80[F].—A. GLODEN, *Nouvelle extension des solutions de la congruence* $x^4 + 1 \equiv 0$ *(mod p) pour p entre* $6 \cdot 10^5$ *et* 10^7. Mss. in possession of the author, 11 rue Jean Jaurès, Luxembourg, and in the Library of Brown University.

This manuscript table gives solutions x of the congruence mentioned in the title for approximately 1300 primes p beyond the limit 600000 set by previous tables [1] and under ten millions. With a few exceptions, only one solution x is given for each p (instead of the usual pair) and in each $x < 40000$. The table is a byproduct of the results of factoring numbers of the form $x^4 + 1$.

D. H. L.

[1] For references to previous tables of this sort see *MTAC* v. 1, p. 6, v. 2, p. 71, 210, v. 3, p. 96.

81[F].—A. GLODEN, *Table de factorisations des nombres* $N^8 + 1$ *pour* $N \leq 400$. Mss. in possession of the author, 11 rue Jean Jaurès, Luxembourg, and in the Library of Brown University.

The table of CUNNINGHAM [1] giving the factors of $N^8 + 1$ for $N \leq 200$ is extended in this manuscript not only by doubling the upper limit of N but also by raising limit 100000 of the smallest prime factor omitted to 600000. Of the 400 entries of the table 140 are complete factorizations, 213 are composite but incompletely factored, while only 47 are of entirely unknown character, beyond the fact that their factors lie above 600000. The smallest number of this latter kind is $\frac{1}{2}(43^8 + 1) = 5844100138801$. The author hopes to raise his 600000 to 800000 by extending his already extensive tables of solutions of the quartic congruence[2] $x^4 + 1 \equiv 0 \pmod{p}$.

A comparison of this table with that of CUNNINGHAM reveals the following errata in the latter.

p. 140	$y = 86$	insert the small factor 61057
	$y = 148$	delete the factor 97
p. 141	$y = 125$	for semicolon read full stop.

D. H. L.

[1] A. J. C. CUNNINGHAM, *Binomial Factorisations*, v. 1, London 1923, p. 140–141.
[2] See *MTAC*, v. 2, p. 71–72, 210–211, 252, 300.

AUTOMATIC COMPUTING MACHINERY

Edited by the Staff of the Machine Development Laboratory of the National Bureau of Standards. Correspondence regarding the Section should be directed to Dr. E. W. CANNON, 418 South Building, National Bureau of Standards, Washington 25, D. C.

TECHNICAL DEVELOPMENTS

Our contribution under this heading, appearing earlier in this issue, is "The solution of simultaneous linear equations with the aid of the 602 calculating punch," by FRANK M. VERZUH.

DISCUSSIONS

A New General Method for Finding Roots of Polynomial Equations

The problem of finding all of the roots of polynomial equations of fairly high degree arises so frequently that a routine for accomplishing this automatically on a high-speed digital computer would be of considerable practical value. However, for every one of the standard methods for finding the roots of a polynomial equation, there are some exceptional cases in which the particular method applied fails to work.

Horner's method and Newton's method require a good initial approximation, to prevent the resulting sequence of approximations from diverging (as in the example $f(x) = x^3 - 5x$, where $x_0 = 1$) or involving division by zero. The method of false position will fail completely for complex roots and for real roots of even multiplicity. Both Lagrange's method and the method of Sturm functions fail for complex roots. The Graeffe method will furnish the absolute value of each root and the total number of roots having each absolute value, but it will not give the roots themselves if there are several complex roots of equal modulus. Since in the case of real polynomials the complex roots always occur in pairs of equal modulus, further refinements of the method are necessary.[1] The refined methods either fail or become extremely complicated in the case of several pairs of roots of equal modulus. In using the Graeffe method efficiently, it is also necessary to exercise considerable judgment to know which one of the possible modifications to use or to know when to stop the squaring process. Bernoulli's method does not converge if there is more than one root of largest absolute value. A modification of Bernoulli's method [2] will give the absolute value if there are n roots having this absolute value, but a different method is required for each different n. Also, considerable judgment must be exercised to know which method to use or when to change methods.

