(a) two equal blunders in successive values or (b) a systematic succession of erroneous values in a table.

It is also proposed to give error patterns, such as that in Table III, for tables of *divided* differences, for use with tables having certain common arrangements of arguments at unequal intervals, for example, with a table having arguments

$$0, \tfrac{1}{4}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{3}{4}, 1, 1\tfrac{1}{4}, 1\tfrac{1}{3}, \cdots$$

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[1] The introduction of this useful distinction in name between rounding-off and true errors is due to C. R. G. COSENS.

[2] It must be remarked that the sequences of errors discussed here can arise from a cause other than the one indicated, though such causes are comparatively less common. For instance, if differencing is done on a calculating machine, a function value may be correctly recorded, but wrongly set on the machine. Likewise, a different sequence of differences indicates blunders of a different type. It is hoped to discuss some of these in Part II of the paper.

[3] In practice, a large blunder shows up well enough for location in earlier orders of differences, in fact, as soon as the largest of the differences due to the blunder sufficiently exceeds the true differences in magnitude, say in the ratio 5 to 1 or 10 to 1. Detection is possible in still earlier differences.

[4] A. VAN WIJNGAARDEN & W. L. SCHEEN of the Mathematisch Centrum of Amsterdam, Holland, have developed the theory independently and have obtained an asymptotic expansion. The result given for 9-th differences in our table was obtained by them and communicated to us for inclusion in this paper. Their 1 percent limit for 10-th differences is 303.

An ENIAC Determination of π and e to more than 2000 Decimal Places

Early in June, 1949, Professor JOHN VON NEUMANN expressed an interest in the possibility that the ENIAC might sometime be employed to determine the value of π and e to many decimal places with a view toward obtaining a statistical measure of the randomness of distribution of the digits, suggesting the employment of one of the formulas:

$$\begin{aligned}\pi/4 &= 4 \arctan 1/5 - \arctan 1/239\\ \pi/4 &= 8 \arctan 1/10 - 4 \arctan 1/515 - \arctan 1/239\\ \pi/4 &= 3 \arctan 1/4 + \arctan 1/20 + \arctan 1/1985\end{aligned}$$

in conjunction with the GREGORY series

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n (2n+1)^{-1} x^{2n+1}.$$

Further interest in the project on π was expressed in July by Dr. NICHOLAS METROPOLIS who offered suggestions about programming the calculation.

Since the possibility of official time was too remote for consideration, permission was obtained to execute these projects during two summer holiday week ends when the ENIAC would otherwise stand idle, and the planning and programming of the projects was undertaken on an extra-curricular basis by the author.

The computation of e was completed over the July 4th week end as a practice job to gain experience and technique for the more difficult and longer project on π. The reciprocal factorial series was employed:

$$e = \sum_{n=0}^{\infty} (n!)^{-1}.$$

The first of the above-mentioned formulas was employed for the computation of π; its advantage over the others will be explained later. The computation of π was completed over the Labor-Day week end through the combined efforts of four members of the ENIAC staff: CLYDE V. HAUFF (who checked the programming for π), Miss HOMÉ S. MCALLISTER (who checked the programming for e), W. BARKLEY FRITZ and the author, taking turns on eight-hour shifts to keep the ENIAC operating continuously throughout the week end.

While the programming for e is valid for a little over 2500 decimal places and, with minor alterations, can be extended to much greater range, and while the programming for π is valid for around 7000 decimal places, the arbitrarily selected limit of 2000+ was a convenient stopping point for e and about all that could be anticipated for a week end's operation for π.

While the details of the programming for each project were completely different, the general pattern of procedure was roughly the same, and both projects will be discussed together. In both projects the ENIAC'S divider was employed to determine a chosen number i of digits of each successive term of the series being computed, the remainder after each division being stored in the ENIAC'S memory and the digits of each term being added to (or subtracted from) the cumulative total. After performing this operation for as many successive terms as practicable, the remainders for these terms were printed on an I.B.M. card (the standard input-output vehicle for the ENIAC), and the process was repeated, continuing through some term beyond which the digits of and remainders for all further terms would be zeros. At this point was printed the cumulative total of the digits of the individual terms, which yielded (after adjustment for carry-over) the actual digits of the series being determined.

The cards bearing the remainders then were fed into the ENIAC reader, and the entire process was repeated for the next i digits, the ENIAC reading each remainder in turn and placing it before the digits of the appropriate term. Each deck of cards bearing remainders was then employed to determine the "next" i digits and the "next" deck of "remainder" cards continuing through the first stopping point beyond the 2000th decimal place. The cards bearing the cumulative totals of sets of i digits of the terms were then adjusted for carry-over into each preceding set of i digits. In the case of e this yielded the final result; in the case of π all the above described operations were performed once for each inverse tangent series, so that each set of "cumulative total" cards, adjusted for carry-over, yielded the digits of one of the series, the final result being determined by the combination of these series in appropriate manner.

