For (15, 2, 1) $(10, 5, 3)_{12}$, $(9, 6, 3)_{11}$, $(8, 7, 3)_2$, $(8, 6, 4)_3$, *read* (15, 2, 1) $(10, 5, 3)_{15}$, $(9, 6, 3)_{13}$, $(8, 7, 3)_3$, $(8, 6, 4)_4$; *for* (14, 3, 1) $(8, 6, 4)_2$, *read* (14, 3, 1) $(8, 6, 4)_3$; *for* (12, 4, 2) $(11, 6, 1)_4$, *read* (12, 4, 2) $(11, 6, 1)_5$. These corrections change the total number of 4×4 magic squares from 539136 to 549504.

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UNPUBLISHED MATHEMATICAL TABLES

82[F].—L. Poletti, *Factor Table and List of Primes for the 30000 natural numbers nearest 15,000,000.* Manuscript table deposited in the library of the American Math. Soc. New York.

This table gives new information for the range 14984970–15000000. The second half from 15000000 to 15015000, is also covered by W. P. Durfee's factor table for the 16th million, a table which is in the same library.

The factor table, which the author calls "Neocribrum," is a "type 3 table" arranged in the usual way modulo 30. On p. 1 are given data on the distribution of the primes in this range. Thus there are 1809 primes which are also classified modulo 30. There are 159 prime pairs. There are 113 consecutive composite numbers following 14996687.

Poletti is the author of *Tavole di Numeri Primi entro Limiti Diversi e Tavole Affini,* Milan, 1920.

D. H. L.

83[G, I].—H. E. Salzer, *Coefficients of the first fifteen General Laguerre Polynomials.* Ms. in possession of the author.

The writer announced previously (*MTAC*, v. 2, p. 89) a manuscript giving the coefficients of Laguerre polynomials, which are a special case of general Laguerre polynomials $L_n^{(\alpha)}(x)$, namely for $\alpha = 0$. The present manuscript gives the polynomials in α which are the coefficients of x^ν in the general Laguerre polynomial

$$L_n^{(\alpha)}(x) \equiv e^x x^{-\alpha} \frac{1}{n!}\left(\frac{d}{dx}\right)^n (e^{-x}x^{n+\alpha}) \equiv \sum_{\nu=0}^{n} \binom{n+\alpha}{n-\nu} \frac{(-x)^\nu}{\nu!}, \text{ for } \nu = 0(1)n,$$

and for $n = 0(1)15$.

H. E. Salzer

AUTOMATIC COMPUTING MACHINERY

Edited by the Staff of the Machine Development Laboratory of the National Bureau of Standards. Correspondence regarding the Section should be directed to Dr. E. W. Cannon, 418 South Building, National Bureau of Standards, Washington 25, D. C.

Technical Developments

Our contribution under this heading, appearing earlier in this issue, is "The California Institute of Technology Electric Analogue Computer" by Prof. G. D. McCann.

Discussions

Procedure for the Machine or Numerical Solution of Ordinary Linear Differential Equations for Two-Point Linear Boundary Values

Introduction. Increased attention is being focused on machine and numerical solutions of differential equations which cannot be solved by ordinary mathematical methods. There is need for more information on this subject for engineers and others who deal with such equations. This paper describes a procedure applicable to numerical or machine solutions of the general non-homogeneous ordinary linear differential equation with variable coefficients where the form of these coefficients does not easily permit of solution by series. The method is based on the well-known properties of linear differential equations.

Ordinary differential equations of order higher than the first commonly describe problems where the known boundary conditions are expressed at two different values of the independent variable. Such problems are known as two-point boundary-value problems. Although a great many linear equations, such as the BESSEL and LEGENDRE equations, may be rigorously handled by the method of FROBENIUS, there are frequently those where the variable coefficients of the derivatives are so complex that a series solution is not feasible. For such equations, recourse to solution by numerical methods or by some type of computing machine or analyzer may be sought. In this event, a difficulty is at once encountered if divided boundary conditions are present. In the case of an equation of order n, the dependent variable and its $n - 1$ derivatives must possess assigned values at some point within the interval in order that a machine or step-by-step solution may proceed from that point. Consider the case of a second-order equation where the two boundary conditions are divided between both ends of the range of the independent variable. Only one of an unlimited number of possible values of the initially unknown dependent variable or one of its derivatives, as the case may be, at one end of the interval of solution will satisfy the boundary condition at the other end. For a fourth-order equation with equally divided boundary conditions, a double latitude of possible initial choices would exist.