In the case of a mathematician working with a desk computer, exceptional cases will not cause much trouble, since they occur rarely, and, by the use of intelligent judgment, they can be detected and an appropriate change of method made. But it would be an imposing task (even without considering the limitations necessitated by the proposed memory sizes of all the digital machines now under construction) to set up routines for detecting and handling every one of the possible combinations of exceptional cases that can arise. It would be of advantage to have a single method which will always give all of the roots, regardless of their positions or multiplicities.

The method outlined below is a first attempt at such a universal method. It is admittedly not a speedy or efficient one, since, in fact, it would require a prohibitive amount of time to carry out the method except on high-speed digital calculating machinery, and even on these machines the process would be time consuming. But it has the advantage that regardless of the nature of the original polynomial, this method will always converge to a root, to as much accuracy as the machine uses. Furthermore, the convergence always takes place within a fixed number of steps (independent of the degree of the polynomial equation).

After finding the "first" root, the equation can be reduced in degree by removal of this root and the other roots found by repeating the same routine. Thus, the entire set of roots and their multiplicities can be found by a machine without the necessity of any human intervention during the problem.

The proposed method is as follows: Given any point p_n which is the nth approximation to a root, the following operations produce the $(n + 1)^{\text{th}}$ approximation. Expand the polynomial around the point p_n by synthetic division. Then, by performing the Graeffe process a fixed number of times, find (to some known degree of accuracy in terms of relative error) R, the absolute value of the root of the transformed equation which is smallest in modulus. Since R is a function of p_n, it will be denoted by $R(p_n)$, and geometrically it is the distance from the point p_n to the nearest root of the original polynomial.

The number actually obtained by applying the Graeffe root-squaring process a fixed number of times will be denoted by $R^*(p_n)$, where $(1 + d_n)R^*(p_n) = R(p_n)$, and by using enough significant figures in the synthetic division and the Graeffe process it is possible to insure that the error term d_n satisfies $|d_n| << 1$.

Since there will be at least one root of the polynomial equation near the circumference C_n of the circle of radius $R^*(p_n)$ about the center p_n, if a suitable set, S_n, of points (such as the vertices of a regular inscribed heptagon or octagon [3]) on C_n are chosen, S_n will contain at least one point s such that $R^*(s) \leq \frac{1}{2}R^*(p_n)$. Choose any one, s (e.g., the first one which is tried) of these points, and call it p_{n+1}.

Starting with the initial approximation $p_0 = 0$ and finding each successive p_n by iterative applications of the above procedure, one can easily verify by induction from $R^*(p_{n+1}) \leq \frac{1}{2}R^*(p_n)$ that $R^*(p_n) \leq 2^{-n}R^*(p_0)$.

Then, if p denotes a root of the original equation which is nearest to p_n, it follows that $|p| \geq R(p_0) = (1 + d_0)R^*(p_0)$.

The relative error of p_n, considered as an approximation to p, is

$$\frac{|p_n - p|}{|p|} = \frac{R(p_n)}{|p|} \leq \frac{(1 + d_n)R^*(p_n)}{(1 + d_0)R^*(p_0)} \leq \frac{(1 + d_n)2^{-n}R^*(p_0)}{(1 + d_0)R^*(p_0)} = \frac{1 + d_n}{1 + d_0}2^{-n} \approx 2^{-n}.$$

It is clear that on a binary computer the nth approximation p_n is good to about n significant figures, if the computer carries several extra significant figures in the calculation to assure that $|d_n|$ is always sufficiently small.

This method is universal in that it does not depend on the location or multiplicities of the roots or on the degree of the original equation.

Since the Graeffe process easily gives the number of roots having the absolute value R, the multiplicity of the root to which p_n converges would be apparent without further calculation.

Complex numbers can arise in the course of the calculation, even when the original coefficients are real numbers. For this reason, and also because very large powers of 2 or 10 can arise as exponents, the use of corresponding complex, floating-point addition and multiplication computer subroutines will be necessary. Such subroutines make it possible for the original equation to have complex coefficients covering a very wide range of magnitudes without involving extra programing.

