| N | 5 | 6 | 8 | 20 *concl.* |
|---|---|---|---|---|
| Latent Root | 1.56705 069 | 1.61889 986 | 1.69593 900 | 0.17661 823 |
| | | | | 0.16582 577 |
| | | | | 0.15633 540 |
| | | | | 0.14791 772 |
| Latent Vector | 1 | 1 | 1 | 0.14039 536 |
| | 0.58056 692 | 0.58862 854 | 0.60050 425 | 0.13362 876 |
| | 0.41880 095 | 0.42832 928 | 0.44267 155 | 0.12750 652 |
| | 0.33006 105 | 0.33966 189 | 0.35437 045 | 0.12193 851 |
| | 0.27325 824 | 0.28252 359 | 0.29691 858 | 0.11685 095 |
| | | 0.24233 781 | 0.25618 093 | |
| | | | 0.22562 937 | |
| | | | 0.20179 019 | |

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# Piecewise Polynomial Approximation for Large-Scale Digital Calculators

**1. Introduction.** Most large-scale digital calculating machines are equipped to perform automatically the arithmetic operations of addition, subtraction, multiplication, division, and in some cases of extracting the square root. All arithmetic processes must be carried out by suitably combining these given operations. But many functions whose evaluation is frequently required, such as the elementary transcendental functions, for example, cannot be represented exactly by any combination of a finite number of the given operations. In order to evaluate such functions, it is necessary to resort to some sort of approximation.

A method frequently employed may be called "piecewise polynomial approximation." This method consists of dividing the interval upon which the required function is to be approximated into a number of sub-intervals upon each of which the function is represented by a polynomial. The coefficients of these polynomials are stored within the machine or external to it in a manner consistent with the machine's construction. When the value of the independent variable is given, the proper sub-range is selected by the machine itself. The operations of addition and multiplication applied to the value of the independent variable and to the stored coefficients are then sufficient to evaluate the appropriate polynomial and hence to obtain an approximation to the required function.

In practice, the range over which the approximation is to hold and the maximum allowable error are usually known in advance. We shall assume that the maximum allowable degree of the approximating polynomials is given. The problem of piecewise polynomial approximation reduces, then, to the determination of the sub-intervals and the coefficients of approximating polynomials so as to be consistent with these specified quantities. This may be stated more precisely as follows:

PROBLEM—Given the function $f(x)$ defined on the interval $[\alpha, \beta]$, a specified constant positive tolerance $T$, and a specified positive integer $N$. Required to divide $[\alpha, \beta]$ into sub-intervals $[c_{i-1}, c_i]$ where, with the number of sub-intervals, $r$, as yet unspecified, $i = 1,2,\cdots r$, and either $\alpha = c_0 < c_1 < \cdots < c_{r-1} < c_r = \beta$, or $\beta = c_0 > c_1 > \cdots > c_{r-1} > c_r = \alpha$, and to determine $n^{\text{th}}$ degree polynomials $P_n{}^i(x)$ with $n \leq N$, such that the upper bound of $|f(x) - P_n{}^i(x)| \leq T$ on $[c_{i-1}, c_i]$.

If the quantities $c_i$ and the polynomials $P_n{}^i(x)$ are determined in such a way that the number of sub-intervals, $r$, shall be a minimum, then they will be said to constitute the best solution of the problem. We suppose that $f(x)$ is a continuous function, possessing as many continuous derivatives as shall be required, and that all of these derivatives shall have a finite number of zeros.

**2. Determination of Sub-Intervals.** We shall restrict ourselves to approximation by $n^{\text{th}}$ degree polynomials agreeing with $f(x)$ at $n + 1$ points on $[c_{i-1}, c_i]$. With the $n + 1$ points of coincidence specified, say $x = x_k{}^i$ $(k = 0,1,\cdots,n)$, any such polynomial $P_n{}^i(x)$ may be expressed by the Lagrange Interpolation Formula,

$$(1) \qquad P_n{}^i(x) = \sum_{k=0}^{n} \frac{Q_{n+1}^i(x) f(x_k{}^i)}{(x - x_k{}^i) Q_{n+1}^{i(1)}(x_k{}^i)},$$

where, $Q_{n+1}^i(x) = (x - x_0{}^i)(x - x_1{}^i)\cdots(x - x_n{}^i)$, and $Q_{n+1}^{i(1)}(x)$ denotes the derivative of $Q_{n+1}^i(x)$. The remainder term, $f(x) - P_n{}^i(x)$, is then given by

$$R_{n+1}^i(x) = Q_{n+1}^i(x) f^{(n+1)}(\xi_i)/(n + 1)!$$

where $\xi_i$ lies on $[c_{i-1}, c_i]$.

