 tabulates sn u, cn u, dn u to 5D, with first differences in u, for $k^2 = 0(.1)1$ and for u at interval .01 from 0 to a limit 2 or more, depending on k^2, but greater than K except at $k^2 = 1$.

F. W. NEWMAN 1 (p. 126), 2 (p. 131) gives $-\log \mathrm{dn}\, \frac{2}{3}K$ to 16D for $\rho = \pi K'/2K = -\frac{1}{2}\ln q = 1(.1)6.4$.

BERGMANN 1 tabulates sn $(u + iv)$ to 2–3D for $\theta = 15°, 20°, 75°$, $u = 0(K/10)K$, $v = 0(K'/10)K'$.

On coefficients in the Fourier expansion of sn u cn u, see Section VII–7 (GLAISHER 1).

S. A. E. WAR ENGINEERING BOARD 1 gives (p. 346) γ to 5D of 1° and sin γ to 7D, where $\sin \frac{1}{2}\gamma = \sin \frac{1}{2}\gamma_0 \,\mathrm{sn}\, u$, where $\gamma_0 = 2\theta = 10°(10°)160°$ and $\tau = \pi u/2K = 0(10°)90°$. More specialized tables of a similar kind follow. See *MTAC*, v. 2, p. 121–122, 1946.

Section XII. WEIERSTRASSIAN FUNCTIONS

Accounts of the Weierstrassian functions are to be found in various texts, for instance in WHITTAKER & WATSON, *Modern Analysis*. Tables relating to such functions are not numerous. WEIERSTRASS & SCHWARZ 1 provides a good collection of formulae and theorems, and on p. 7 tabulates coefficients in the expansion of the sigma function $\sigma(u)$. If

$$\sigma(u) = \sum_{m,n} a_{mn}(\tfrac{1}{2}g_2)^m(2g_3)^n u^{4m+6n+1}/(4m + 6n + 1)!\ (m, n = 0, 1, 2, \cdots)$$

the table gives the (integral) values of a_{mn} for the 33 combinations of m and n for which $4m + 6n + 1 \leqslant 35$. A number of functions are expanded to a considerable length in the lists in TANNERY & MOLK 1, v. 4, p. 88f.

Several Weierstrassian formulae for evaluating half-periods and incomplete integrals (WEIERSTRASS & SCHWARZ 1, p. 67–71) involve quantities $\mathfrak{L}_{0,1}$, $\mathfrak{L}_{0,2}$, $\mathfrak{L}_{0,3}$ defined by omitting from the series

$$\mathfrak{L}_0 = 1 + (\tfrac{1}{2})^2 l^4 + \left(\frac{1.3}{2.4}\right)^2 l^8 + \left(\frac{1.3.5}{2.4.6}\right)^2 l^{12} + \cdots$$

its first, first two, and first three terms respectively. The three quantities are tabulated, when appreciable to the number of decimals given (10 or more), with first differences in the case of $\mathfrak{L}_{0,1}$, in NAGAOKA & SAKURAI 1, p. 56, for $k^2 = 0(.001).5$, where $l = (1 - \sqrt{k'})/(1 + \sqrt{k'})$.

GREENHILL published two tables, both calculated by A. G. HADCOCK, relating to the Weierstrassian function $\wp(u)$ for which

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3$$

in the "equianharmonic" case $g_2 = 0$, the discriminant $g_2^3 - 27g_3^2$ being negative and consequently two roots of the cubic complex.

GREENHILL 1 tabulates $\wp(u)$ to 5–7S when $g_3 = 4$, for arguments $u = r\omega_2/180$ and $u = r\omega_2'/180$, where $r = 0(1)180$ in both cases. Here ω_2 is the real half-period, and $\omega_2' = \omega_3 - \omega_1 = i\omega_2\sqrt{3}$. The corresponding Jacobian modulus is sin 15°, and $\omega_2 = K/3^{\frac{1}{4}}$. The last figures are very inaccurate (see Part III).

GREENHILL 2 tabulates $\wp'(u)$, $\wp(u)$, $\zeta(u)$ and $\sigma(u)$ to about 5D when $g_3 = 1$, for $u = r\omega_2/180$, $r = 0(1)240$. The Jacobian modulus is still sin 15°, and the real half-period ω_2 is now

$$\omega_2 = 2^{\frac{2}{3}}K/3^{\frac{1}{4}} = \tfrac{1}{2}\{\Gamma(\tfrac{1}{3})\}^2/[3^{\frac{1}{2}}\Gamma(\tfrac{2}{3})] = 1.52995.$$

The values of $\wp(u)$ are $2^{-\frac{2}{3}}$ times those in GREENHILL 1. The table is reproduced in GOSSOT 1, and to not more than 4D in JAHNKE & EMDE 1.

In the case when g_2, g_3 are real and the discriminant $g_2^3 - 27g_3^2$ is positive, so that the cubic has real roots, I have encountered no tables of $\wp(u)$. But owing to the occurrence of this case of real roots in Gauss's elliptic ring method of calculating secular perturbations of planets, there have been several tables from astronomical sources for calculating the quantities ω and η, in terms of which various real complete integrals involving $\sqrt{S} = \sqrt{(4s^3 - g_2 s - g_3)}$ may be expressed.

Supposing the roots of the cubic in the order $e_1 > e_2 > e_3$, and all surds positive, we have

$$\omega = \omega_1 = \int_{e_3}^{e_2} \frac{ds}{\sqrt{S}} = \int_{e_1}^{\infty} \frac{ds}{\sqrt{S}},$$

$$\eta = \eta_1 = -\int_{e_3}^{e_2} \frac{sds}{\sqrt{S}}.$$

The expressions in terms of complete elliptic integrals of modulus given by $k^2 = (e_2 - e_3)/(e_1 - e_3)$ are

$$\omega = K/\sqrt{(e_1 - e_3)}, \qquad \eta = \sqrt{(e_1 - e_3)}\{E - \tfrac{1}{3}(2 - k^2)K\},$$

but it was pointed out by H. Bruns[1] that ω and η may be evaluated in terms of hypergeometric functions, without first finding the roots e_1, e_2, e_3. If an absolute invariant g is defined by

$$g = \frac{g_2^3}{27g_3^2} = \frac{4(1 - k^2k'^2)^3}{\{(2 - k^2)(1 + k^2)(1 - 2k^2)\}^2},$$

and $x = 1 - g^{-1}$, then, if $g_3 \geqslant 0$,

$$\omega(12g_2)^{\frac{1}{4}}/\pi = F_\omega = F(\tfrac{1}{12}, \tfrac{5}{12}, 1, x),$$
$$12\eta(12g_2)^{-\frac{1}{4}}/\pi = F_\eta = F(-\tfrac{1}{12}, \tfrac{7}{12}, 1, x).$$

Arndt 1 tabulates F_ω and F_η to 7D with Δ for $x = 0(.001)1$. Interpolation is difficult when $x > .98$; Arndt recommends instead the use of formulae involving four hypergeometric series, for which he tabulates coefficients of powers of $1 - x$.

If a different absolute invariant γ is defined by

$$\cos 3\gamma = g_3\sqrt{(27/g_2^3)}, \qquad 0 \leqslant 3\gamma \leqslant 180°,$$

$\cos 3\gamma$ having the same sign as g_3, then whichever this sign is, we have[2]

$$F_\omega = F(\tfrac{1}{6}, \tfrac{5}{6}, 1, \sin^2 \tfrac{3}{2}\gamma) = P_{-\frac{1}{6}}(\cos 3\gamma),$$
$$F_\eta = F(-\tfrac{1}{6}, \tfrac{7}{6}, 1, \sin^2 \tfrac{3}{2}\gamma) = P_{\frac{1}{6}}(\cos 3\gamma),$$

P as usual denoting a Legendre function. Witt 3 supplements Arndt 1 by tabulating F_ω and F_η to 8D with Δ for $\xi = .4(.001).5$, where

$$\xi = \tfrac{1}{2}\{1 - \sqrt{(1 - x)}\} = \tfrac{1}{2}(1 - g^{-\frac{1}{2}}) = \sin^2 \tfrac{3}{2}\gamma.$$

The range in ξ corresponds to $.96 \leqslant x \leqslant 1$, the region of difficulty in Arndt 1. Witt 3 tabulates also, logarithmically, coefficients in continued fractions for F_ω and F_η.

The tables of Arndt 1 give ω_1 and η_1 only when $g_3 \geqslant 0$, corresponding to $0 \leqslant 3\gamma \leqslant 90°$, and Witt 3 does not extend the range. When $g_3 < 0$, they may be used to determine ω_3 and η_3. For, connected with a given cubic expression, $S = 4s^3 - g_2s - g_3$, having real g_2 and g_3 such that $g_2^3 > 27g_3^2$, there are four positive quantities, ω, η and what I propose here to call ω^*, η^*, such that

$$\omega_1 = \omega, \qquad \omega_3 = i\omega^*, \qquad \eta_1 = \eta, \qquad \eta_3 = -i\eta^*, \qquad \eta\omega^* + \eta^*\omega = \tfrac{1}{2}\pi.$$

ω and η are as defined above, while

$$\omega^* = \int_{e_2}^{e_1} \frac{ds}{\sqrt{(-S)}} = \int_{-\infty}^{e_3} \frac{ds}{\sqrt{(-S)}}, \qquad \eta^* = \int_{e_2}^{e_1} \frac{sds}{\sqrt{(-S)}},$$

all square roots being taken positively. For given g_2, we have

$$\omega^*(g_3) = \omega(-g_3), \qquad \eta^*(g_3) = \eta(-g_3).$$

Thus it is immaterial whether we consider two functions, say ω and η, in the whole range $0 \leqslant 3\gamma \leqslant 180°$, or four functions in half the range, say $0 \leqslant 3\gamma \leqslant 90°$. This is similar to the well-known choice between quadrantal and semi-quadrantal tabulation of the trigonometric functions or of the complete integrals K, E, K', E'. However the matter is regarded, the tables of Arndt 1 give only half the desirable information, the sign of g_3 being lost in the argument $x = 1 - 27g_3^2/g_2^3$.