The time required to find one root by this method (as calculated from the number of multiplications and additions) will be proportional to the square of the degree of the original polynomial. It follows that the time required to find all roots will be proportional to the cube of the degree of the polynomial.

Several modifications of this method could be made which would speed up convergence, but probably not sufficiently to make the method actually feasible for extensive machine use. The constant $\frac{1}{2}$, the number of points in S_n, and the degree of accuracy to which R should be calculated, could be adjusted to minimize the time. Also, it would save time to test for convergence during the computation rather than to iterate a fixed number of times.

The author wishes to express his appreciation to Dr. L. B. Tuckerman, Jr., Dr. E. W. Cannon, & Mr. George Gourrich, all of the NBS, for their suggestions and assistance in developing this method.

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[1] E. Bodewig, "On Graeffe's method for solving algebraic equations," *Quart. Appl. Math.*, v. 4, 1946, p. 177–190.

[2] Bernard Dimsdale, "On Bernoulli's method for solving algebraic equations," *Quart. Appl. Math.*, v. 6, 1948, p. 77–81.

[3] The choice of an octagon will allow enough overlap to permit this method to converge even if $|d_n|$ is fairly large.

Bibliography Z-VIII

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2. Andrew D. Booth & Kathleen H. V. Britten, *General Considerations in the Design of an All-Purpose Electronic Digital Computer*. Second edition, Aug. 1947, 24 leaves, mimeographed. British Rubber Producers' Research Association, Welwyn Garden City. 21.6 × 27.9 cm. Not available for distribution. Copies of nos. 1 and 2 may be consulted in the Library of The Institute for Advanced Study.
3. Arnold A. Cohen & William R. Keye, "Selective alteration of digital data in a magnetic drum computer memory," transcript of a paper presented at the Institute of Radio Engineers, National Convention, May 1,

1948; available from Engineering Research Associates, Inc., St. Paul, Minn., 10 p., illustr. 27.5 × 21.4 cm.

4. J. M. COOMBS, "Storage of numbers on magnetic tape," National Electronics Conference, *Proc.*, v. 3, 1947, p. 201–209. 15.3 × 22.7 cm.

ABSTRACT: This paper describes a memory system for storing digital information on magnetic tapes. The tapes are bonded to the surface of an aluminum drum. Associated with each tape are three heads for reading, writing, and erasing magnetized spots on the tapes. This equipment allows numbers to be stored indefinitely, to be inspected as often as required, and to be removed when no longer needed. The system will store 200,000 magnetized spots on a drum 34 inches in diameter and 10 inches wide.

5. ANDREW HAMILTON, "Brains that click," *Popular Mechanics Mag.*, Mar. 1949, p. 162–167, 256, 258, illustrs. 22.5 × 24.1 cm.

6. G. G. HOBERG and J. N. ULMAN, JR., *Glossary of Computer Terms*, Special Devices Center, Office of Naval Research, Port Washington, Long Island, N. Y., May 1948, 12 p. 21.6 × 27.9 cm.

From the Foreword: The reporter of a new art is always confronted with the problem of describing previously unknown concepts and devices for which there are no words in our general vocabulary. Project Whirlwind reports inevitably contain a considerable number of specialized terms used in new senses. This Glossary has been prepared for distribution to the recipients of these reports in the hope that it will clarify any terms whose meanings might be strange or in doubt.

7. C. H. PAGE, "Digital computer switching circuits," *Electronics*, v. 21, 1948, p. 110–118. 21.6 × 27.9 cm.

Basic operational requirements of digital computers and fundamentals of the means for obtaining them are set forth. For the most part familiar switching circuits can be used, but they must meet the special requirements of positive action described in this paper.