Suppose that $f^{(n+1)}(x) \cdot f^{(n+2)}(x) \leq 0$ on $[\alpha, \beta]$. In this case, designate the end-points of the sub-intervals by $c_0, c_1, \cdots, c_r$ in order of increasing subscripts from left to right. The upper bound of $|f^{(n+1)}(x)|$ on $[c_{i-1}, c_i]$ occurs at $x = c_{i-1}$. Denote by $Q_{n+1}^{i\,\max}$ the upper bound of $|Q_{n+1}^i(x)|$ on $[c_{i-1}, c_i]$. Then

$$(2) \qquad |R_{n+1}^i(x)| \leq |Q_{n+1}^{i\,\max} f^{(n+1)}(c_{i-1})|/(n + 1)!.$$

Let us transform the independent variable in such a way that the interval $[c_{i-1}, c_i]$ becomes $[-1,1]$.

$$(3) \quad x = \tfrac{1}{2}(c_i - c_{i-1})u + \tfrac{1}{2}(c_i + c_{i-1}); \qquad u = (2x - c_i - c_{i-1})/(c_i - c_{i-1}).$$

Denote the transform of $Q_{n+1}^i(x)$ by $[\frac{1}{2}(c_i - c_{i-1})]^{n+1} L_{n+1}(u)$. Since $Q_{n+1}^i(x)$ is of leading coefficient unity, so is $L_{n+1}(u)$. In fact,

$$L_{n+1}(u) = (u - u_0)(u - u_1)\cdots(u - u_n),$$

where $u_0, u_1, \cdots, u_n$ are the points into which $x_0{}^i, x_1{}^i, \cdots, x_n{}^i$, respectively, are transformed, and $u$ ranges on the interval $[-1,1]$. Denote by $L_{n+1}^{\max}$ the upper bound of $|L_{n+1}(u)|$ on $[-1,1]$. Now

$$Q_{n+1}^{i\,\max} = [\tfrac{1}{2}(c_i - c_{i-1})]^{n+1} L_{n+1}^{\max},$$

and therefore from (2),

$$|R^i_{n+1}(x)| \leq |[\tfrac{1}{2}(c_i - c_{i-1})]^{n+1} L^{\max}_{n+1} f^{(n+1)}(c_{i-1})| / (n+1)!.$$

We wish to determine the division points $c_i (i = 0, 1, \cdots, r)$ in such a way that

$$|R^i_{n+1}(x)| \leq T \quad \text{on} \quad [c_{i-1}, c_i].$$

This condition will surely be satisfied if

$$(4) \qquad |[\tfrac{1}{2}(c_i - c_{i-1})]^{n+1} L^{\max}_{n+1} f^{(n+1)}(c_{i-1})| / (n+1)! \leq T \quad \text{on} \quad [c_{i-1}, c_i].$$

Solving (4) for $c_i$, we obtain

$$(5) \qquad c_i \leq c_{i-1} + 2|(n+1)!T/[L^{\max}_{n+1} f^{(n+1)}(c_{i-1})]|^{\frac{1}{n+1}},$$

a condition which may be used to generate successive end-points from left to right. If the equality in expression (5) holds, $[c_{i-1}, c_i]$ will be called a complete sub-interval. If the inequality holds, it will be called an incomplete sub-interval. Note that it is in general impossible to derive from (4) a condition for generating the end-points from right to left, since $c_{i-1}$ does not enter algebraically in this expression.