INNES used an absolute invariant ι (iota), sometimes with numerical suffix, identical with 3γ above. As ι is an inconvenient symbol, and the angle γ (rather than thrice it) is the simplest quantity to use in writing down the trigonometric solution of the cubic, I shall describe the INNES-ROBBINS-MERFIELD tables in the γ notation. These tables between them cover the whole range $0 \leqslant 3\gamma \leqslant 180°$, and so form a complete set of tables for calculating ω, η, ω^*, η^* for any γ. The formulae for this purpose are set out in INNES 4. They involve the hypergeometric series

$$F(-\tfrac{1}{6}, \tfrac{7}{6}, 2, \sin^2 \tfrac{3}{2}\gamma) \qquad \text{and} \qquad F(\tfrac{1}{6}, \tfrac{5}{6}, 2, \sin^2 \tfrac{3}{2}\gamma),$$

which, unlike the series for $P_{\pm\frac{1}{6}}$ (cos 3γ), remain finite (and in fact vary little) over the whole range $0 \leqslant 3\gamma \leqslant 180°$. Logarithms of these two hypergeometric functions are tabulated to 7D with Δ^2 for $3\gamma = 0(1°)90°$ in both ROBBINS 1 and INNES 4, and for $3\gamma = 90°(1°)180°$ in MERFIELD 2. Logarithms of the first 20 coefficients in each series are given to 8D in MERFIELD 1.

If the four quantities ω, η, ω^*, η^* are to be calculated for given g_2, g_3, whichever the sign of g_3, two hypergeometric functions are taken from the INNES-ROBBINS table, and two from the table of MERFIELD 2; the latter is an important complement to the former. A small illustrative table of $2\omega/\pi$, $2\eta/\pi$, etc. is given for 16 values of γ in INNES 4 (p. 363). This interesting table covers the whole range, but the use of either γ (or 3γ) or $g_3\sqrt{(27/g_2^3)}$ as equidistant argument would have been better; the table would have been symmetrical, and moreover it is curious to see the roots of the cubic figuring as primary arguments when the work aims at avoiding their determination.

The above tables are astronomically related to those of HILL 1, CALLANDREAU 1 and INNES 1. Those of HILL 1, having $\theta = \sin^{-1} k$ as argument, have been mentioned in Section VII–5. CALLANDREAU 1 and INNES 1 are too specialized to be of much general mathematical interest, and all three may probably be regarded as superseded by the later work described above. Discussions of the various methods from an astronomical point of view are given in INNES 3 and by E. DOOLITTLE.[3]

[1] H. BRUNS, *Ueber die Perioden der elliptischen Integrale erster und zweiter Gattung*, Dorpat, 1875. (I have seen only the reprint in *Math. Ann.*, v. 27, 1886, p. 234–252. It is here stated that this reprint is "einer im Jahre 1875 zum Doctorjubiläum von V. J. Buniakowsky von der physiko-mathematischen Facultät der Universität Dorpat dargebrachten Festschrift.")

[2] H. C. PLUMMER, *An Introductory Treatise on Dynamical Astronomy*, Cambridge, 1918, p. 214.

[3] E. DOOLITTLE, *The Secular Variations of the Elements of the Orbits of the Four Inner Planets*, Amer. Phil. Soc., *Trans.*, n.s., v. 22, part 2, Philadelphia, 1912.

Section XIII. DIXON FUNCTIONS

A theory of elliptic functions based upon the cubic curve $x^3 + y^3 - 3\alpha xy = 1$ was developed by A. C. DIXON.[1] The case $\alpha = 0$, which corresponds to the "equianharmonic" case of Weierstrassian functions, has been used in O. S. ADAMS 1 in calculations concerning map projections. When $\alpha = 0$, functions sm u and cm u are defined by

$$x = \operatorname{sm} u, \qquad y = \operatorname{cm} u, \qquad \operatorname{sm}^3 u + \operatorname{cm}^3 u = 1,$$

$$u = \int_0^x (1 - x^3)^{-\frac{2}{3}} dx = \int_y^1 (1 - y^3)^{-\frac{2}{3}} dy,$$

and a real period 3λ is given by

$$\lambda = \int_0^1 (1 - x^3)^{-\frac{2}{3}} dx = \tfrac{1}{3}\{\Gamma(\tfrac{1}{3})\}^2/\Gamma(\tfrac{2}{3}) = 1.76663\ 875.$$

ADAMS 1 writes K instead of λ, but DIXON's notation is more convenient here.

The connection with the equianharmonic Weierstrassian functions tabulated in GREENHILL 2 and JAHNKE & EMDE 1 (see Section XII) is

$$\operatorname{sm} u = \frac{2\sqrt{3}\wp(u/\sqrt{3})}{\sqrt{3} - \wp'(u/\sqrt{3})}, \qquad \operatorname{cm} u = \frac{\wp'(u/\sqrt{3}) + \sqrt{3}}{\wp'(u/\sqrt{3}) - \sqrt{3}}.$$

$u = \lambda$ corresponds to $r = 120$ in these Weierstrassian tables, and ADAMS 1 constructs his table with arguments $u = 0(\lambda/120)\lambda$. He gives sm u, cm u and sm u/cm u to 4D. In addition, he gives $1 -$ cm u to 8–5D for $u = 0(\lambda/120)\lambda/3$. Values in the remaining two-thirds of the period are given by

$$\text{sm}\,(\lambda + u) = 1/\text{cm}\,u, \qquad \text{cm}\,(\lambda + u) = -\,\text{sm}\,u/\text{cm}\,u,$$
$$\text{sm}\,(2\lambda + u) = -\,\text{cm}\,u/\text{sm}\,u, \qquad \text{cm}\,(2\lambda + u) = 1/\text{sm}\,u.$$

[1] A. C. DIXON, *Quart. Jn. of Math.*, v. 24, 1890, p. 167–233.

PART II

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PART III

Errors in Elliptic Tables

Note. Absence of a table from this part does not mean that it contains no errors. I have included all information pointed out by others and known to me, but **Part III** describes mainly my own investigations. My program related only to single-entry tables, mainly of a basic and many-figure character. Inclusion of an author in **Part III** does not mean that all his elliptic tables have been examined; only those mentioned have been examined. I have checked no double-entry tables whatever; but for the convenience of the user I have placed together under LEGENDRE the known errors in his great double-entry tables of $F(\theta, \phi)$ and $E(\theta, \phi)$. Anyone using one of the many abbreviations of these double-entry tables should see whether they are affected by the errors mentioned under LEGENDRE.

AIREY 1.

(α) K_1, K_2, k'^2 = [0(.00001).00010; 13D].

Differencing shows freedom from gross error, in both function values and second differences. I have also confirmed the values at .00010 by direct calculation. In most of these tables, Airey gives smoothly-varying second differences (presumably rounded from values computed with extra decimals).

(β) K_1, K_2, k'^2 = [0(.0001).0010; 12D].

Differencing shows complete freedom from gross error. I have also confirmed the values at .0010 by direct computation.

(γ) K_1, K_2, k'^2 = [0(.001).170; 10D].

There is a gross error in K_2 at $k'^2 = .014$; *for* 352, *read* 353. When this is corrected, both K_1 and K_2 difference well. Airey's smoothed second differences agree well with the second differences of his function values; the two quantities rarely differ by two final units.

(δ) E_1, E_2, k'^2 = [0(.00001).00010; 12D].

At $k'^2 = .00010$, E_1 *should end in* 75, *not* 57. Apart from this, differencing shows complete freedom from gross error. At .00010 I have also confirmed E_1 (as corrected) and E_2 by direct computation.

(ϵ) E_1, E_2, k'^2 = [0(.0001).0010; 11D].

Differencing shows complete freedom from gross error. I have also confirmed nearly half the values by direct computation.

(ζ) E_1, E_2, k'^2 = [0(.001).100; 10D].

Differencing shows complete freedom from gross error, Airey's second differences again agreeing well with the second differences of his printed values. But recalculation with extra decimals for $k'^2 = .001(.001).010$, while it confirms the values of E_2, shows that the values of E_1 for $k'^2 = .003(.001).009$ average about a unit too low, the deduced values of E being (since E_1 has to be multiplied by about 4) too low by from 2 to 5 units of the tenth decimal. The errors cannot be due to Airey's having used incorrect coefficients in his power series, for his E_1 is correct for much higher values of k'^2, for instance, for $k'^2 = .1$ at the end of the table.

BERTRAND 1.

(α) $\log q$, θ = [0(5′)90°; 5D].

I have compared this table with MEISSEL's 8D table. There are only three errors in MEISSEL 1 which could affect Bertrand's abridgment; all three (marked M below) occur in Bertrand, and are proof that Bertrand's table was extracted without acknowledgment from MEISSEL's. In addition, BERTRAND (or his collaborator THOMAN) has rounded MEISSEL's value incorrectly in two cases. At 8°45′ and 51°15′ MEISSEL's values end in 500, but I have not thought it necessary to investigate more closely.

	θ	D	*For*	*Read*
M	17°20′	4	0	1
M	46 25	5	1	2
	51 20	5	3	4
	55 0	5	1	2
M	65 5	5	4	5

(β) $\log \vartheta_3$ [called $\log \Theta(\omega)$], θ = [0(30′)90°; 9D].

The function tabulated is $\frac{1}{2}\log(2K/\pi)$. I have recalculated the table, using 12D values of $\log K$ from LEGENDRE (slightly corrected where necessary). There are the following errors:

θ	D	*For*	*Read*	θ	D	*For*	*Read*
6°30′	7	1	6	52°30′	4	6	5
7 30	9	1	7	67 30	0	9	0
8 0	8–9	81	08	69 30	9	1	2
41 0	9	4	3	78 0	9	0	1
51 30	9	0	1	82 30	9	6	7

DALE 1.

(α) K, E, θ = [0(1°)90°; 5D].

There is one rounding error; in K at 88°, *for final* 1, *read* 2.

DWIGHT 2.

(α) $K/\ln(4 \sec\theta)$, θ = [86°(1′)90°; 7 or more D].

Differencing shows that the table is free from gross error up to at least 89°50′, and the values at 89°50′(1′)90° I have confirmed by recalculation. Dwight's differences agree with his function values.

(β) K, θ = [86°(1′)90°; 5D].

Differencing shows that the table is free from gross error up to at least 89°50′, and the values at 89°50′(1′)90° I have confirmed by recalculation. Dwight's differences agree with his function values. Comparison of Dwight's values at θ = 86°(6′)90° with antilogarithms of LEGENDRE's log K shows all such values correct, except for one very fine rounding error; at θ = 87°24′, *for* 4.48114, *read* 4.48115. There may be other unimportant rounding errors, but both tables are evidently of high accuracy.