The number of places i chosen for each interval of computation, the maximum magnitude of each remainder, the amount of memory space available, and the details of divider operation (the number of places to which division can be performed to yield a positive remainder, and the necessary conditions of relative and absolute positioning of numerator and denominator) all were interrelated, and where opportunity for selection existed, that selection was made which provided maximum efficiency of computation. In the case of π there was imposed the additional requirement that identical programming apply for all series employed, and for this reason the formula:

$$\pi/4 = 4 \arctan 1/5 - \arctan 1/239$$

was superior to the other two.

In order to insure absolute digital accuracy, the programming was arranged so that one half applied to computation and the other half to checking. Before any deck of "remainder" cards was employed to determine the next i digits, the cards were reversed and employed in the checking sequence to confirm each division by a multiplication and each addition by a subtraction and vice versa, reproducing the previous deck of "remainder" cards and insuring that the cumulative total reduced to zero. (In the case of e this was a simple inversion of the computation; in the case of π the factor $(2n + 1)^{-1}$ in each term made it a more complicated affair.) After the correctness of each deck was established through this checking, the "remainder" cards were rereversed, and the computation proceeded for the next i digits.

Since the determination of each i digits was not begun until the determination of the previous i digits had been confirmed by checking, the ENIAC stood idle during the reversals and rereversals and comparisons of the decks in the computation of e; in the case of π, however, the ENIAC was never idle, for operation on each series was alternated with operation on the other, card-handling on either being accomplished while the other was being operated upon by the ENIAC. In the case of e, insurance against any undiscovered accidental misalignment of cards was provided by rerunning the entire computation without checking, i.e., without card reversals, confirming the original results; in the case of π, the same assurance was provided by a programmed check upon the identification numbers of each successive card in both computation and checking.

In the case of e, there was printed (in addition to each "remainder" card) a card containing the current i digits of $(n!)^{-1}$ for $n = 20K; K = 1, 2, 3 \cdots$; in the case of π only remainder and final total cards were printed.

The ENIAC determinations of both π and e confirm the 808—place determination of e published in *MTAC*, v. 2, 1946, p. 69, and the 808—place determination of π published in *MTAC*, v. 2, 1947, p. 245, as corrected in *MTAC*, v. 3, 1948, p. 18–19.

Only the following minor observation is offered at this time concerning the randomness of the distribution of the digits. Publication on this subject will, however, be forthcoming soon. A preliminary investigation has indicated that the digits of e deviate significantly from randomness (in the sense of staying closer to their expectation values than a random sequence of this length normally would) while for π no significant deviations have so far been detected.

The programming was checked and the first few hundred decimal places of each constant were determined on a Sunday before each holiday week end mentioned above, the principal effort being made on the longer week end. The actual required machine running time for both computation and checking in the case of e was around 11 hours, though card-handling time approximately doubled this, and the recomputation without checking added about 6 hours more; actual required machine running time (including card-handling time) for π was around 70 hours.

The following values of π and e have been rounded off to 2035D and 2010D respectively.

π =	3.14159	26535	89793	23846	26433	83279	50288	41971	69399	37510
	58209	74944	59230	78164	06286	20899	86280	34825	34211	70679
	82148	08651	32823	06647	09384	46095	50582	23172	53594	08128
	48111	74502	84102	70193	85211	05559	64462	29489	54930	38196
	44288	10975	66593	34461	28475	64823	37867	83165	27120	19091
	45648	56692	34603	48610	45432	66482	13393	60726	02491	41273
	72458	70066	06315	58817	48815	20920	96282	92540	91715	36436
	78925	90360	01133	05305	48820	46652	13841	46951	94151	16094
	33057	27036	57595	91953	09218	61173	81932	61179	31051	18548
	07446	23799	62749	56735	18857	52724	89122	79381	83011	94912
	98336	73362	44065	66430	86021	39494	63952	24737	19070	21798
	60943	70277	05392	17176	29317	67523	84674	81846	76694	05132
	00056	81271	45263	56082	77857	71342	75778	96091	73637	17872
	14684	40901	22495	34301	46549	58537	10507	92279	68925	89235
	42019	95611	21290	21960	86403	44181	59813	62977	47713	09960
	51870	72113	49999	99837	29780	49951	05973	17328	16096	31859
	50244	59455	34690	83026	42522	30825	33446	85035	26193	11881
	71010	00313	78387	52886	58753	32083	81420	61717	76691	47303
	59825	34904	28755	46873	11595	62863	88235	37875	93751	95778
	18577	80532	17122	68066	13001	92787	66111	95909	21642	01989
	38095	25720	10654	85863	27886	59361	53381	82796	82303	01952
	03530	18529	68995	77362	25994	13891	24972	17752	83479	13151
	55748	57242	45415	06959	50829	53311	68617	27855	88907	50983
	81754	63746	49393	19255	06040	09277	01671	13900	98488	24012
	85836	16035	63707	66010	47101	81942	95559	61989	46767	83744
	94482	55379	77472	68471	04047	53464	62080	46684	25906	94912
	93313	67702	89891	52104	75216	20569	66024	05803	81501	93511
	25338	24300	35587	64024	74964	73263	91419	92726	04269	92279
	67823	54781	63600	93417	21641	21992	45863	15030	28618	29745
	55706	74983	85054	94588	58692	69956	90927	21079	75093	02955
	32116	53449	87202	75596	02364	80665	49911	98818	34797	75356
	63698	07426	54252	78625	51818	41757	46728	90977	77279	38000
	81647	06001	61452	49192	17321	72147	72350	14144	19735	68548
	16136	11573	52552	13347	57418	49468	43852	33239	07394	14333
	45477	62416	86251	89835	69485	56209	92192	22184	27255	02542
	56887	67179	04946	01653	46680	49886	27232	79178	60857	84383
	82796	79766	81454	10095	38837	86360	95068	00642	25125	20511
	73929	84896	08412	84886	26945	60424	19652	85022	21066	11863
	06744	27862	20391	94945	04712	37137	86960	95636	43719	17287
	46776	46575	73962	41389	08658	32645	99581	33904	78027	59009
	94657	64078	95126	94683	98352	59570	98258			