If $f^{(n+1)}(x) \cdot f^{(n+2)}(x) \geq 0$ on $[\alpha, \beta]$, we designate the end-points of the sub-intervals by $c_r, c_{r-1}, \cdots, c_0$ from left to right. An inequality analogous to (5) may be derived. In either case, the condition which successive end-points must satisfy suggests a procedure for the determination of the sub-intervals. This procedure may be stated as follows:

RULE: Generate the quantities $c_i$ by the recurrence formula

$$(6) \qquad c_i = c_{i-1} \pm K/|f^{(n+1)}(c_{i-1})|^{\frac{1}{n+1}}, \text{ where}$$

$$(7) \qquad K = 2\{[(n+1)!T]/L^{\max}_{n+1}\}^{\frac{1}{n+1}}.$$

If $f^{(n+1)}(x) \cdot f^{(n+2)}(x) \leq 0$ on $[\alpha, \beta]$, start with $c_0 = \alpha$, use the plus sign in (6), and continue the recurrence process until some quantity, say $c_s$, greater than or equal to $\beta$ is obtained. $c_{r-1}$ is then taken to be $c_{s-1}$ and $c_r$ taken to be $\beta$. If $f^{(n+1)}(x) \cdot f^{(n+2)}(x) \geq 0$ on $[\alpha, \beta]$, start with $c_0 = \beta$, use the minus sign in (6) and continue the recurrence process until some quantity, say $c_s$, less than or equal to $\alpha$ is obtained. $c_{r-1}$ is then taken to be $c_{s-1}$ and $c_r$ taken to be $\alpha$.

If neither $f^{(n+1)}(x) \cdot f^{(n+2)}(x) \leq 0$ nor $f^{(n+1)}(x) \cdot f^{(n+2)}(x) \geq 0$ over the entire range $[\alpha, \beta]$, we may divide $[\alpha, \beta]$ into sub-ranges upon which these conditions will hold alternately. This is always possible. We may then apply the foregoing rule to each sub-range separately, taking for $c_0$ one of the end-points of the sub-range in question. This procedure will result in several incomplete sub-intervals of the type $[c_{r-1}, c_r]$ being employed upon the range $[\alpha, \beta]$. All but one of the incomplete sub-intervals could be eliminated by choosing the $c_0$'s in a less naive manner, but the saving achieved seems hardly worth the additional complication.

**3. A First Order Approximation to the Number of Sub-Intervals Required.** Let $h_i = c_i - c_{i-1}$. We have from (6),

$$(8) \qquad |h_i| = K/|f^{(n+1)}(c_{i-1})|^{\frac{1}{n+1}}.$$

When $f^{(n+1)}(x) \cdot f^{(n+2)}(x) \leq 0$, $|f^{(n+1)}(x)|$ decreases with increasing $x$, and hence, since the sub-intervals are generated from left to right, $|f^{(n+1)}(c_{i-1})|$ decreases with increasing $i$. When $f^{(n+1)}(x) \cdot f^{(n+2)}(x) \geq 0$, $|f^{(n+1)}(x)|$ increases with increasing $x$. But since the sub-intervals are in this case generated from right to left, $|f^{(n+1)}(c_{i-1})|$ again decreases with increasing $i$. In either case, the following theorem follows directly from (8).

THEOREM I. Over an interval in which the sign of $f^{(n+1)}(x) \cdot f^{(n+2)}(x)$ does not change, the length of each complete sub-interval generated is greater than or equal to the length of the immediately preceding sub-interval.
Also from (8)

THEOREM II. When $f(x)$ is an $(n + 1)^{\text{th}}$ degree polynomial, all complete sub-intervals are of equal length.

Again, for the case in which $f^{(n+1)}(x) \cdot f^{(n+2)}(x)$ does not alternate in sign throughout the interval $[\alpha, \beta]$, let

$$h_{\min} = K/|f^{(n+1)}(c_0)|^{\frac{1}{n+1}}, \text{ and} \tag{9}$$

$$h_{\max} = K/|f^{(n+1)}(c_r - h_1)|^{\frac{1}{n+1}}. \tag{10}$$

From Theorem I, it follows that

$$h_{\min} \leq |h_i| \leq h_{\max}, \tag{11}$$

where $|h_i|$ is the length of any complete sub-interval. Now

$$|c_j - c_0| = \sum_{i=1}^{j} |h_i|,$$

and hence from (11)

$$jh_{\min} \leq |c_j - c_0| \leq jh_{\max}.$$

Replacing $j$ by $r$, the number of sub-intervals required to cover the entire range $[\alpha, \beta]$, and recalling that $|c_r - c_0| = \beta - \alpha$, we have

$$(\beta - \alpha)/h_{\max} \leq r \leq 1 + (\beta - \alpha)/h_{\min}, \tag{12}$$

where the quantity, 1, on the right-hand side of (12) enters by virtue of the fact that $r$ must be an integer. We formulate our results as follows:

THEOREM III. The number of sub-intervals, $r$, required to represent $f(x)$ on $[\alpha, \beta]$ is bounded by the quantities $(\beta - \alpha)/h_{\max}$ and $1 + (\beta - \alpha)/h_{\min}$ where $h_{\min}$ and $h_{\max}$ are given by (9) and (10), respectively.