GAUSS 2

The argument in Gauss's table on p. 403 will be called γ. If θ is the usual modular angle, $\gamma = 180° - 2\theta$. Thus Gauss's γ = 1°(1°)180° is equivalent to θ = 89°.5(−0°.5)0.

(α) $M = \pi/2K$, γ = [1°(1°)180°; 7D].

There are six errors which are not merely rounding errors:

γ	θ	D	*For*	*Read*
2°	89°.0	5–7	201	198
13	83 .5	7	4	5
41	69 .5	7	4	6
82	49 .0	7	1	2
156	12 .0	7	8	6
179	0 .5	6	0	1

There are also some 25 rounding errors.

(β) log M, γ = [1°(1°)180°; 7D].

There are five errors which are not merely rounding errors:

γ	θ	D	*For*	*Read*
2°	89°.0	6–7	80	75
7	86 .5	7	1	0
13	83 .5	5–7	699	701
41	69 .5	7	2	3
45	67 .5	7	4	5

There are also 15 rounding errors.

(γ) On a number of errors in the lemniscate constants on p. 413–432, see J. W. WRENCH, JR., *MTAC*, v. 3, p. 201, 1948.

GLAISHER 1.

(α) q, θ = [1°(1°)89°; 8D].

Before receiving the SPENCELEY tables I had differenced Glaisher's table and found it free from gross error up to at least 80°. I have since compared the two tables, and found Glaisher in complete agreement with the SPENCELEY values rounded to 8D. Glaisher thus appears free from error of any kind.

(β) $\log q$, $\theta = [1°(1°)89°; 8D]$.

Two very slight rounding errors, exactly as in MEISSEL: at 33°, *for* 819, *read* 820, and at 42°, *for* 332, *read* 333.

I have not examined the other tables.

GOSSOT, see GREENHILL 2.

GREENHILL 1.

(α) $s = \wp(r\omega_2/180)$, $S = \wp(r\omega_2'/180)$, $r = 0(1)180$, 5 or more figures, case $\wp'^2 = 4\wp^3 - 4$.

I have not examined the table in detail, but it may be useful to warn that the final digits are demonstrably unreliable. The values near the end of the table should be derivable from those near the beginning by the relations

$$s(180 - r) - 1 = 3/[s(r) - 1], \quad S(180 - r) - 1 = 3/[S(r) - 1],$$

but some of Greenhill's values do not satisfy this test fully. The table could easily be recalculated, using HIPPISLEY or SPENCELEY at $\theta = 15°$ and 75°, since for $0 \leqslant r \leqslant 90$

$$s(r) = 1 + \sqrt{3}[D(r) + B(r)]/[D(r) - B(r)] = 1 + \sqrt{3}\cot^2 \tfrac{1}{2}\phi \qquad (\theta = 15°)$$
$$S(r) = 1 - \sqrt{3}[D(r) + B(r)]/[D(r) - B(r)] = 1 - \sqrt{3}\cot^2 \tfrac{1}{2}\phi \qquad (\theta = 75°)$$

with obvious modifications when $90 \leqslant r \leqslant 180$.

GREENHILL 2.

(α) $\wp(r\omega_2/180)$, $\wp'(r\omega_2/180)$, $\zeta(r\omega_2/180)$, $\sigma(r\omega_2/180)$, $r=0(1)240$, about 5D, case $\wp'^2 = 4\wp^3 - 1$.

I have not examined the tables for errors. Two errors were pointed out in O. S. ADAMS, and one by MILLER:

Function	r	*For*	*Read*
$\wp(u)$	3	1468.820	1537.9625
$\wp'(u)$	35	−73.4302	−75.9603
$\sigma(u)$	123	1.0441	1.0438

See *MTAC*, v. 1, p. 109, 1943, and v. 1, p. 395, 1945. The errors occur also in the reprints in GOSSOT and in JAHNKE & EMDE 1_1–1_3, but are corrected in JAHNKE & EMDE 1_4.

HANCOCK 1.

(α) K, E, $\theta = [0(1°)70°(30')80°(12')89°(6')90°; 5D]$.

With one exception, the table is accurately copied from LÉVY 1, none of the errors (which I list under LÉVY) being corrected. The additional error introduced by Hancock is in E at 89°54′; in the fourth decimal, *for* 1, *read* 0.

HAYASHI 1.

This contains a great variety of elliptic tables with k^2 as argument. I have examined only those tables mentioned below, which appeared to me to be the most important. The volume was published in 1930, and the *Berichtigungen* sheet of 1932 will be lacking in copies bought before its appearance. Thus I have thought it useful to include in the lists of corrections those given by Hayashi himself in relation to the tables which I have examined, especially since Hayashi's corrections require occasional emendation. I have not reprinted his corrections relating to unexamined tables.

(α) K, $k^2 = [0(.001)1; 10\text{–}12D]$.

There are the following errors:

p.	k^2	D	*For*	*Read*
72	.999	8–9	06	60
72	.013	8–9	68	86
72	.987	11–12	53	47
73	.939	12	4	1
73	.095	0	0	1
74	.877	11–12	61	57
77	.737	9–10	56	65
81	.493	9	4	9

Those at .999, .737 and .493 were corrected in the 1932 *Berichtigungen* sheet, which also gives a false and unnecessary correction at .956. The obvious error at .095 was pointed out by SAMOĬLOVA-ĪAKHONTOVA, and the error at .013 by KAPLAN. See also *MTAC*, v. 2, p. 127, 1946, and COMRIE, *Math. Rev.*, v. 7, 1946, p. 485. As far as I am aware, the errors of 6, 4 and 3 final units at .987, .877 and .939 respectively have not previously been published.

When the above corrections are made, differencing (up to twentieth differences at the end) shows freedom from gross error up to at least $k^2 = .984$. The last 15 values, for $k^2 = .985(.001).999$, I have recomputed to about 14 decimals; except for the errors at .987 and .999, mentioned above, Hayashi is completely correct.

(β) $\log q$, k^2 = [0(.001)1;10D].

The first four of the following corrections were given in Hayashi's 1932 *Berichtigungen* sheet (the argument .232 being misprinted .223), and the fifth in SAMOĬLOVA-ĪAKHONTOVA 1.

p.	k^2	*D*	*For*	*Read*
73	.050	3	3	5
74	.874	10	8	7
76	.232	4–5	62	26
78	.688	10	8	9
81	.503	1–2	36	63

When these corrections are made, differencing shows freedom from gross error, except that some ten or twelve values at each end of the table cannot be completely checked by differencing. I have, however, recomputed 28 values, for $k^2 = .001(.001).014$ and $k^2 = .986(.001).999$, to about 13D, and find no important error (my values suggest rounding errors at $k^2 = .010, .012, .013$, where *for final* 8, 1, 7, I find 7, 0, 6 respectively).

(γ) q, k^2 = [0(.001).5;8D].

Hayashi's 1932 *Berichtigungen* sheet gives the following corrections:

p.	k^2	*D*	*For*	*Read*
94	.279	8	7	8
98	.395	5	4	6
102	.482	3	0	1

When these corrections are made, differencing shows freedom from gross error. The last decimal appears to be good, which is not surprising, as Hayashi also tabulates 10D logarithms; see (β) above. Every tenth value has been compared with the 10D value given in MILNE-THOMSON 1; this failed to disclose any rounding errors in Hayashi at $k^2 = 0(.01).5$, and computation shows that Hayashi is correctly rounded at $k^2 = .47$, where MILNE-THOMSON ends in 50.

(δ) $q^{\frac{1}{4}}$, k^2 = [0(.001).5;8D].

Hayashi's 1932 *Berichtigungen* sheet gives the following corrections (that at .118 is illegible, but should be as below):

p.	k^2	*D*	*For*	*Read*
84	.008	8	7	8
84	.024	3–5	648	739
86	.068	3–8	816360	756984
86	.077	4	1	0
86	.081	8	6	3
88	.118	4–8	58203	63000
88	.125	7–8	70	66
88	.142	5–7	928	784
90	.161	6–8	493	049
92	.236	0–5	011	0.36011
94	.256	6–8	728	073
96	.315	4–5	60	06

The above leaves the following errors uncorrected:

p.	k^2	*D*	*For*	*Read*
84	.028	5	6	5
88	.120	7	2	0
90	.193	0–5	022	0.34022

When all the above corrections have been made, differencing shows freedom from gross error, except that about 8 values at the beginning of the table cannot be checked to the full 8 decimals by differencing. I have, however, recomputed the first 14 values, for $k^2 = .001$ (.001).014, to 10 decimals, and confirmed Hayashi's results completely.

(ϵ) $\pi\vartheta_1'$, k^2 = [0(.001).5;8D].

The function is called $\vartheta_1'(0)$ in Hayashi. His 1932 *Berichtigungen* sheet gives the following corrections:

p.	k^2	*D*	*For*	*Read*
84	.049	3	1	7
92	.207	8	6	5
94	.292	5	1	2
96	.312	5	4	6
96	.342	4	2	1
102	.498	5	3	8

This leaves one gross error uncorrected: p. 102, k^2 = .465, 5D, *for* 8, *read* 9.
When all these corrections have been applied, differencing shows freedom from gross error, except that the first 9 values cannot be fully checked by differencing. I have, however, recomputed the first 10 values, for k^2 = .001(.001).010, to 9 or more decimals, and confirmed Hayashi's values (apart from one or two possible slight rounding errors).

(ζ) ϑ_2, k^2 = [0(.001).5;8D].

Hayashi's 1932 *Berichtigungen* sheet gives the following corrections:

p.	k^2	*D*	*For*	*Read*
97	.328	5	9	4
101	.442	4–5	37	73

When these are applied, differencing shows that no gross error remains, except that the first 8 values cannot be fully checked by differencing. I have, however, recomputed the first 10 values, for k^2 = .001(.001).010, to about 10 decimals, and confirmed Hayashi's results.

(η) ϑ_3, k^2 = [0(.001).5;8D].

Hayashi's 1932 *Berichtigungen* sheet gives the following correction:

p. 85, k^2 = .026, 7D, *for* 8, *read* 9.

When this is applied, differencing shows freedom from gross error throughout the table.

(θ) ϑ_4, k^2 = [0(.001).5;8D].