The theory of ordinary linear differential equations appears extensively in the mathematical literature.[1, 2, 3] Application to the two-point boundary problem where numerical or machine solutions are involved does not appear to be generally well known. The purpose of this paper is to show in relatively simple mathematical terms and by graphical illustration how the two-point boundary problem may be handled. The method applies to any ordinary linear differential equation of order n where the boundary conditions are expressible in terms of linear combinations of the dependent variable and its $n - 1$ derivatives and where K boundary conditions are known at one point of the interval of solution and the remaining $n - K$ conditions are known at some other point of the interval. It is shown that for $K \leq n/2$ only $K + 1$ trial solutions with arbitrarily selected values for those derivatives which are initially unknown are required to determine the unique solution satisfying all n boundary conditions. It is also shown that $n + 1$ trial solutions will give data for any solution of the non-homogeneous equation. In addition, a possible graphical procedure is suggested for converging on the solution of non-linear equations.

Theory. Consider an ordinary non-homogeneous linear differential equation of order n with variable coefficients. This is

$$(1) \qquad f_0(d^ny/dx^n) + f_1(d^{n-1}y/dx^{n-1}) + \cdots + f_{n-1}(dy/dx) + f_ny = f.$$

Assume $f_0, f_1, \cdots f_n, f$ are finite, one-valued, and continuous functions of x in the interval $a_0 \leq x \leq b_0$, and that f_0 does not vanish at any point in the interval. Under these conditions, there is known to exist a solution y such that y and its first n derivatives are continuous and have unique values at every point in the interval. The reduced or homogeneous equation corresponding to equation (1) is

$$f_0(d^n y/dx^n) + f_1(d^{n-1}y/dx^{n-1}) + \cdots + f_{n-1}(dy/dx) + f_n y = 0 \tag{2}$$

and is known to have n and only n linearly independent solutions,[4] $y_1, y_2, \cdots y_n$. The known complete solution of equation (1) is then

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n + y_p = y_p + \sum_{i=1}^{i=n} c_i y_i, \tag{3}$$

where y_p is any particular solution of equation (1) and $c_1, c_2, \cdots c_n$ are arbitrary constants to be determined by the n boundary conditions.

If the quantities B_k which assume boundary values are linear,[5] we may express them in the following manner:

$$B_k = \sum_{j=0}^{j=n-1} a_{k,j}(d^j y/dx^j), \qquad k = 1, 2, \cdots n, \tag{4}$$

where each of the coefficients, $a_{k,j}$, represents a constant or some known function of x, and where $|a_{k,j}| \neq 0$ since the boundary conditions are linearly independent. Substituting the general solution, equation (3), into equation (4) and factoring out the c's, we have for the particular point, $x = x_k$, at which B_k has a known value,

$$\left(B_k = \sum_{i=1}^{i=n} c_i \sum_{j=0}^{j=n-1} a_{k,j}(d^j y_i/dx^j) + \sum_{j=0}^{j=n-1} a_{k,j}(d^j y_p/dx^j) \right)_{x=x_k}, \quad k = 1, 2, 3, \cdots n.$$

In this expression for B_k, all terms are of fixed value. Since there are n such linear expressions for the B's, and the product $|a_{k,j}| \cdot |d^j y_i/dx^j|$ does not vanish, we may solve explicitly for the n values of the c's. Carrying this out one obtains linear equations of the form,

$$c_i = b_i + \sum_{k=1}^{k=n} b_{i,k} B_k, \qquad i = 1, 2, \cdots n. \tag{5}$$

The term b_i is a combination of the constant values of y_p and its derivatives for the particular x involved, and $b_{i,k}$ is a combination of the constants $a_{k,j}$ and the fixed values of y_i and its derivatives. Substituting equation (5) into equation (3) one obtains

$$y = y_p + \sum_{i=1}^{i=n} b_i y_i + \sum_{k=1}^{k=n} B_k \sum_{i=1}^{i=n} b_{i,k} y_i. \tag{6}$$

This is a formal statement of the fact that the general solution may be expressed directly in terms of linear combinations of the n boundary parameters. It now becomes evident that for a given equation of order n, if n arbitrary but linearly independent solutions y_i of equation (2) and any particular solution y_p of equation (1) are obtained, then the solution to the problem with any desired values of divided boundary parameters, $B_1, B_2, \cdots, B_n$, known at any points within the interval (a_0, b_0), may be obtained by direct substitution into equation (6). It should be emphasized that y_i and y_p are *any* solutions to their respective equations, barring linearly dependent

y_i's, without restriction on initial or final values. It is seen from equation (6) that, if the boundary conditions are all homogeneous, then each B equals zero and the solution is simply

$$y = y_p + \sum_{i=1}^{i=n} b_i y_i.$$

Also, if the equation is homogeneous but not all of the boundary conditions are homogeneous, then

$$y = \sum_{k=1}^{k=n} B_k \sum_{i=1}^{i=n} b_{i,k} y_i.$$

The term in $b_i y_i$ is zero since b_i is a linear combination of the fixed boundary values of y_p and its $n - 1$ derivatives, and these quantities exist only in the non-homogeneous equation. If the equation and boundary conditions are all homogeneous, then it is apparent that $y = 0$, and there is no problem.