**4. A Second Order Approximation to the Number of Sub-Intervals Required.** Expressions for determining the lengths of the sub-intervals in terms of $f^{(n+2)}(\xi)$ can also be derived, but due to the indefiniteness of the quantity $\xi$, they are not appreciably more accurate than those developed in the last section, and hence are of little practical value in estimating the number of sub-intervals required. They are, however, of some theoretical interest.

Solving (8) for $|f^{(n+1)}(c_{i-1})|$ and subtracting from the resulting expression a similar expression for $|f^{(n+1)}(c_{i-2})|$, we obtain

$$|f^{(n+1)}(c_{i-1})| - |f^{(n+1)}(c_{i-2})| = K^{n+1}[|h_i|^{-(n+1)} - |h_{i-1}|^{-(n+1)}]. \tag{13}$$

We may, by use of the law of the mean, write

$$(14) \qquad |f^{(n+1)}(c_{i-1})| - |f^{(n+1)}(c_{i-2})| = -|h_{i-1}f^{(n+2)}(\xi_{i-1})|,$$

where $\xi_{i-1}$ lies on $[c_{i-2}, c_{i-1}]$. Substituting for the left-hand side of (13) its value as given by (14), and solving for $h_i$, we obtain

$$(15) \qquad |h_i| = |h_{i-1}| \{1/[1 - K^{-(n+1)} |h_{i-1}^{n+2} f^{(n+2)}(\xi_{i-1})|]\}^{1/(n+1)}.$$

Theorem II is an immediate consequence of this expression. Equation (15) may be written in the form

$$(16) \qquad K^{-(n+1)} |h_{i-1}^{n+2} f^{(n+2)}(\xi_{i-1})| = 1 - |h_{i-1}/h_i|^{n+1}.$$

By Theorem I, the quantity $|h_{i-1}/h_i|^{n+1}$ is less than or equal to unity. Hence the quantity on the right-hand side of (16) is less than unity and greater than or equal to zero. If $|f^{(n+2)}(\xi_{i-1})|$ increases with increasing $i$, the quantity in braces in (15) increases with increasing $i$. We may therefore state

THEOREM IV. If $f^{(n+1)}(x) \cdot f^{(n+3)}(x) \le 0$, the ratio of the length of any complete sub-interval to the length of the previous one increases with each sub-interval generated.

The converse of this theorem is not, in general, true.

Let

$$f_{\min}^{(n+2)} = \begin{cases} |f^{(n+2)}(c_0)| & \text{when} \quad f^{(n+1)}(x) \cdot f^{(n+3)}(x) \leqslant 0 \\ |f^{(n+2)}(c_r - h_{\min})| & \text{when} \quad f^{(n+1)}(x) \cdot f^{(n+3)}(x) \geqslant 0 \end{cases}$$

and

$$(17) \qquad M_{\min} = (h_{\min})^{n+2} f_{\min}^{(n+2)} / K^{n+1}.$$

From (15), it follows that

$$|h_i| \ge |h_{i-1}| \{1/(1 - M_{\min})\}^{\frac{1}{n+1}}$$

and, by recurrence

$$|h_i| \ge |h_0| \{1/(1 - M_{\min})\}^{\frac{i-1}{n+1}} \ge h_{\min} + \frac{i-1}{n+1} M_{\min} h_{\min}.$$

Summing from $i = 1$ to $j$,

$$|c_j - c_0| \ge j h_{\min} + \tfrac{1}{2} j(j-1) M_{\min} h_{\min}/(n+1).$$

Replacing $j$ by $r$ and $|c_j - c_0|$ by $\beta - \alpha$, we have

THEOREM V. The number of sub-intervals, $r$, required to represent $f(x)$ on $[\alpha, \beta]$ must satisfy the inequality

$$(18) \qquad \beta - \alpha \ge r h_{\min} + \tfrac{1}{2} r(r-1) M_{\min} h_{\min}/(n+1),$$

where $M_{\min}$ is given by (17). Since $r$ enters quadratically in (18), an upper bound to the number of sub-intervals required can easily be determined. For $f(x)$ an $(n+1)^{\text{st}}$ degree polynomial, the second term on the right of (18) vanishes.