This is called $\vartheta_0(0)$ in Hayashi. Differencing reveals one gross error:

p. 88, k^2 = .112, 6D, *for* 2, *read* 3.

Otherwise there are no gross errors.

Hayashi 3.

(α) *E*, k^2 = [0(.001)1;10D].

There are known errors as follows (Kaplan; also Comrie, *Math. Rev.*, v. 7, 1946, p. 485):

p.	k^2	*D*	*For*	*Read*
62	.052	9	4	5
62	.939	9	4	3
62	.936	9	9	8
63	.201	3–4	65	86
63	.732	10	6	8
64	.668	0	0	1

At .669, the first two digits of the argument are missing. SAMOĬLOVA-͡IAKHONTOVA gave a faulty correction at .201, misprinting the argument as .151 and failing to correct the fourth decimal.

When the above corrections are made, differencing shows freedom from gross error up to at least $k^2 = .991$. I have recomputed the values for $k^2 = .990(.001).999$, and found only two rounding errors; at $k^2 = .991$, *for final* 1, *read* 0, and at $k^2 = .996$, *for final* 8, *read* 9.

Comparison with my 12D calculations at $k = .1(.1).9$, with SCHMIDT 1 at $k' = .1(.1).9$ and with SPENCELEY 1 at $\theta = 30°, 45°, 60°$ reveals only one very fine rounding error; at $k^2 = .360$, *for final* 5, *read* 4. Slight roughness in differences near $k^2 = .938$ points to at least one other last-figure error, but in general the final digit appears to be good.

HEUMAN 1.

It is not surprising that the 6D tables of $F_0(\alpha)$ and $E_0(\alpha)$, our $2K/\pi$ and $2E/\pi$, are very exact, for Heuman says that he calculated to 10D, from LEGENDRE'S logarithms. As these tables are, as far as I know, the only *natural* tables of, substantially, K and E at interval 0°.1 throughout the quadrant, I have examined them with care.

(α) $F_0(\alpha) = 2K/\pi$, α = [0(0°.1)90°;6D].

Assuming the values at 67°.5 and 76°.8 corrected to 1.527948 and 1.838639, as shown in Heuman's errata, the table is then certainly free from anything like gross error. I have differenced up to 70° and also recomputed to at least 8D the values at 0(0°.5)70°(0°.1)90°. In the values recomputed, I have found only one certain rounding error; at 5°.5, *for final* 8, *read* 9. I have not investigated further cases where my seventh and eighth decimals were 50. All Heuman's differences accurately correspond to his function values (corrected at 67°.5 and 76°.8).

(β) Auxiliary function $G_0(\alpha)$, α = [65°(0°.1)90°;6D].

Differencing shows that the table is free from gross error, and that Heuman's differences accurately correspond to his function values.

(γ) $E_0(\alpha) = 2E/\pi$, α = [0(0°.1)90°;6D].

Differencing shows that the table is free from gross error, and that Heuman's differences accurately correspond to his function values. Here also the rounding to 6D appears to be highly accurate. I have recomputed the values at 0(1°)90° to nine decimals, and find only one very slight rounding error; at 16°, *for final* 7, *read* 6.

(δ) $\Lambda_0(\alpha, \beta)$, α = 0(1°)90°, β = 0(1°)90°, etc., 6D.

I have not examined this important table, as it is of double entry and so falls outside my program. Heuman himself gives two corrections:

p.	α	β	*For*	*Read*
185	71°	15°	94145	84145
196	0°.3	87°	8523	8623

HIPPISLEY 1.

I have not examined the tables for errors. Seven corrections are given in *MTAC*, v. 1, p. 325, 1944.

INNES 2.

(α) $\log q$, θ = [0(1°)45°;10D].

I have derived 10D logarithms (uncertain by about a unit in the tenth place) of the values of q given in SPENCELEY 1. These never differ from the values of Innes by more than four units in the tenth place. The table is thus free from gross error. Innes did not expect his tenth decimal to be exact.

(β) $\log \log (1/q)$, θ = [0(1°)45°;10D].

The table is correctly abbreviated from VERHULST 1. The only error in Innes is therefore at 21°; in the ninth and tenth decimals, *for* 23, *read* 17.

(γ) $\log q$, θ = [0(1′)10′,44°50′(1′)45°;8D].

At 44°55′, *for* 380981, *read* 380891. Also on p. 496 are little tables of q and log log $(1/q)$ for the same arguments; these are free from gross error.

JAHNKE & EMDE 1.

I have read only the tables of $\log q$, K, E as functions of θ; and only in the latest edition, which seems the simplest way to contribute to the perfection of future editions. I add a few corrections pointed out by others in various editions. I am concerned here only with tables; on errors in formulae, see *MTAC*, v. 1, p. 108, 198, 391–399.

(α) $\log q$, θ = [0(5′)90°;4D].

The table is abbreviated from BERTRAND's 5D table with the same arguments. It thus descends ultimately from MEISSEL's 8D table.

BOWMAN, *MTAC*, v. 3, p. 41, 1948, has pointed out the following errors in the characteristics of the logarithms in all editions of Jahnke & Emde: at 2°55′, 3°55′, 4°55′, 5°55′, *for* $\bar{5}$, *read* $\bar{4}$; at 37°50′, *for* $\bar{5}$, *read* $\bar{2}$.

All other errors in 1_4 are errors of 1 unit in the fourth place, and were found by reading against MEISSEL.

There are first the following three errors:

	θ	*For*	*Read*
M	17°20′	$\bar{3}.7640$	$\bar{3}.7641$
	38 55	$\bar{2}.4960$	$\bar{2}.4959$
M	65 5	$\bar{1}.0270$	$\bar{1}.0271$

The first and last, marked M, are due ultimately to errors in MEISSEL. The second corresponds to no error in BERTRAND (if all copies agree with mine), but could have been caused by an error in LÉVY.

There are also 56 further errors, all of the same type. BERTRAND's last figure is 5, and Jahnke & Emde wrongly round upwards. In all 56 cases their value should be reduced by one unit of the fourth decimal. The arguments concerned are:

2° 5′	11°10′	17°25′	28°30′	45°10′	53°25′	71° 5′	83°50′
5 25	11 15	18 20	29 15	45 45	63 0	73 5	85 10
6 45	11 40	18 30	34 55	49 20	64 45	76 50	85 40
7 40	12 5	18 55	35 35	49 25	66 0	78 40	86 50
8 15	13 45	19 25	37 0	49 35	68 0	79 20	86 55
9 45	13 50	20 20	41 50	49 40	68 15	80 30	87 30
10 0	15 20	27 25	43 5	50 25	69 10	83 20	89 30

The above are all the errors in 1_4. COMRIE, in his list in *MTAC*, v. 1, p. 392, 1945, gives two errors in 1_1 only; at 64°5′, *for* 4087, *read* 0087, and at 69°55′, *for* 1195, *read* 1159. I observe that both are corrected in the 1923 reprint.

(β) K, E, θ = [0(1°)70°(30′)80°(12′)89°(6′)90°;4D].

The errors surviving in 1_4 are:

Function	θ	*For*	*Read*
K	35° 0′	1.7313	1.7312
	46 0	1.8692	1.8691
	84 12	3.6853	3.6852
	86 48	4.2746	4.2744
	87 24	4.4812	4.4811
	88 12	4.8479	4.8478
E	7 0	1.5650	1.5649
	22 0	1.5142	1.5141
	54 0	1.2682	1.2681
	62 0	1.1921	1.1920

Function	θ	*For*	*Read*
E	66 0	1.1546	1.1545
	67 0	1.1454	1.1453
	69 0	1.1273	1.1272
	81 24	1.0313	1.0314
	86 0	1.0087	1.0086
	88 12	1.0022	1.0021
	89 18	1.0005	1.0004

The following errors, in earlier editions only, are given in *MTAC*, v. 1, p. 198, 1944, from Glazenap 1:

Function	θ	*For*	*Read*	
K	86°48′	4.2744	4.2746	1_1, 1_2, 1_3
	87 36	4.5619	4.5609	1_1, 1_2, 1_3
	89 36	6.3504	6.3509	1_1, 1_2, 1_3
E	8 0	1.5630	1.5632	1_1, 1_2

(γ) $F(\theta, \phi)$, $E(\theta, \phi)$

See the note at the beginning of Part III. Also *MTAC*, v. 1, p. 198, 1944, and v. 1, p. 395, 1945.

(δ) $\wp(u)$, $\wp'(u)$, $\zeta(u)$, $\sigma(u)$, $r = 0(1)240$, case $\wp'^2 = 4\wp^3 - 1$.

See under Greenhill 2.

MTAC, v. 1, p. 392, 1945, gives, from Airey, a correction to ω_2 on p. 72 of $\mathbf{1}_1$ only (correct in the 1923 reprint); *for* 1.53995, *read* 1.52995.

Kaplan 1.

(α) K, E, $\log k'^2 = [\bar{1}(.005)\bar{2}(.01)\bar{6};10D]$.

Differencing shows the function values in these useful tables to be free from gross error. I have not examined Kaplan's differences.

Legendre (general remarks).

I have had the modern photographic reprints, Legendre 6 (Potin), 7 (Emde) and 8 (Pearson) continuously available. The original publications, Legendre 1–5, I have inspected only in libraries. Legendre 6 and 7 reproduce Legendre 3, while Legendre 8 reproduces Legendre 5. The lists of errors will relate primarily to the reprints on which I have worked. The contents of the reprints are not identical, and in fact natural values of K and E are reproduced only from 3 (in 7), and logarithmic values of K and E only from 5 (in 8). I state whether the errors listed for K and E also occur in 5, but have not otherwise examined K and E in 5; similarly for log K and log E and 3.

Legendre's great double-entry tables of $F(\theta, \phi)$ and $E(\theta, \phi)$ are reproduced in all three reprints, so that it is easy for a modern worker to compare Legendre 3 (through 6 or 7) with Legendre 5 (through 8). Double-entry tables have fallen outside my own program of differencing and recomputation, but in view of the importance of Legendre's double-entry tables, a rearrangement of an errata list due to E. L. Kaplan is included. Complete checking of the double-entry tables would be such a large task that, as far as I know, it has never been performed, but Kaplan's list contains many more items than any previous list of errors.