e =	2.71828	18284	59045	23536	02874	71352	66249	77572	47093	69995
	95749	66967	62772	40766	30353	54759	45713	82178	52516	64274
	27466	39193	20030	59921	81741	35966	29043	57290	03342	95260
	59563	07381	32328	62794	34907	63233	82988	07531	95251	01901
	15738	34187	93070	21540	89149	93488	41675	09244	76146	06680
	82264	80016	84774	11853	74234	54424	37107	53907	77449	92069
	55170	27618	38606	26133	13845	83000	75204	49338	26560	29760

67371	13200	70932	87091	27443	74704	72306	96977	20931	01416
92836	81902	55151	08657	46377	21112	52389	78442	50569	53696
77078	54499	69967	94686	44549	05987	93163	68892	30098	79312
77361	78215	42499	92295	76351	48220	82698	95193	66803	31825
28869	39849	64651	05820	93923	98294	88793	32036	25094	43117
30123	81970	68416	14039	70198	37679	32068	32823	76464	80429
53118	02328	78250	98194	55815	30175	67173	61332	06981	12509
96181	88159	30416	90351	59888	85193	45807	27386	67385	89422
87922	84998	92086	80582	57492	79610	48419	84443	63463	24496
84875	60233	62482	70419	78623	20900	21609	90235	30436	99418
49146	31409	34317	38143	64054	62531	52096	18369	08887	07016
76839	64243	78140	59271	45635	49061	30310	72085	10383	75051
01157	47704	17189	86106	87396	96552	12671	54688	95703	50354
02123	40784	98193	34321	06817	01210	05627	88023	51930	33224
74501	58539	04730	41995	77770	93503	66041	69973	29725	08868
76966	40355	57071	62268	44716	25607	98826	51787	13419	51246
65201	03059	21236	67719	43252	78675	39855	89448	96970	96409
75459	18569	56380	23637	01621	12047	74272	28364	89613	42251
64450	78182	44235	29486	36372	14174	02388	93441	24796	35743
70263	75529	44483	37998	01612	54922	78509	25778	25620	92622
64832	62779	33386	56648	16277	25164	01910	59004	91644	99828
93150	56604	72580	27786	31864	15519	56532	44258	69829	46959
30801	91529	87211	72556	34754	63964	47910	14590	40905	86298
49679	12874	06870	50489	58586	71747	98546	67757	57320	56812
88459	20541	33405	39220	00113	78630	09455	60688	16674	00169
84205	58040	33637	95376	45203	04024	32256	61352	78369	51177
88386	38744	39662	53224	98506	54995	88623	42818	99707	73327
61717	83928	03494	65014	34558	89707	19425	86398	77275	47109
62953	74152	11151	36835	06275	26023	26484	72870	39207	64310
05958	41166	12054	52970	30236	47254	92966	69381	15137	32275
36450	98889	03136	02057	24817	65851	18063	03644	28123	14965
50704	75102	54465	01172	72115	55194	86685	08003	68532	28183
15219	60037	35625	27944	95158	28418	82947	87610	85263	98139
55990	06738								

Values of the auxiliary numbers arccot 5 and arccot 239 to 2035D are in the possession of the author and also have been deposited in the library of Brown University and the UMT FILE [1] of *MTAC*.

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[1] See *MTAC*, v. 4, p. 29.

RECENT MATHEMATICAL TABLES

691[A].—M. LOTKIN, "Table of the first 200 factorials to 20 places," Ballistic Research Laboratories, Aberdeen Proving Ground, *Technical Note* no. 106, 1949, 11 p. mimeograph, 21.7 × 27.8 cm.

The table gives the first 20 significant figures of $n!$ for $n = 1(1)\,200$ together with the exponent of the power of 10 by which the figure should be multiplied to give the approximate value of $n!$ The author was unaware of a previous table by UHLER[1] giving the exact values of these factorials.