**5. Approximation by Particular Types of Polynomials.** If, in $\underset{n+1}{L}(u)$ we let $u_0 = u_1 = \cdots = u_n = 0$, we obtain

$$L_{n+1}(u) = u^{n+1}, \qquad \text{and} \qquad (19) \quad L_{n+1}^{\max} = 1.$$

In this case, the polynomials $P_n{}^i(x)$ given by (1) assume indeterminate forms. The indetermination may be resolved by rearranging terms, setting

$$x_0{}^i = (c_i + c_{i-1})/2,\ x_1{}^i = x_0{}^i + \epsilon,\ x_2{}^i = x_0{}^i + 2\epsilon, \text{ etc.},$$

and passing to the limit.[1] For a given $i$, $P_n{}^i(x)$ reduces then to the $n^{\text{th}}$ degree polynomial consisting of the first $n + 1$ terms of the Taylor's Series expansion about the point $(c_i + c_{i-1})/2$.

If we take $u_k = \cos\left[\frac{1}{2}(2k + 1)/(n + 1)\right]\pi$, $k = 0, 1, \cdots, n$, we obtain $L_{n+1}(u) = T_{n+1}(u)$, and $L_{n+1}^{\max} = 1/2^n$, where $T_{n+1}(u)$ is the Chebyshev Polynomial [2] of the first kind of order $n + 1$, defined by the formula

$$T_0(u) = 1; \quad T_n(u) = 2^{1-n}\cos(n \cdot \cos^{-1} u); \quad n = 1, 2, 3, \cdots.$$

Of all the $n^{\text{th}}$ degree polynomials of leading coefficient unity, $T_n(u)$ is known to be the one whose absolute value on the interval $[-1,1]$ has the smallest upper bound.[3] From this property, we may deduce

THEOREM VI. The best of all sets of sub-intervals generated by the fundamental rule is that set obtained by taking $L_{n+1}(u)$ to be the Chebyshev Polynomial of the first kind of order $n + 1$.

But if $f^{(n+1)}(\xi)$ is constant on $[\alpha, \beta]$, the best set of sub-intervals generated by the fundamental rule will be the best of all sets of sub-intervals generated in any manner whatsoever. We thus have

THEOREM VII. For $f(x)$ an $(n + 1)$st degree polynomial, the best solution to the problem of piecewise polynomial approximation is obtained by applying the fundamental rule, taking for $L_{n+1}(u)$ the Chebyshev Polynomial of the first kind of order $n + 1$.

**6. Numerical Example.** Consider the following numerical example:

EXAMPLE. Required to approximate the function $\sin x$ piecewise by cubic polynomials on the interval $[0, \frac{1}{2}\pi]$ in such a way that $\sin x$ is everywhere on the interval represented to an accuracy of $1 \times 10^{-6}$.

We have here $f(x) = \sin x$ (footnote 4), $[\alpha, \beta] = [0, \pi/2]$, $n = 3$, $T = 1 \times 10^{-6}$. Since $f^{(n+1)}(x) \cdot f^{(n+2)}(x) \geq 0$ on $[\alpha, \beta]$, the sub-intervals are to be generated from right to left starting with $c_0 = \frac{1}{2}\pi$.

For the Taylor's Series representation, we have from (7) $K = 2(4! \times 10^{-6})^{\frac{1}{4}} = 0.13998$, and from (6)

$$c_i = c_{i-1} - 0.13998\,(\sin c_{i-1})^{-\frac{1}{4}}. \tag{20}$$

The values of $c_i$ obtained by repeated use of (20) are listed in the second column of Table I. Eleven sub-intervals are required. This is consistent with the bounds given by Theorem III; namely, $r \leq 12.23$; $r \geq 6.85$.

Column 3 of Table I gives values of $c_i$ rounded to two decimals in such a way that $|h_i|$ is always on the small side. For the tabulation of the coefficients, it is convenient to refer each polynomial to the interval $[-1, 1]$. The approximating polynomials are then expressed explicitly as functions of $u$, where $u$ and $x$ are related by (3). The coefficients of these polynomials are given in the first part of Table II. Table III gives values of each approximating polynomial at the end-points of the sub-interval upon which it is to be used. The remainder should be greatest at these points. As was to be expected, the absolute value of the remainder is in all cases less than $1 \times 10^{-6}$.