The original tables of Legendre are now so rare that I believe that the reprints alone have now much general working value. Moreover, photographic reproductions of this kind give a guarantee against variations from one copy to another, so that a complete list of errors in one copy would enable all other copies to be completely corrected with certainty. I trust that my investigations concerning the single-entry tables in the reprints will form a useful contribution. It will be evident that much work remains to be done even in this limited sphere, in connection with the tables of log K and log E.

LEGENDRE 3, 5.

Table VI (p. 323–332 of Legendre 3, p. 269–278 of Legendre 5), containing logarithms of the moduli in LANDEN's scale (see **Part** I, Section I), has not been reprinted. The following errors were pointed out by C. J. MERFIELD, *Astron. Jn.*, v. 30, p. 190, 1917, in relation to Legendre 5; I observe that the last two are absent in Legendre 3. I have corrected the last two arguments, which MERFIELD misprinted.

θ	Function	*For*	*Read*
5°.1	log c	$\underline{5}.94887$	$\underline{8}.94887$
9 .5	log b^0	$9.9999\underline{9}$ 96	$9.9999\underline{8}$ 96
34 .0	log c	$9.\underline{9}4756$	$9.\underline{7}4756$

LEGENDRE 7.

The tables were reproduced by EMDE from Legendre 3. I have not examined the unimportant first part (p. 338–341) of T.VIII.

(α) $K, E, \theta = [0(1°)90°;12D]$.

The complete integrals in the second part (p. 342–344) of T.VIII are highly important. Before receiving the SPENCELEY tables I had differenced Legendre's values of K and E, and also recomputed these functions for $\theta = 5°(5°)70°(1°)89°$, and for various other arguments where errors were suspected. The following lists should be complete; they are based on SPENCELEY, and I agree with the correction in every case which I have recalculated. Besides the erroneous and corrected digits in the eleventh and twelfth decimals, the corrections (SPENCELEY minus Legendre) are also shown; *r* indicates that the error is merely one of rounding, Legendre's value being within 1 final unit of the true (unrounded) value; +1 and −1 consequently indicate departures of between 1 and 1½ final units from the truth. It will be observed that the maximum correction is 18 units of the twelfth decimal in $K(85°)$, and that the corrections in the table of E are all small.

There are two gross errors in Legendre's differences:

$\Delta^2E(10°)$, *for* $\underline{2}5070$, *read* $\underline{1}5070$; $\Delta^2K(33°)$, *for* $820\underline{0}0$, *read* $820\underline{8}0$.

Otherwise his differences are consistent with his function values, except for errors in the last two figures of $\Delta^6E(36°)$, $\Delta^5E(37°)$, $\Delta^6E(37°)$, $\Delta K(62°)$, $\Delta K(63°)$, $\Delta K(65°)$.

	K				E		
θ	*For*	*Read*	corr.	θ	*For*	*Read*	corr.
11°	75	76	r	5°	77	78	r
12	27	28	r	23	15	17	+2
24	13	12	r	24	53	54	r
39	04	06	+2	27	43	44	+1
46	31	26	−5	28	26	27	+1
47	84	80	−4	39	85	86	r
48	65	64	−1	40	23	24	r
49	38	36	−2	46	31	30	−1
50	06	05	−1	47	18	17	r
53	35	34	r	48	82	81	r
54	01	05	+4	50	97	98	r
62	81	79	−2	54	64	65	+1
81	43	44	r	55	78	80	+2
83	73	72	−1	66	25	24	r
85	66	84	+18	69	57	58	r
89	25	26	r	73	36	38	+2
				74	74	73	−1
				79	45	44	r

All the above errors, except one in the differences, also appear in the second part (p. 288–290) of T.VIII in Legendre 5. The error in $\Delta^2K(33°)$ is present neither in Legendre 5 ***nor in 6*** *copies of Legendre 3 inspected,* although Legendre 7 (EMDE) reproduces another copy of Legendre 3! This illustrates the advisability of concentrating research on the reprints.

LEGENDRE 8.

The tables were reproduced by PEARSON from Legendre 5. The extremely important tables of log K and log E will be considered in six parts. From 0 to 15° and from 75° to 90° the function values have 14D, but the differences throughout relate to 12D values. In the 14D portions, I have aimed primarily at finding errors seriously affecting the first 12D, but in the case of log K I have gone further, and the corrections given will turn Legendre's 14D tables into reasonably good 13D tables.

In general, I have not read Legendre's differences systematically against my difference sheets, and the only statement I make concerning them will be found under (β). See also KAPLAN 2, and my comment in RMT **568**.

(α) log K, $\theta = [0(0°.1)15°;14D]$.

I have differenced the values to the full 14D. The largest error is undoubtedly one of 14 units of the last place at $\theta = 5°.0$. I have recalculated a few values at arguments suggested by the differencing. When the following corrections to the end figures are made, it is probable that no further correction will exceed some four units of the fourteenth decimal.

θ	*For*	*Read*
1°.1	35	30
5 .0	26	40
5 .3	06	11
5 .5	01	05

All the above errors also occur in Legendre 3.

(β) log K, $\theta = [15°(0°.1)75°;12D]$.

I have differenced the values. The only gross errors in the function values are: $\theta = 24°.9$, 7D, *for* 2, *read* 3; $\theta = 30°.9$, 0D, *for* 8, *read* 0.

These corrections have previously been pointed out by HEUMAN; the differences correspond to the corrected values, and, as has been pointed out by ARCHIBALD (*MTAC*, v. 2, p. 181, 1946), the errors are absent in Legendre 3. When the corrections are made, the table differences very well; the sixth difference never exceeds 17, so that even the twelfth decimal must be quite good.

I have examined the *first* differences between 15° and 75°, and found one gross error; at 74°.5, *for* 955, *read* 953. There are no other gross errors, but the last digit is one unit out of correspondence with Legendre's function values at 37°.0 and 75°.0. The first of these three errors is absent in Legendre 3.

(γ) log K, $\theta = [75°(0°.1)90°;14D]$.

As in (α), function values are to 14D, differences to 12D. Differencing to the full 14D proved the later decimals to be unreliable, although the first 9D are everywhere correct and only in one instance (at 89°.3) is the 10th affected. Attempts to detect errors affecting the twelfth decimal quickly showed that more than differencing would be required, and in order to make a satisfactory first contribution to the correction of the table I have lavished more time on it than on any other table investigated for this *Guide*; yet much more could still be done. I recalculated to 14D (with slight uncertainty in the last place) values of log K for 81 of the 150 arguments, including $\theta = 75°(0°.5)84°(0°.1)85°.5(0°.5)88°(0°.1)89°.9$, and found disagreements of more than three final units in the 27 cases listed below. These 27 cases I have recalculated to 15 or 16D, in order to try to give definitive corrections for them. When the corrections are made, differencing now suffices to show freedom from other than last-figure error up to at least 88°. The list is not very likely to be complete in respect of errors exceeding three final units, but from the difference tables I judge it to include all errors of ten or more final units, and it may even be found ultimately to include all errors of six or more final units. A noteworthy feature of the results is that the errors are largely in the twelfth decimal, the next two being frequently correct or nearly so. To exhibit this more clearly, the list gives, besides the erroneous and corrected values in the tenth to fourteenth places, the correction (true value minus Legendre). There is clearly an unexpected number of errors approximately multiples of 100; in particular, nine of ten successive values (between 84°.3 and 85°.2) are too small by about two units in the twelfth place. The attempt to correct the table has convinced me (as other computations convinced KARL PEARSON) of the vast labor which Legendre, without calculating machines, must have put into his tables.

θ	*For*	*Read*	corr.
78°.6	19291	19301	+10
79 .1	38162	38189	+27
79 .4	45626	45631	+5
79 .5	34061	34067	+6
80 .1	26294	26289	−5
80 .3	60617	60515	−102
83 .0	63180	63170	−10
83 .2	57234	57269	+35
83 .7	47733	47632	−101
84 .3	92765	92965	+200
84 .4	52700	52900	+200
84 .5	32799	33002	+203
84 .6	87556	87623	+67
84 .7	08925	09130	+205
84 .8	81351	81552	+201
84 .9	67275	67475	+200
85 .0	31833 / 31823	32033	+200 / +210
85 .1	08187	08385	+198
85 .2	09462	09661	+199
85 .3	93083	93116	+33
85 .7	48980	48989	+9
86 .8	54662	51623	−3039
86 .9	34046	34188	+142
87 .1	30193	30198	+5
88 .7	05699	06099	+400
88 .9	76200	76194	−6
89 .3	21710	31710	+10000

All the above errors also occur in Legendre 3.

(δ) $\log E$, θ = [0(0°.1)15°;14D].

I have differenced the 14D values. The maximum fifth difference is 162, so that, as in (α), no error will seriously affect the twelfth decimal.

(ϵ) $\log E$, θ = [15°(0°.1)75°;12D].

I have differenced the values. The only gross errors in the function values are:

θ	*D*	*For*	*Read*
22°.4	12	0	9
34 .4	9	8	0
36 .2	10	7	5
47 .7	11	1	0

The correction at 34°.4 has already been pointed out by HEUMAN; the others have recently been published in KAPLAN 2, and I found them independently. All four also occur in Legendre 3. In all four cases Legendre's differences agree with function values corrected as above, so that the mistakes are simply misprints. When the corrections have been made, the table differences extremely well; the fifth difference never exceeds 11, and the twelfth decimal must be quite good. In $\Delta \log E$ (37°.0), *for* 181, *read* 151 (MERFIELD, *loc. cit.*).

(ζ) $\log E$, θ = [75°(0°.1)90°;14D].

Legendre differences to only 12D. Differencing of the full 14D values shows that the table is not reliable to more than 12D. A gross error, pointed out by HEUMAN, is: θ = 86°.0, 9D, *for* 6, *read* 3. Legendre's differences again agree with the corrected value, and, as pointed out by ARCHIBALD (*MTAC*, v. 2, p. 181, 1946), the error is absent in Legendre 3. At θ = 84°.4, a decimal point should replace the letter x. Between 75° and 89°, the differencing shows that values rounded to 12D will be free from gross error, once that at 86°.0 has been corrected. The table deserves detailed investigation of the kind described in (γ) for the corresponding table of $\log K$.

LEGENDRE 6, 7, 8.