The advantage of expressing the solution directly in terms of the boundary parameters will now be illustrated for a fourth-order equation with boundary values B_1 and B_2 existing at $x = a$ and B_3 and B_4 existing at $x = b$. The limits a, b are ordinary points of the interval (a_0, b_0). For $n = 4$, the solution y and its first $n - 1$ derivatives may be written from equation (6) as

$$y = B_1 Y_1 + B_2 Y_2 + B_3 Y_3 + B_4 Y_4 + Y$$
$$dy/dx = B_1(dY_1/dx) + \text{etc.}, \qquad d^2y/dx^2 = B_1(d^2 Y_1/dx^2) + \text{etc.},$$
$$d^3y/dx^3 = B_1(d^3 Y_1/dx^3) + \text{etc.},$$

where Y_1, Y_2, Y_3, Y_4 are linear combinations of y_1, y_2, y_3, y_4, and Y is a linear combination of the same y's and y_p. Now consider the problem of a machine or step-by-step solution starting at $x = a$ where the two boundary parameters have the desired values $B_1 = \beta_1$ and $B_2 = \beta_2$. From the two boundary relations at $x = a$ given by equation (4) we may write

$$\beta_1 = [a_{1,0}y + a_{1,1}(dy/dx) + a_{1,2}(d^2y/dx^2) + a_{1,3}(d^3y/dx^3)]_{x=a}$$
$$\beta_2 = [a_{2,0}y + a_{2,1}(dy/dx) + a_{2,2}(d^2y/dx^2) + a_{2,3}(d^3y/dx^3)]_{x=a}$$

from which two of the initial values of the derivatives may be solved in terms of the remaining two. Thus, for any arbitrary values assigned to $y_{x=a}$ and $(dy/dx)_{x=a}$, the values of the second and third derivatives at $x = a$ may be calculated.[6] With B_1 and B_2 assigned the values β_1 and β_2, respectively, the solution and its first derivative become

$$y = B_3 Y_3 + B_4 Y_4 + Y_5, \qquad dy/dx = B_3 dY_3/dx + B_4 dY_4/dx + dY_5/dx,$$

where Y_5 is the new function, $Y + \beta_1 Y_1 + \beta_2 Y_2$. At $x = a$, Y_j and dY_j/dx (where $j = 3, 4, 5$) assume fixed values so that

$$(7) \qquad y_{x=a} = d_3 B_3 + d_4 B_4 + d_5, \qquad (dy/dx)_{x=a} = e_3 B_3 + e_4 B_4 + e_5,$$

where the d's and e's are constants. Equations (7) show that the initial values of y and dy/dx are related linearly to the boundary parameters at $x = b$. This linearity may be represented graphically as shown in Fig. 1. Although the existence of these linear families of boundary parameters is now established, their determination is still unknown for any trial solution.

Assume now that the problem at hand requires that $B_3 = \beta_3$ and $B_4 = \beta_4$ at $x = b$. It is readily seen that there exists a point S which defines the proper initial choice of $y_{x=a}$ and $(dy/dx)_{x=a}$. It is also evident from the linearity of equations (7) that, in proceeding on a straight line joining any two points, such as 1 and 2 in Fig. 1, the values of B_3 and B_4 will vary linearly with the distance measured along that line. With these facts in mind, let us make three separate trial solutions with any arbitrary combination of $y_{x=a}$ and $(dy/dx)_{x=a}$ which define three non-collinear points 1, 2, 3. The initial values of the second and third derivatives for each trial are of course determined so as to satisfy $B_1 = \beta_1$ and $B_2 = \beta_2$ at $x = a$. When these trial solutions have reached $x = b$, the values of y and its three derivatives at this point are used to calculate B_3 and B_4. By linear interpolation point P_1 which represents a point where B_3 equals the desired value β_3 may be located on line 1–2. Likewise points P_2 and P_3 which lie on the desired β_3 line are located by linear interpolation and extrapolation along lines 2–3 and 3–1. Similarly points Q_1, Q_2, Q_3 are located on the desired β_4 line. These two parameter lines may now be constructed and the desired solution point S determined by their intersection.