For the Chebyshev approximation,

$$K = 2(2^3 \times 4! \times 10^{-6})^{\frac{1}{4}} = 0.23541, \quad \text{and} \quad c_i = c_{i-1} - 0.23541\,(\sin c_{i-1})^{-\frac{1}{4}}.$$

The unrounded values of $c_i$ are given in Column 4 and the rounded values in Column 5 of Table I. Seven sub-intervals are required. This again is in agreement with values predicted by Theorem III, $r \leq 7.68$; $r \geq 4.63$. Again the approximating polynomials are tabulated as functions of $u$. Their coefficients are given in the second part of Table II. Table IV gives the value of each approximating polynomial at the end-points and at the mid-point of the sub-interval upon which it is to be used. For the fourth sub-interval, values of $P_3{}^i(x)$ are also tabulated at the points $u = \cos \frac{1}{4}k\pi (k = 1, 2, 3)$ at which the remainder should be zero, and at the points $u = \cos \frac{1}{8}(2k + 1)\pi$, $k = 0, 1, 2, 3$, at which the absolute value of the remainder should be a maximum. As before, the remainders are all less in absolute value than the prescribed tolerance of $1 \times 10^{-6}$.

TABLE I—ENDPOINTS OF SUB-INTERVALS

| Sub-Interval | Taylor's Series | | Chebyshev Polynomials | |
|---|---|---|---|---|
| | Unrounded | Rounded | Unrounded | Rounded |
| $i$ | $c_i$ | $c_i$ | $c_i$ | $c_i$ |
| 0 | 1.5708 | 1.58 | 1.5708 | 1.58 |
| 1 | 1.4308 | 1.45 | 1.3354 | 1.35 |
| 2 | 1.2905 | 1.31 | 1.0983 | 1.12 |
| 3 | 1.1491 | 1.17 | 0.8559 | 0.88 |
| 4 | 1.0059 | 1.03 | 0.6034 | 0.63 |
| 5 | 0.8599 | 0.89 | 0.3321 | 0.36 |
| 6 | 0.7098 | 0.74 | 0.0206 | 0.05 |
| 7 | 0.5541 | 0.59 | 0.0000 | 0.00 |
| 8 | 0.3897 | 0.43 | | |
| 9 | 0.2114 | 0.26 | | |
| 10 | 0.0047 | 0.06 | | |
| 11 | 0.0000 | 0.00 | | |

TABLE II—COEFFICIENTS OF APPROXIMATING POLYNOMIALS

$$P_3{}^i(x) = a_0{}^i + a_1{}^i u + a_2{}^i u^2 + a_3{}^i u^3, \text{ where } u = (2x - c_i - c_{i-1})/(c_i - c_{i-1})$$

Approximation by Taylor's Series

| $i$ | $[c_{i-1}, c_i]$ | | $a_0{}^i$ | $a_1{}^i$ | $a_2{}^i$ | $a_3{}^i$ |
|---|---|---|---|---|---|---|
| 1 | 1.58 | 1.45 | 0.9984 4379 | −0.0036 2488 | −0.0021 0921 | 0.0000 0256 |
| 2 | 1.45 | 1.31 | 0.9818 5353 | −0.0132 7486 | −0.0024 0554 | 0.0000 1084 |
| 3 | 1.31 | 1.17 | 0.9457 8400 | −0.0227 3574 | −0.0023 1717 | 0.0000 1857 |
| 4 | 1.17 | 1.03 | 0.8912 0736 | −0.0317 5173 | −0.0021 8346 | 0.0000 2593 |
| 5 | 1.03 | 0.89 | 0.8191 9157 | −0.0401 4640 | −0.0020 0702 | 0.0000 3279 |
| 6 | 0.89 | 0.74 | 0.7277 2560 | −0.0514 4013 | −0.0020 4673 | 0.0000 4823 |
| 7 | 0.74 | 0.59 | 0.6170 5913 | −0.0590 1876 | −0.0017 3548 | 0.0000 5533 |
| 8 | 0.59 | 0.43 | 0.4881 7725 | −0.0698 1956 | −0.0015 6217 | 0.0000 7447 |
| 9 | 0.43 | 0.26 | 0.3381 9668 | −0.0799 9141 | −0.0012 2174 | 0.0000 9632 |
| 10 | 0.26 | 0.06 | 0.1593 1821 | −0.0987 2273 | −0.0007 9659 | 0.0001 6454 |
| 11 | 0.06 | 0.00 | 0.0299 9550 | −0.0299 8650 | −0.0000 1350 | 0.0000 0450 |