I have not examined Legendre's great double-entry tables for errors by differencing or recomputation, but for the convenience of the reader I had, before receiving KAPLAN 2, drawn up lists of the errors stated by BOHLIN, MERFIELD, GLAZENAP, SAMOĬLOVA-ÎAKHONTOVA, HEUMAN, WITT and ARCHIBALD; for references, see RMT 568 (in the present number). As I found KAPLAN 2 to contain all such known errors, and also many more found by KAPLAN, I have, by kind permission, incorporated all KAPLAN's material on p. 18–19 of his paper, as well as his information on p. 15 about $F(76°, \phi)$, into fresh lists, separating errors in $F(\theta, \phi)$ from errors in $E(\theta, \phi)$, and arranging each list in order of increasing θ. The lists below also contain three known errors in $E(\theta, \phi)$ of 1816, absent from PEARSON's reprint of the 1826 values, and hence outside the scope of KAPLAN's list.

There are two versions of the tables; Legendre 3 (1816) is reproduced in Legendre 6 and 7, while Legendre 5 (1826) is reproduced in Legendre 8. Legendre 6 and 7 no doubt reproduce different copies of Legendre 3, but these two reprints give identical values at all arguments mentioned below. An index, 1 or 2, attached to an erroneous value indicates that the error occurs only in the 1816 or 1826 version respectively. The values labeled "obscure" in KAPLAN 2 are so only in Legendre 8, not in the beautifully printed table in Legendre 5, hence I have called them "Obscure$^{2\ \mathrm{reprint}}$."

Owing to repetition of first and last columns and lines, tabular values are given twice if θ is one of 5°(5°)85°, twice if ϕ = 45°, and four times if both hold. Errors are included in the lists if they occur at all, even if the multiple presentation includes the correct value in another place.

The errors of one or two units given on p. 15 of KAPLAN 2 have not been incorporated; KAPLAN remarks that they are presumably only typical of considerable portions of the double-entry tables.

(α) $F(\theta, \phi)$, θ = 0(1°)90°, ϕ = [0(1°)90°;9–10D].

θ	ϕ	*For*	*Read*
2°	42°	0.73311 09968	0.73311 00968
3	32	0.55858 03004	0.55858 01004
	64	1.11750 64382^{2}	1.11750 63482
5	67	Obscure$^{2\ \mathrm{reprint}}$	1.17091 31870
	73	1.27598 58448^{2}	1.27598 56448
7	5	0.08729 81052	0.08726 81052
	54	Obscure$^{2\ \mathrm{reprint}}$	0.94421 96925
9	20	0.34293 53662	0.34923 53662
12	89	1.57035 40839	1.57035 40439
14	1	0.01743 33444^{2}	0.01745 33444
	6	0.11473 09355^{2}	0.10473 09355
20	32	0.56174 72078	0.56174 52078
21	88	1.58584 47725	1.58784 47725
23	90	1.63631 74093	1.63651 74093

θ	φ	*For*	*Read*
26	89	1.63627 86937	1.63627 85937
27	42	0.74754 01816	0.74574 01816
	59	1.06251 39233	1.06251 29233
	85	1.56840 67133	1.56480 67133
30	35	0.62203 01811[2]	0.62003 01811
31	12	0.20984 42225[2]	0.20984 42245
34	84	1.59518 05343	1.59518 05543
44	80	1.58906 49280	1.59806 49280
46	51	0.95158 7953	0.95157 7953
	90	1.86914 7545	1.86914 7546
57	90	2.08035 8167	2.08035 8067
59	65	1.34195 7808	1.34196 7808
60	60	1.21253 6614	1.21259 6614
64	79	1.84693 8718	1.84793 8718
66	89	2.30100 5177	2.30100 5157
71	61	1.29179 6493	1.29719 6493
74	85	2.39615 7101	2.39615 6101
75	37	0.69130 9761[2]	0.69190 9761
	84–88	Increase by 1×10^{-9}	
76	55	1.13561 6079	1.13561 6100
	56	1.16468 3555	1.16468 3376
	57	1.19438 4925	1.19438 4946
	58	1.22475 3902	1.22475 3923
	59	1.25582 5454	1.25582 5475
	60	1.28763 6943	1.28763 6964
	61	1.32022 8066	1.32022 8088
	62	1.35364 1002	1.35364 1024
	63	1.38792 0552	1.38792 0575
	64	1.42311 4279	1.42311 4302
	65	1.45927 2632	1.45927 2656
	66	1.49644 9058	1.49644 9083
	67	1.53470 0075	1.53470 0101
	68	1.57408 5305	1.57408 5331
	69	1.61466 7446	1.61466 7473
	70	1.65651 2149	1.65651 2176
	71	1.69968 7781	1.69968 7809
	72	1.74426 5022	1.74426 5051
	73	1.79031 6254	1.79031 6284
	74	1.83791 4673	1.83791 4703
	75	1.88713 3051	1.88713 3082
	76	1.93804 2064	1.93804 2095
	77	1.99070 8091	1.99070 8120
	78	2.04519 0371	2.04519 0398
	79	2.10153 7463	2.10153 7488
	80	2.15978 2927	2.15978 2950
	81	2.21994 0244	2.21994 0265
	82	2.28199 7092	2.28199 7111
	83	2.34590 9231	2.34590 9247
	84	2.41159 4470	2.41159 4485
	85	2.47892 7378	2.47892 7391
	86	2.54773 5614	2.54773 5625
	87	2.61779 8834	2.61779 8843
	88	2.68885 1060	2.68885 1066

θ	*φ*	*For*	*Read*
	89	2.76058 7039	2.76058 7042
78	90	2.97856 8952	2.97856 8951
79	50	1.00317 2141	1.00317 2041
83	16	0.28289 5780	0.28289 7580
	31	0.55907 5915[2]	0.56907 5915
	74	1.92525 6351	1.92515 6351
84	18	0.31936 7699	0.31939 7699
86	86	3.17204 1744	3.17030 9981
87	49	0.98238 1667	0.98328 1667
	87	3.45644 5172	3.45667 6092
88	4	0.06936 9880[2]	0.06986 9880
90	31	0.55956 2733[2]	0.56956 2733

(β) $E(\theta, \phi)$, $\theta = 0(1°)90°$, $\phi = [0(1°)90°;9\text{–}10D]$.

θ	*φ*	*For*	*Read*
4°	51°	0.88952 98250	0.88962 98250
6	44	0.76721 15513	0.76721 15313
8	90	1.56296 22295	1.56316 22295
10	5	0.00237 31277[2]	0.08726 31277
	41	0.71391 98746	0.71391 97846
	72	Obscure[2] reprint	1.24934 56503
11	7	0.12215 20142[2]	0.12216 20142
15	54	0.93450 23938[2]	0.93460 23938
16	45	0.77994 47234[2]	0.77994 47334
20	14	0.24406 47559	0.24406 47569
	50	Obscure[2] reprint	0.86142 06177
22	78	1.31072 85082	1.31972 85082
23	62	Obscure[2] reprint	1.05610 64197
24	20	0.34791 80029	0.34791 80019
	40	0.68953 08235	0.68953 08335
33	48	0.81181 56955	0.81181 56965
34	41	0.69797 33784	0.69797 33384
35	45	0.76188 30381	0.76128 30381
36	45	0.76003 72623	0.76003 72625
40	69	1.10811 86991	1.10811 86891
42	27	0.46376 10915	0.46366 10915
45	73	1.13785 83451	1.13785 43451
51	23	0.33502 9979[1]	0.39502 9979
64	90	1.17317 9382	1.17317 9383
75	45	0.71289 3043	0.71289 3045
79	32	0.53101 7384	0.53101 1384
82	14	0.24195 9269[1]	0.24196 9269
85	21	Obscure[2] reprint	0.35843 1155
86	2	0.03489 9351[1]	0.03489 9531
	6	0.10462 9396[2]	0.10452 9396

LÉVY 1

(α) K, $\theta = [0(1°)70°(30')80°(12')89°(6')90°;5D]$.

Comparison with an 8D table, partly deduced from LEGENDRE and partly computed independently, shows the following errors:

θ	*D*	*For*	*Read*
23° 0′	4	3	5
70 30	4–5	29	30
71 30	4–5	90	89

θ	*D*	*For*	*Read*
74 30	5	2	3
82 48	5	2	3
83 12	5	4	5
85 36	5	7	6
86 12	5	6	7
87 36	3–5	190	091
89 36	4–5	38	89

(β) $E, \theta = [0(1°)70°(30')80°(12')89°(6')90°;5D]$.

Comparison with an accurate table deduced from LEGENDRE shows the following errors:

θ	*D*	*For*	*Read*
8° 0'	3–4	29	31
81 24	4	2	3
89 18	4	4	3
89 54	4	1	0

(γ) $\log q, \theta = [0(5')90°;5D]$.

This table was copied with acknowledgment from BERTRAND, who abbreviated from MEISSEL without acknowledgment. In addition to the errors listed under BERTRAND (α), Lévy has seven more through faulty copying:

θ	*D*	*For*	*Read*
1°30'	2	8	3
2 55	0	$\overline{5}$	$\overline{4}$
33 45	1	4	3
38 55	5	5	3
59 35	1	7	9
64 5	1	4	0
69 55	3–4	95	59

MEISSEL 1.

(α) $\log q, \theta = [0(1')90°;8D]$.

This is the largest single-entry table relating to elliptic functions in existence. It is not so well known as many slighter works, and copies are somewhat rare. Lack of acknowledgment, especially in BERTRAND 1, has hindered recognition of the fact that it is the source of several better known abbreviations. Three errors obviously derived from Meissel show that the 5D table at interval 5' in BERTRAND 1 was extracted from Meissel 1. Bertrand said (p. xii) that his tables were due to F. THOMAN. Bertrand's book was published in 1870; if in the later part of the year (on which point I lack information), it is possible that the Franco-Prussian war made inopportune a candid acknowledgment of the German origin of the table of $\log q$. Subsequent copyists and abbreviators (LÉVY, POTIN, even JAHNKE & EMDE) have followed BERTRAND, not Meissel.

I have not differenced Meissel's table completely, but in order to check several other tables I have differenced (i) the values at 0(6')90°, (ii) the values at 0(5')90°. In this way one-third of the values (ten in every half-degree) have been tested for gross errors.

In addition, I have differenced the values at 0(1')1°10' and 89°(1')90°; moreover, I have checked the values at 0(1')10' and 89°50'(1')90°, the former by comparison with the little table on p. 496 of INNES 2, and the latter by simple calculation therefrom. This supplementary work revealed no gross error in the regions at the beginning and end where differencing at interval 6' or 5' fails as a check.