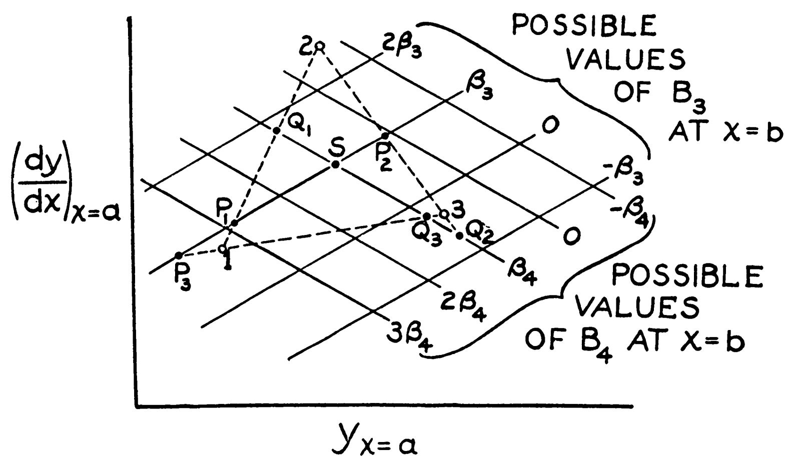

FIG. 1. Control plot for solution of fourth-order linear differential equation with equally divided boundary values.

Although the desired boundary conditions can now be satisfied with a fourth solution which begins with the correct combination of $y_{x=a}$ and $(dy/dx)_{x=a}$, it is not necessary to effect this fourth solution. For any particular value of x in the interval (a, b), the correct values of y and its derivatives may be obtained by linear interpolation and extrapolation to the solution point S from the corresponding values at points 1, 2, 3. This further linearity is at once evident from equations (7) if we replace B_3 and B_4 by any two of the derivatives of y at this particular value of x.

Consider next a sixth-order equation with three boundary conditions known at $x = a$ and the other three at $x = b$. Here, there is a triple latitude of possible choices of the three initially unknown values of y and its five derivatives at $x = a$ and only one unique combination will satisfy the given conditions at $x = b$. To solve this problem, we may visualize, in place of Fig. 1, three families of parallel planes in a three-dimensional space defined by the three initial values of y and its derivatives which are unknown. To determine the solution point S representing the common intersection of the three boundary-value planes will require four different trial solutions, each satisfying the three conditions at $x = a$. These trials will define four non-coplanar points in the three-dimensional space. Linear interpolation and extrapolation along any three non-coplanar lines joining these points will determine three sets of three points, each set uniquely defining one of the desired boundary-value planes. Again the values of y and its derivatives can be obtained at point S for any value of x by a three-dimensional linear interpolation and extrapolation from the four points.

In the case of a second-order equation with its two boundary conditions divided, only two trials, each satisfying the initial condition at $x = a$, are necessary to establish the solution. The relationship is shown graphically in Fig. 2 where the solution point S is determined by the linearity along the line joining the trial points 1 and 2.

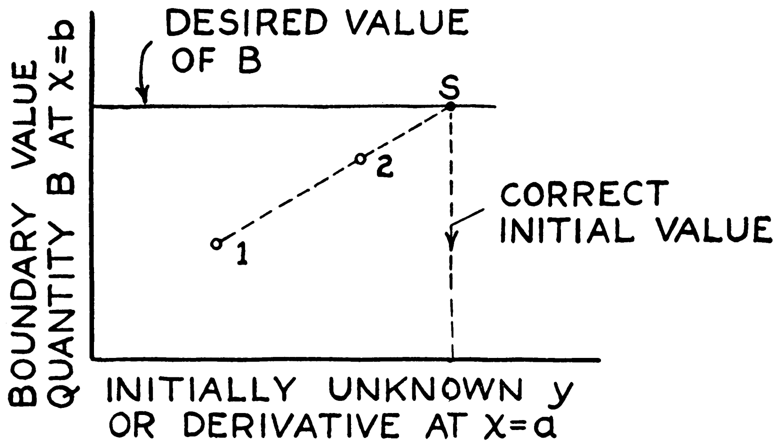

FIG. 2. Control plot for solution of second order linear differential equation with equally divided boundary values.

So far, only cases of equally divided boundary conditions have been discussed. If unequal division occurs, the solution should be started at the point where the greater number of known conditions exists. In the case of a fourth-order equation where three conditions are known for $x = a$, only one derivative of y is unknown, and two trials will be sufficient for solution.