Approximation by Chebyshev's Polynomials

| $i$ | $[c_{i-1}, c_i]$ | | $a_0{}^i$ | $a_1{}^i$ | $a_2{}^i$ | $a_3{}^i$ |
|---|---|---|---|---|---|---|
| 1 | 1.58 | 1.35 | 0.9944 0787 | −0.0121 4388 | −0.0065 6828 | 0.0000 2674 |
| 2 | 1.35 | 1.12 | 0.9441 4734 | −0.0378 9492 | −0.0062 3630 | 0.0000 8348 |
| 3 | 1.12 | 0.88 | 0.8414 7008 | −0.0648 3626 | −0.0060 5133 | 0.0001 5551 |
| 4 | 0.88 | 0.63 | 0.6852 8780 | −0.0910 3392 | −0.0053 4686 | 0.0002 3688 |
| 5 | 0.63 | 0.36 | 0.4750 3083 | −0.1187 9574 | −0.0043 2215 | 0.0003 6050 |
| 6 | 0.36 | 0.05 | 0.2035 6655 | −0.1517 5436 | −0.0024 4046 | 0.0006 0692 |
| 7 | 0.05 | 0.00 | 0.0249 9740 | −0.0249 9221 | −0.0000 0781 | 0.0000 0262 |

TABLE III—COMPARISON OF TAYLOR'S SERIES APPROXIMATION WITH TRUE VALUE OF SIN $x$

| $x$ | $i$ | $u$ | $P_3^i(x)$ | true value of sin $x$ | $P_3^i(x) - \sin x$ |
|---|---|---|---|---|---|
| 1.58 | 1 | −1 | 0.9999 5690 | 0.9999 5765 | −0.0000 0075 |
| 1.45 | 1 | 1 | 0.9927 1226 | 0.9927 1299 | −0.0000 0073 |
| 1.45 | 2 | −1 | 0.9927 1201 | 0.9927 1299 | −0.0000 0098 |
| 1.31 | 2 | 1 | 0.9661 8397 | 0.9661 8495 | −0.0000 0098 |
| 1.31 | 3 | −1 | 0.9661 8400 | 0.9661 8495 | −0.0000 0095 |
| 1.17 | 3 | 1 | 0.9207 4966 | 0.9207 5060 | −0.0000 0094 |
| 1.17 | 4 | −1 | 0.9207 4970 | 0.9207 5060 | −0.0000 0090 |
| 1.03 | 4 | 1 | 0.8572 9810 | 0.8572 9899 | −0.0000 0089 |
| 1.03 | 5 | −1 | 0.8572 9816 | 0.8572 9899 | −0.0000 0083 |
| 0.89 | 5 | 1 | 0.7770 7094 | 0.7770 7175 | −0.0000 0081 |
| 0.89 | 6 | −1 | 0.7770 7077 | 0.7770 7175 | −0.0000 0098 |
| 0.74 | 6 | 1 | 0.6742 8697 | 0.6742 8791 | −0.0000 0094 |
| 0.74 | 7 | −1 | 0.6742 8708 | 0.6742 8791 | −0.0000 0083 |
| 0.59 | 7 | 1 | 0.5563 6022 | 0.5563 6102 | −0.0000 0080 |
| 0.59 | 8 | −1 | 0.5563 6017 | 0.5563 6102 | −0.0000 0085 |
| 0.43 | 8 | 1 | 0.4168 6999 | 0.4168 7080 | −0.0000 0081 |
| 0.43 | 9 | −1 | 0.4168 7003 | 0.4168 7080 | −0.0000 0077 |
| 0.26 | 9 | 1 | 0.2570 7985 | 0.2570 8055 | −0.0000 0070 |
| 0.26 | 10 | −1 | 0.2570 7981 | 0.2570 8055 | −0.0000 0074 |
| 0.06 | 10 | 1 | 0.0599 6345 | 0.0599 6401 | −0.0000 0056 |
| 0.06 | 11 | −1 | 0.0599 6400 | 0.0599 6401 | −0.0000 0001 |
| 0.00 | 11 | 1 | 0.0000 0000 | 0.0000 0000 | 0.0000 0000 |