The values at 0(6')90° are entirely free from gross error. I have compared the values at 0(1°)90° with logarithms, to 10 or more D, of the values of q given in SPENCELEY 1. This revealed only two rounding errors, both very fine; at 33°, *for* 819, *read* 820 (rounding 819507), and at 42°, *for* 332, *read* 333 (rounding 332520).

Differencing at interval 5′ does reveal three gross errors and one rather large last-figure error:

θ	D	*For*	*Read*
4°50′	8	4	9
17 20	4	0	1
46 25	5	0	1
65 5	5	4	5

In confirming the corrections at 46°25′ and 65°5′ by differencing at interval 1′, it was found that in each case a neighboring value is affected by gross error: θ = 46°26′, 5D, *for* 6, *read* 7; θ = 65°4′, 5D, *for* 3, *read* 4.

On p. 14 the right-hand argument 11 is erroneously printed as 12.

In general the table is evidently accurate, but its 5400 values may perhaps contain 15 or 20 gross errors. Differencing of the whole table at interval 1′ is highly desirable.

MILNE-THOMSON 1.

(α) q, k^2 = [0(.01)1;10D].

Differencing suffices to show freedom from gross error up to at least $k^2 = .88$. For $k^2 = .88(.01).99$ I have deduced 10D natural values from the 10D table of log q in HAYASHI 1; in all cases I found agreement within one unit of the tenth decimal. Thus all the function values are free from gross error. Moreover, Milne-Thomson's differences accurately correspond to his function values.

MILNE-THOMSON 2.

(α) K, E, k^2 = [0(.01)1;9D].

"About 70 errors (in 200 values) of a unit each, one of two units, and none greater" (COMRIE, *Math. Rev.*, v. 7, p. 485). The error of two units is in K', at $m = k^2 = .01$, where *for final* 5, *read* 3. Comparison with the 10D values in HAYASHI 1, 3 confirms that in all other cases Milne-Thomson's values are within $1\frac{1}{2}$ units of the truth, the greatest remaining discrepancy between the two values being one of 15 units of the tenth decimal, in K' at $m = .37$, where Milne-Thomson ends in 81, HAYASHI ends in 795, and I find 79528. I have not checked the differences, nor the table of $1/\vartheta_3^2 = \pi/2K$.

MILNE-THOMSON 3.

(α) K, E, k^2 = [0(.01)1;7D].

Free from error. I have read against both HAYASHI 1, 3 and Milne-Thomson 2.

(β) q, k^2 = [0(.01)1;8D].

Free from error, assuming the table correct in a few cases in which the 10D table in Milne-Thomson 1 ends in 50.

MOORE 1.

(α) $2E$, k' = [0(.01)1;4D].

Since FMR *Index* was published, I have seen MOORE's table in the British Museum. It gives $2E$ (not $4E$, as erroneously stated on p. 317 of FMR *Index*). The last two decimals are often false, and the table is of no interest apart from its early date.

NAGAOKA & SAKURAI 1.

(α) K, E, k^2 = [0(.001)1;6D].

"In comparing the first 250 values of K [with the 7D values of SAMOĬLOVA-ÎAKHONTOVA] only slight last figure differences were found in about a score of cases" (ARCHIBALD, *Scripta Math.*, v. 3, 1935, p. 365).

PLANA 1.

(α) log $(1/q)$, θ = [0(0°.1)45°(1°)90°;10D].

The present writer was responsible for printing Plana 1848 in bold type on p. 321 of FMR *Index*. Further experience has unfortunately shown the implied recommendation to be quite unwarranted. Differencing immediately proved the table to be scandalously inaccurate. While it would not be difficult to recompute the whole 10D table (for instance, mainly by taking twelve-figure antilogarithms of VERHULST'S values, corrected where necessary), I have lacked time, and shall confine myself to stating the numerous corrections necessary to

θ	PLANA (8D)	MEISSEL	θ	PLANA (8D)	MEISSEL
1°.2	4.56207 605	4.56197 100	35°.4	1.59296 681	1.59296 451
4 .2	3.47347 021	3.47347 740	35 .6	1.58771 573	1.58771 568
5 .5	3.23897 197	3.23897 195	40 .1	1.47743 564	1.47563 204
9 .4	2.77213 727	2.77215 327	40 .9	1.45678 520	1.45678 185
10 .1	2.70946 393	2.70946 258	41 .1	1.45211 467	1.45211 450
13 .4	2.46216 418	2.46216 318	41 .6	1.44052 385	1.44052 284
13 .6	2.44917 384	2.44917 385	42 .0	1.43132 888	1.43132 667*
14 .1	2.41750 352	2.41750 252	42 .1	1.42900 828	1.42903 813
15 .2	2.35152 745	2.35152 845	42 .2	1.42675 335	1.42675 374
15 .7	2.32306 816	2.32306 793	42 .3	1.42417 316	1.42447 347
15 .8	2.31748 196	2.31748 197	42 .4	1.42219 736	1.42219 730
16 .0	2.30641 298	2.30641 257	42 .5	1.41992 512	1.41992 520
16 .1	2.30092 808	2.30092 827	42 .6	1.41765 690	1.41765 715
16 .3	2.29006 836	2.29005 836	42 .7	1.41539 320	1.41539 312
16 .9	2.25789 743	2.25820 939	42 .8	1.41313 315	1.41313 309
18 .0	2.20256 652	2.20256 662	42 .9	1.41087 671	1.41087 704
18 .3	2.18794 474	2.18796 174	43 .7	1.39296 873	1.39296 884
18 .4	2.18314 464	2.18314 474	44 .0	1.38631 594	1.38631 596
18 .5	2.17835 396	2.17835 290	44 .2	1.38195 353	1.38189 922
18 .8	2.16412 567	2.16412 568	44 .8	1.36873 522	1.36873 581
20 .0	2.10931 747	2.10931 745	45 .0	1.36438 892	1.36437 635
20 .3	2.09610 811	2.09610 832	46	1.34278 395	1.34278 396
20 .6	2.08308 433	2.08308 444	48	1.30055 663	1.30055 763
22 .0	2.02460 039	2.02460 036	49	1.27988 537	1.27988 450
22 .7	1.99667 024	1.99667 036	50	1.25949 994	1.25947 994
26 .2	1.86812 129	1.86812 124	53	1.19970 369	1.19970 347
26 .4	1.86126 771	1.86126 797	54	1.18020 131	1.18020 158
26 .5	1.85785 945	1.85785 926	55	1.16088 275	1.16088 475
26 .9	1.84434 183	1.84434 182	60	1.06653 416	1.06653 431
27 .0	1.84099 060	1.84099 133	63	1.01117 242	1.01115 242
27 .4	1.82769 382	1.82770 224	86	0.52945 956	0.52945 958
27 .5	1.82440 592	1.82440 774	88	0.45221 546	0.45202 155
27 .6	1.82112 718	1.82112 418			
27 .7	1.81784 242	1.81785 147			
28 .1	1.80486 789	1.80486 753			
28 .2	1.80091 548	1.80164 786			
28 .7	1.78520 349	1.78570 349			
29 .4	1.76379 582	1.76379 882			
29 .7	1.75455 482	1.75455 478			
30 .2	1.73933 275	1.73933 272			
30 .9	1.71839 436	1.71839 602			
34 .9	1.60617 525	1.60619 744			

Obvious errors in integral part only:

	For	*Read*
29°.9	7	1
64	1	0
65	1	0
66	1	0
67	1	0

For argument 86.6 read 36.6.

* At 42°.0, Meissel would wrongly give 668.

turn the table into a reasonably good 8D table. I have compared Plana's 10D values with MEISSEL's 8D values, or rather with their complements, since Plana tabulates log $(1/q)$ and MEISSEL log q. MEISSEL's values at arguments which are multiples of $6'$ are entirely free from gross error and the eighth decimal is generally very accurate. On p. 277 are listed the 79 cases (out of 495 values!) in which Plana's value differs by more than 100 units (in fact, 122 or more) of the tenth decimal from MEISSEL's 8D value. Plana's values are given rounded to 8D. These are all gross errors in Plana. There are also more than a score of cases, not listed, in which Plana's value differs from MEISSEL's by between 51 and 100 units of the tenth decimal. Some of these will be due to errors of rounding in MEISSEL, but many will be gross errors in Plana, and there will probably be other gross errors in Plana not affecting the eighth decimal. The list on p. 277 aims partly at salvaging something from the chaos of Plana's table, which is fairly accessible, and partly at making more widely available to scientists at least the gist of MEISSEL's rare work.

POTIN 1.

(α) $\log q,\ \theta = [0(5')90°;5D]$.

Contains all the errors given under BERTRAND 1 (α) and none of those given under LÉVY 1 (γ). I have not read the table against BERTRAND to see whether Potin has introduced any copying errors.

(β) $\log \vartheta_3$ [called $\log \theta_2(0)$], $\theta = [0(30')90°;9D]$.

The function tabulated is $\frac{1}{2}\log(2K/\pi)$. Except that the integral part at 67°30′ is corrected, the table is faithfully copied from BERTRAND, and has the same errors as those already given in BERTRAND 1 (β).

(γ) $F(\theta,\phi),\ E(\theta,\phi),\ \theta = 0(1°)90°,\ \phi = [0(1°)90°;9\text{–}10D]$.

See under LEGENDRE.

ROSENBACH, WHITMAN & MOSKOVITZ 1.

I have not examined the tables for errors. On two errors, see ARCHIBALD, *MTAC*, v. 2, p. 217, 1947.

RUNKLE 1.

Not too bad up to $\alpha = .5$, rather unreliable from .5 to .7, and so unreliable as to be almost useless between .7 and .75.

SAMOĬLOVA-ÎAKHONTOVA 1.

I have not checked the extensive double-entry tables.

(α) $K,\ E,\ k^2 = [0(.001)1;7D]$.

The values are based on HAYASHI 1, 3, which give at least three further decimals over the whole range. I have examined the tables only for those arguments at which HAYASHI has errors, and for two arguments at which Samoĭlova-Îakhontova herself points out errors in the errata slip. The first and fourth of the following errors are due to errors in the source; the second and third are from the errata slip.

p.	k^2	Function	D	*For*	*Read*
90	.201	E	4	5	6
92	.310	K	2	2	1
94	.439	E	1–7	387428	3807428
105	.999	K	7	5	6

ARCHIBALD (*Scripta Math.*, v. 3, 1935, p. 365) has checked that the values of K for $k^2 = 0(.01)1$ are in agreement with the 9D values of MILNE-THOMSON 2.