Boundary conditions need not exist at the extremities of the interval of solution. In such cases the solution may begin at a point within the interval where at least half of the conditions are known. It will be necessary to reverse the direction of the independent variable in order to cover the complete range, but otherwise the procedure will be the same.

The following rules may be stated from the foregoing development. They hold for any ordinary linear differential equation of order n defined by equation (1) where K boundary conditions are known at one point within the interval of solution and the remaining $n - K$ conditions are known at some other point within the interval. If $K \leq n/2$, only $K + 1$ trial solutions with $K + 1$ arbitrarily selected initial values of the initially unknown derivatives are required to determine uniquely the desired solution which satisfies all boundary conditions. If data for the solution for any or all possible combinations of the n boundary conditions are required, then any n solutions, barring linearly dependent ones, of the homogeneous equation without reference to initial conditions and any one solution of the complete equation, or a total of $n + 1$ arbitrary trial solutions, must be made.

Discussion. Although the $n + 1$ trial solutions are sufficient to solve all boundary value problems for a given equation, in most practical cases, where interest is centered on a given set of boundary values, the method described using $K + 1$ trials, where $K \leq n/2$, will be the simpler procedure. Even in this case $K + 1$ families of solutions may be had with the $K + 1$ trials, since any boundary value at $x = b$ for each of the K boundary parameters and any value of b within the interval (a_0, b_0) may be used. All of these solutions must satisfy the same conditions at $x = a$.

Graphical illustration has been used to describe the linearities involved in order to aid in visualizing the problem. The solution point and the values of the various functions at this point may be determined by direct solution of the linear relationships involved. And indeed, if it is necessary to discuss an equation of order higher than the sixth, our spatial visualization which is limited to three dimensions would not aid in this problem. If the method of the $n + 1$ trials is adopted, the use of determinants will facilitate the necessary computation of b_i and $b_{i,k}$ in equation (6).

There are several practical considerations which place some limits on the success of these methods. It is essential that the distances between the trial points be of the same order of magnitude as the distance from any point to the solution point. Although the danger of interpolation and extrapolation on curved lines is absent, still the errors inherent in any machine or numerical solution will limit the extent of accurate extrapolation. It is usually possible in most physical problems by approximation, comparison, and reasonable guessing to predict the general region of solution and thus choose trial points which are not too far removed from the solution point. If a reasonably close estimate cannot be made, any $K + 1$ trials will point to the approximate location of S whereupon $K + 1$ additional trials in the neighborhood of S will yield the solution.

In order to insure sufficient accuracy in the results, a reasonable estimate of the range of magnitudes of y and its $n - 1$ derivatives must be made in the case of a machine solution. It may be necessary to repeat one or more trials with adjusted scale factors if the estimate is far in error.

Although $K + 1$ trial solutions are mathematically sufficient for solution, one or two more may be desirable in the case of a machine solution as a check on the accuracy of the work. In the case of the fourth-order equation with equally divided boundary values, the triangular configuration of trial points shown in Fig. 1 might well be replaced by four points representing the corners of a square. Any three of the six possible line segments joining the four points may be considered as locating these points. These three segments then determine independently three points on the desired parameter line. The remaining three segments may be considered as dependent on the first three and hence will determine three dependent points on the same parameter line.

The method described in this paper has been used successfully by the author in the solution of a problem in the theory of shells involving a fourth-order equation with equally divided boundary conditions. Triangular sets of trial points in the machine solution used were adequate.

In the case of non-linear equations or linear equations where the boundary conditions are non-linear a procedure similar to that described in the foregoing paragraphs is suggested. In this case graphical representation would be indicated, and, in a control plot corresponding to Fig. 1, the families of lines representing boundary parameters would no longer be straight lines or linearly spaced. However, by the principle of uniqueness of solution, it is evident that any one boundary parameter line of one family will not cross any other boundary parameter line of the other family more than once. Also, if a reasonably good initial estimate is made, it should be possible to converge on the solution with a few successive sets of trial points.

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[4] The criterion for linear independence of these solutions is that the Wronskian of the y_i's (where $i = 1, 2, 3, \cdots n$) and their $n - 1$ derivatives does not vanish.

[5] Boundary values are said to be linear if they can be expressed as linear combinations of the dependent variable and its $n - 1$ derivatives.

[6] If in some actual problem $y_{x=a}$ and $(dy/dx)_{x=a}$, for example, took on assigned boundary values, then these may not be changed. Thus $(d^2y/dx^2)_{x=a}$ and $(d^3y/dx^3)_{x=a}$ would be the unknown initial quantities.

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