8. G. W. PATTERSON, R. L. SNYDER, L. P. TABOR, & IRVEN TRAVIS, *The EDVAC, a Preliminary Report on Logic and Design*, Moore School of Electrical Engineering, University of Pennsylvania, Feb. 16, 1948, iv, 100 leaves; 3 folding plates in pocket. 21.6 × 27.4 cm.

Two reports on the EDVAC [Electronic Discrete Variable Computer] dated September 30, 1945, and June 30, 1946 [see *MTAC*, v. 3, p. 379–380], discussed at length the relative merits of various projected machines, circuits, and computer techniques but did not describe in detail any one machine plan. Since then the plan of the EDVAC has been almost entirely frozen, and the construction is far advanced. This report is mostly devoted to a careful description of exactly how orders and numbers are represented in the EDVAC, how it treats them, and what basic operations it can perform. Therefore, this report is required reading for anyone who expects to prepare problems for the EDVAC. Considerations dictating particular choices in logical design are seldom weighed.

After a three-page historical introduction, the general organization of the EDVAC is explained in fifteen pages supplemented by a system block diagram. The reader need not be familiar with the EDVAC or with prior reports on it in order to understand the report under review. Seven pages deal cursorily with design problems, the real purpose being to give the unfamiliar reader a picture of the EDVAC's physical structure.

Mathematicians are familiar with the important logical distinction between a set containing only one element and the element itself. Digital computers abound in just such distinct objects which stand in "one-to-one" correspondence and are confused in common discourse. Consider for example: (1) a sequence of 44 characters (pulses or spaces) which the machine interprets as an order, (2) the 10-character segment of the order called the second address, (3) the number into which this address itself is interpreted, (4) the position in the memory denoted by this address, (5) the sequence of 44 characters stored in this memory position and called a word, and (6) the number into which this word is interpreted for arithmetic operations. This report introduces an elaborate symbolic functional notation to make such distinctions compactly and unambiguously. Letter symbols are chosen with mnemonic values so that the reviewer was able to follow the descriptions quite easily; however, he has been intimately engaged in planning the NBS Interim Computer which is very similar to the EDVAC in logical design. Readers less familiar with this design complained that the use made of the notation does not justify the effort of learning it. Perhaps a discussion less "scrupulously accurate" and containing more "clumsy locutions" would be easier for beginners; however, the reviewer approves the introduction of the symbolism, because he feels that a compact precise language for exact communication between specialists is needed. Furthermore he hopes that ultimately both the invention and checking of logical designs of computing machines will be facilitated by a suitably formalized logical calculus. The report should, however, have included a list of terminology and symbols for ready location of forgotten definitions.

A large drawing shows the control panel of the EDVAC. Its use is described in six pages. A bare introduction to the reading of logical block diagrams at the level where tubes, gates, flipflops, and delay lines are shown as blocks is given in seven pages which trace through just that part of the computer which performs addition and subtraction without checking. The final pages of the report treat briefly the diagnosis of errors and give design suggestions for future machines of the EDVAC type.

This report was written primarily for persons who must prepare problems for the EDVAC. It gives the data which are indispensable to them, namely, what the EDVAC will do in response to any possible coded order. The engineering discussions are calculated to supply background and to satisfy the curiosity of these people whose training is more in mathematics than in engineering. It would be a grave error to employ these oversimplified arguments as the bases of important engineering decisions.

9. JOHN PFEIFFER, "The machine that plays gin rummy," *Science Illustrated*, v. 4, Mar. 1949, p. 46–48, 84, 86–88, illustr. 27.9 × 20.3 cm. See also "Machine plays chess," *Science News Letter*, v. 54, 21 Aug. 1948, p. 123.

10. M. V. WILKES & W. RENWICK, "An ultrasonic memory unit for the EDSAC," [Electronic Delay Storage Automatic Calculator], *Electronic Engin.*, London, v. 20, 1948, p. 208–213, illustr. 19 × 25.4 cm.

11. M. V. WILKES, "The design of a practical high-speed computing machine. The EDSAC," [Electronic Delay Storage Automatic Calculator], R. Soc. London, *Proc.*, v. 195 A, 1948, p. 274–279. 17.2 × 25.4 cm.