TABLE IV—COMPARISON OF APPROXIMATION BY CHEBYSHEV'S POLYNOMIALS WITH TRUE VALUE OF SIN $x$

| $x$ | $i$ | $u$ | $P_3^i(x)$ | true value of sin $x$ | $P_3^i(x) - \sin x$ |
|---|---|---|---|---|---|
| 1.58 | 1 | −1.0 | 0.9999 5673 | 0.9999 5767 | −0.0000 0094 |
| 1.465 | 1 | 0.0 | 0.9944 0787 | 0.9944 0879 | −0.0000 0092 |
| 1.35 | 1 | 1.0 | 0.9757 2245 | 0.9757 2336 | −0.0000 0091 |
| 1.35 | 2 | −1.0 | 0.9757 2248 | 0.9757 2336 | −0.0000 0088 |
| 1.235 | 2 | 0.0 | 0.9441 4734 | 0.9441 4820 | −0.0000 0086 |
| 1.12 | 2 | 1.0 | 0.9000 9960 | 0.9001 0044 | −0.0000 0084 |
| 1.12 | 3 | −1.0 | 0.9000 9950 | 0.9001 0044 | −0.0000 0094 |
| 1.000 | 3 | 0.0 | 0.8414 7008 | 0.8414 7098 | −0.0000 0090 |
| 0.88 | 3 | 1.0 | 0.7707 3800 | 0.7707 3888 | −0.0000 0088 |
| 0.88 | 4 | −1.0 | 0.7707 3798 | 0.7707 3888 | −0.0000 0090 |
| 0.8704 8494 | 4 | −0.9238 7953 | 0.7646 4155 | 0.7646 4155 | 0.0000 0000 |
| 0.8433 8835 | 4 | −0.7071 0678 | 0.7469 0132 | 0.7469 0044 | 0.0000 0088 |
| 0.8028 3543 | 4 | −0.3826 8343 | 0.7193 2866 | 0.7193 2867 | −0.0000 0001 |
| 0.755 | 4 | 0.0 | 0.6852 8780 | 0.6852 8867 | −0.0000 0087 |
| 0.7071 6457 | 4 | 0.3826 8343 | 0.6496 8088 | 0.6496 8087 | 0.0000 0001 |
| 0.6666 1165 | 4 | 0.7071 0678 | 0.6183 2742 | 0.6183 2656 | 0.0000 0086 |
| 0.6395 1506 | 4 | 0.9238 7953 | 0.5968 0639 | 0.5968 0640 | −0.0000 0001 |
| 0.63 | 4 | 1.0 | 0.5891 4390 | 0.5891 4476 | −0.0000 0086 |
| 0.63 | 5 | −1.0 | 0.5891 4392 | 0.5891 4476 | −0.0000 0084 |
| 0.495 | 5 | 0.0 | 0.4750 3083 | 0.4750 3165 | −0.0000 0082 |
| 0.36 | 5 | 1.0 | 0.3522 7344 | 0.3522 7423 | −0.0000 0079 |
| 0.36 | 6 | −1.0 | 0.3522 7353 | 0.3522 7423 | −0.0000 0070 |
| 0.205 | 6 | 0.0 | 0.2035 6655 | 0.2035 6716 | −0.0000 0061 |
| 0.05 | 6 | 1.0 | 0.0499 7865 | 0.0499 7917 | −0.0000 0052 |
| 0.05 | 7 | −1.0 | 0.0499 7918 | 0.0499 7917 | 0.0000 0001 |
| 0.025 | 7 | 0.0 | 0.0249 9740 | 0.0249 9740 | 0.0000 0000 |
| 0.00 | 7 | 1.0 | 0.0000 0000 | 0.0000 0000 | 0.0000 0000 |

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[1] J. F. STEFFENSEN, *Interpolation*, Baltimore, 1927, p. 22.

[2] For a working list of coefficients and formulae relating to the Chebyshev Polynomials, see C. W. JONES, J. C. P. MILLER, J. F. C. CONN, R. C. PANKHURST, "Tables of Chebyshev polynomials," R. Soc. Edinb., *Proc.*, v. 62A, 1946, p. 187–203. See *MTAC*, v. 2, p. 262.

[3] For proof see STEFAN KACMARZ and HUGO STEINHAUS, *Theorie der Orthogonalreihen*, Warsaw, 1935, p. 111–112.

[4] True values of sin $x$ were obtained from NBSCL, *Tables of Circular and Hyperbolic Sines and Cosines*, New York, 1940.