(β) $q,\ k^2 = [0(.001)1;8D]$.

This table falls for purposes of checking into two parts.

For $k^2 = 0(.001).5$, the function values were reprinted from HAYASHI 1, first differences being added. I have confirmed that the function values were correctly copied, the major errors at $k^2 = .395$ and $.482$ being corrected; at $k^2 = .279$, the last-figure correction (*for final* 7, *read* 8) also given by HAYASHI in 1932 has, however, not been made. See HAYASHI 1 (γ). The function values are thus free from gross error. At $k^2 = .403$, the last digit is almost illegible in my copy, but the differences show that 5 is intended.

For $k^2 = .501(.001)1$, the table is original. Differencing shows freedom from gross error up to at least $k^2 = .988$, and I have recomputed the values at $k^2 = .986(.001).999$ to 10D, finding only two rounding errors. Comparison of every tenth value with the ten-decimal table at interval .01 in MILNE-THOMSON 1 also revealed two rounding errors, as well as two cases (at .58 and .79) in which MILNE-THOMSON 1 ends in 50, and in both of which recomputation showed HAYASHI to be wrongly rounded. Recomputation of a small stretch with rough differences round $k^2 = .977$ discovered three more rounding errors. The various rounding errors referred to are:

k^2	.580	.670	.790	.975	.977	.979	.980	.992	.997
for	8	0	8	7	8	4	8	6	2
read	9	1	7	8	9	5	7	5	3

Further rounding errors doubtless exist, but in general the final digit is evidently accurate.

The differences are printed in small and occasionally defective type, a few digits being barely legible, but, as far as I can see, the differences in the whole table of q accurately correspond to the printed function values.

SCHLÖMILCH 1.

(α) E, $k' = [0(.01)1;5D]$.

Comparison with the accurate 11D table of SCHMIDT 1 shows that Schlömilch's values are all correct, if his last-digit underlinings are disregarded. The last digits should be underlined (to denote tabular values in excess of the true values) at .07 and .60.

SCHMIDT 1.

I have not examined the extensive double-entry tables.

(α) E, $k' = [0(.01)1;11D]$.

These are quadrants of ellipses with semi-axes 1 and k'. The table is by far the most important of its kind, and it is unfortunate that it has not been more widely known, as it is thoroughly reliable. Differencing, supplemented by recomputation at $k' = 0(.01).10$, shows that there are no gross errors. It should be noted that, in all cases where my information goes beyond the eleventh decimal, Schmidt's value has been obtained by rounding downwards, that is, he has not raised the digit in the eleventh place, even when the following digit exceeds 4. Presumably this holds throughout the table.

SPENCELEY 1.

I have not examined the double-entry tables. In the page headings are given three transcendental functions, namely K, E, q. The quantities $D(90)$ and $1/D(90)$ also given in the headings are not transcendental, being equal to $1/\sqrt{k'}$ and $\sqrt{k'}$ respectively, and thus merely algebraic functions of the modulus. I have examined the values of K, E, q as given (each twice, without any disagreements) on p. 2–181; the values are given another twice in the second half of the table, which I have not examined.

(α) K, $\theta = [0(1°)90°;15D]$.

I have differenced the values for $0(1°)81°$; at the end this requires twenty-second differences. The process proves the Spenceley values to be free from gross error for $\theta = 0(1°)69°$; beyond 69°, the table cannot be checked to the full 15 decimals by differencing. For $\theta = 70°(1°)89°$, 13D values of my own agree with the Spenceley values to that number of decimals, so that the whole table is certainly accurate enough for all practical purposes.

(β) E, $\theta = [0(1°)90°;15D]$.

I have differenced the values for $\theta = 0(1°)83°$, and so proved freedom from gross error up to at least 73°. For $70°(1°)89°$, my own calculations check the Spenceley values to at least 13D. Thus the whole table is again of great accuracy.

(γ) q, θ = [0(1°)90°;16S].

I first differenced the values for $\theta = 0(1°)71°$ to 16D, rounding to this number for $\theta = 0(1°)63°$, where further decimals are given. This proved freedom from gross error to 16D up to at least 63°. The process could be pushed a few degrees higher. I have, so far, checked the values above 63° only to the extent that I have derived 10D logarithms of all the Spenceley values of q, and these agree with 9D logarithms which I had earlier deduced from the K and K' of Legendre 7, using $\log q = - CK'/K$, where $C = \pi \log e$. This check will suffice for most purposes.

Later, out of curiosity, I differenced the Spenceley values of q for $\theta = 0(1°)22°$ to 18D, and to my surprise found incontestable evidence of errors in the eighteenth decimal. I have, so far, recomputed q to 20D for $\theta = 1°(1°)12°$. These values difference well, and give the following corrections, the first digit mentioned being in the sixteenth decimal in each case:

θ	*For*	*Read*	θ	*For*	*Read*
3°	4727	4708	9°	192	198
5	8574	8571	10	284	286
6	8341	8369	11	765	770

At the other six arguments I agree completely with the Spenceleys. The largest error so revealed is 6 units in the eighteenth decimal at 9°; the largest error expressed in units of the last place kept by the Spenceleys is 28 final units at 6°.

I have not yet differenced the values up to 63° to 17D. Anyone not content with 16D in this range should difference or recalculate. The errors which I have found are unlikely to affect the main Spenceley tables.

VERHULST 1.

Verhulst says (p. 252) that the table of $\log \log (1/q)$ was calculated for him by LOXHAY, a young Belgian mathematician, from LEGENDRE'S tables. The formula is

$$\log \log (1/q) = \log K' - \log K + 0.13493\ 41839\ 94670\ 6.$$

I shall deal with the table in two parts.

(α) $\log \log (1/q)$, θ = [0(0°.1)15°;14D].

After differencing, I first recalculated all values, correcting $\log K$ as in LEGENDRE 8 (α) and $\log K'$ as in LEGENDRE 8 (γ); in the latter case LEGENDRE'S table is, of course, now regarded as a table of $\log K'$ for $\theta = 0(0°.1)15°$ rather than as a table of $\log K$ for $\theta = 75°(0°.1)90°$. As $\log K$ and $\log K'$ so obtained are still liable in most cases to considerable last-figure errors, my results are in general very uncertain in the 14th decimal, but are sufficient to correct Verhulst into a table to 13 working decimals.

There are two gross errors due presumably to slips by LOXHAY: $\theta = 1°.2$, 5D, *for* 6, *read* 5; $\theta = 4°.2$, 6–7D, *for* 35, *read* 44. These are the only errors due to Loxhay.

θ	*For*	*Read*	θ	*For*	*Read*
0°.7	795573	805573	5°.4	86983	87050
1 .1	35632	35631	5 .5	12965	13165
1 .3	95056	95456	5 .6	83793	83994
2 .9	56794	56799	5 .7	30859	31059
3 .1	79255	79397	6 .3	81232	81131
3 .2	60913	57874	6 .8	71013	71048
4 .3	74463	74472	7 .0	91125	91115
4 .7	25150	25183	9 .7	27826	27723
4 .8	57083	57282	9 .9	92341	92336
4 .9	33199	33397	10 .5	99630	99632
5 .0	83774	83960	10 .6	51339	51341
5 .1	83060	83260	10 .9	13635	13660
5 .2	85112	85311	11 .4	33851	33861
5 .3	11686	11886			

The errors at the 27 arguments corresponding to those listed under Legendre 8 (α, γ) are given at the bottom of p. 280, the last digit being always in the fourteenth place. The list probably omits no error of more than ten final units in Verhulst. The corrections are slightly provisional, but are liable to errors of only a unit or so in the last place, as I later applied a few small corrections to log K not listed under LEGENDRE 8 (α), and kept extra decimals in the calculations when I knew them. At 1°.1 LEGENDRE'S errors in log K and log K' nearly cancel, so that Verhulst is nearly correct.

(β) log log $(1/q)$, θ = [15°(0°.1)45°;12D].

There are exactly four gross errors:

θ	D	*For*	*Read*
15°.5	11	4	3
21 .0	9–10	22	16
30 .9	8	6	5
37 .7	12	9	0

The correction to the last figure at 37°.7 is approximate, being that which brings Verhulst's value into agreement with LEGENDRE'S log K and log K', and also rectifies the differences. When the above corrections are made, the table differences very well, especially when it is remembered that it was formed by subtracting Legendre's log K from his log K' (and adding a constant, which, however, will not affect the differences); even if LEGENDRE had no rounding errors, Verhulst's values could frequently be wrong by one final unit. A similar remark applies to the 14D portion considered in (α) above. Verhulst's values are unaffected by the two gross errors (one obvious) at 24°.9 and 30°.9 in LEGENDRE 5, listed under LEGENDRE 8 (β), his error at 30°.9 being quite different; thus he probably used LEGENDRE 3.

WAYNE 1.

(α) E, k' = [0(.01)1;3D].

Comparison with SCHLÖMILCH 1 and SCHMIDT 1 shows two slight rounding errors; at k' = .20, *for* 1.050, *read* 1.051, and at k' = .81, *for* 1.426, *read* 1.425.

The Memory Tube and Its Application to Electronic Computation

In a paper presented on 10 June 1947 at the IRE Electron Tube Conference, the author described a novel electronic device which is called the Memory Tube.[1] The Memory Tube has the ability to record electrical signals in the form of a charge pattern deposited by an electron beam on a dielectric target, to store this information for any desired period of time, and to reproduce the stored information in the form of electrical signals as many times as desired, without erasing the recorded information. The recorded charge pattern or any part of it can be erased whenever desired. These characteristics of the Memory Tube make it useful for many applications, including electronic computation. The purpose of this paper is to review the basic principles of operation of the Memory Tube and to describe a special form suitable for use as a memory device in an electronic computer.

The Memory Tube resembles a conventional cathode-ray tube. At one end of an elongated envelope are placed one or more electron guns. At the other end is placed a dielectric target, which ordinarily consists of a glass plate coated with a suitable material, such as phosphor. In front of the target a fine-mesh metal screen is placed at a distance of a few thousandths of an inch from the target. The screen serves to control the electric field at the