NEWS

Association for Computing Machinery.—The balloting resulted in the election of the Council of the Association as listed in *MTAC*, v. 3, p. 380. The Council, according to the provisional Constitution and Bylaws, holds office until May 31, 1949. The new Council met in New York on Jan. 7. Messrs. E. C. BERKELEY and R. V. D. CAMPBELL were formally elected to serve as secretary and treasurer, respectively, until May 31, 1950.

At present, a committee under the chairmanship of Mr. E. G. ANDREWS is compiling a nomenclature list for large-scale computers. The list is to include items relating to: (1) general computer terminology (i.e., terms relating to digital computers, special-purpose computers, etc.); (2) computer components (i.e., storage, arithmetical organ, etc.); (3) notation (modified binary systems, etc.); and (4) operation (i.e., programing, etc.).

Institute of Radio Engineers.—The program of the National Convention of the IRE, March 7–10, 1949, New York City, included, on March 8, two sessions concerned with large-scale computers. The first session, on *Electronic Computers*, E. W. CANNON, chairman, included the following papers: "A dynamically regenerated memory tube," by J. P. ECKERT, JR., H. LUKOFF, & G. SMOLIAR, Eckert-Mauchly Computer Corp., Philadelphia; "An electronic differential analyzer," by ALAN B. MACNEE, Massachusetts Institute of Technology; "An analogue computer for the solution of linear simultaneous equations," by ROBERT M. WALKER, Watson Scientific Computing Laboratory; "The electronic isograph for a rapid analogue solution of algebraic equations," by BYRON O. MARSHALL, JR., Cambridge Field Station, Air Matériel Command; and "Parametric electronic computer," by CHARLES J. HIRSCH, Hazeltine Electronics Corporation, Little Neck, L. I.

In the first of these papers a description was presented of the work of the Eckert-Mauchly Computer Corp., and of the University of Manchester group under F. C. WILLIAMS, on the use of a nearly standard cathode-ray tube to provide storage of a large number of binary digits at a relatively low cost per digit, and more particularly with an access time of not more than a few microseconds. The system is advantageous in that it embodies the ability to move from one part of the memory to any other at high speed. Secondly, the low cost and easy maintenance allowed by the use of completely standard high-production electronic components with no special tubes make it an extremely desirable high-speed memory system. A detailed analysis of a computer based on such a system is now possible, and, in this connection, models of the memory system are being set up to further assess its reliability and practicality.

Mr. MacNee's talk concerned an electronic differential analyzer, capable of solving up to sixth order ordinary differential equations, both linear and nonlinear. Its high operation speed and extreme flexibility permit rapid investigation of wide ranges of equation solutions with regard to periodicity, instability, and discontinuities. Two new computing elements, an electronic function generator and an electronic multiplier, are employed.

The electrical analogue computer described by Mr. Walker accepts information for systems of linear equations of up to 12 unknowns in digital form from a set of punched cards thus facilitating the preparation, checking, and insertion of input data. Solutions of well-determined problems are easily and rapidly obtained, and may be refined to any desired accuracy by a simple iteration procedure.

Following this was a discussion of an electronic analogue computer known as an isograph, with which the complex plane may be rapidly investigated for roots of up to 10th-degree polynomials. The value of any polynomial is given for all values of the complex variable. It is believed that the isograph, which may be used alone or in conjunction with a large-scale computer, will be of special value in servo-mechanisms and that, in general, theoretical analyses of engineering problems will be furthered by its use.

The last talk dealt with a novel computer that operates on the principle of an alignment chart wherein data voltages are aligned in time in the same manner that data quantities are aligned in distance on a slide rule. Since the "time scales" of this electronic slide rule may be calibrated according to any function of time which can be electrically realized, a large variety of operations can be performed. Because of the rapidity of operation, they can be repetitive and performed on variable parameters. This computer can be made as accurate as desired if high enough voltages and rapid enough samplings are taken.

The second session, on *Electronic Computing Machines*, Dr. E. U. CONDON, chairman, included the following papers: "Results of tests on the BINAC," by J. W. MAUCHLY, Eckert-Mauchly Computer Corp.; "The Mark III computer," by H. H. AIKEN, Harvard University; "The IBM type 604 electronic calculator," by RALPH PALMER, IBM, New York; "Electrostatic memory for a binary computer," by F. C. WILLIAMS, University of Manchester, England; "Counting computers," by G. R. STIBITZ, Burlington, Vt.; and "Programing a computer for playing chess," by CLAUDE E. SHANNON, Bell Telephone Laboratories, New York.

The operating characteristics of the BINAC, which represents the practical application of computer elements discussed at last year's convention (i.e., a mercury delay-line memory and magnetic tape input and output equipment), were discussed by Dr. Mauchly. This device is of particular interest because, except for input and output equipment, it is entirely electronic.

The Mark III, the latest of the large-scale computers developed under Prof. Aiken's direction, has been designed for greater speed and reliability, more flexible memory facilities, and greater ease of preparation of input data than were found in the earlier computers.

The type 604 electronic calculator described by Mr. Palmer combines an electronic arithmetic element, including a 13-digit electronic counter, with punched-card input and output equipment and additional mechanical storage registers, with the possibility of carrying out automatically a "program" of as many as 20 arithmetic operations.

Dr. Williams treated the electrostatic memory on which he has carried out extensive research. These memories probably show the greatest promise of any of the basic types thus far proposed for computers, since they combine the high reading and writing speed of the delay-line type of memory with a very short "access time."

Dr. Stibitz proposed a new type of computer which combines the more accurate elements of the familiar differential analyzer, such as gears and differentials, with a new type of "function unit," resulting in a computer having the simplicity and low cost of an analogue computer and the higher accuracy of the digital type.

Dr. Shannon discussed the programing of a chess game on a large-scale computer. While the possibility of such applications was early recognized, this probably represents the first serious attempt to analyze the programing of such an operation.

Massachusetts Institute of Technology.—A Special Course in Analogue Computation, designed particularly to meet the needs of users of industrial types of analogue computing machines, was initiated at the Massachusetts Institute of Technology on June 20, 1949, to last for three weeks. The course was presented by Dr. SAMUEL H. CALDWELL, professor of electrical engineering and director of the Institute's Center of Analysis, and dealt especially with the treatment of engineering problems by machines designed for the solution of differential equations. The objective of the course was to provide a broader understanding of the uses and potentialities of analogue computers. The increasing availability of these machines throughout industry makes it important that trained personnel be prepared fully to exploit their benefits. Demonstrations were arranged using the MIT Differential Analyzer, as well as various types of electronic differential analyzers available or under development at the Institute. The course included a unified treatment of the following subject matter: mathematics refresher, basic analogue processes, machine solution of differential equations, calculation of scale factors, and electronic analogue machines.

Swedish State Board for Computing Machinery.—The Swedish State Board for Computing Machinery has recently been formed with Admiral STIG ERICSSON as president and Professors T. LAURENT, E. VELANDER, N. ZEILON (of Lund), and permanent secretary G. A. WIDELL (of Stockholm) as members. The Board's secretary is GÖSTA MALMBERG (Ecklesiastikdepartementet, Stockholm).

The immediate plans for the future include the construction of a relay computer in agreement with a project by Dr. CONNY PALM at Stockholm Institute of Technology. No definite plans exist regarding the design and construction of an electronic computer.

Any inquiries should be addressed to the secretary, or to Dr. C. E. FRÖBERG, Institute of Mechanics and Mathematical Physics, Lund.

OTHER AIDS TO COMPUTATION

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1. SIDNEY G. HACKER, *Arithmetical View Points. An Introduction to Mathematical Thinking.* Pullman, Wash., State College of Washington Bookstore, 1948, x, 144 p. + 14 plates. 21 × 27.4 cm. $1.50. Offset print.

This interesting *potpourri* by a professor of mathematics represents a course of lectures, which might be read with interest by senior undergraduates. There are 6 main headings.