every basic system function is included. Tests of every operating function under every expected condition have been performed with a repeated reliability which confirms the adequacy of the selected design standards. Reliability of the basic circuits and of the magnetic and mechanical components has been further established in an extensive laboratory program of component research and in the development of other types of magnetic drum storage systems during the past two years.

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DISCUSSIONS

Notes on Modern Numerical Analysis—I

EDITORIAL NOTE: There is a general feeling that, once the problems of construction and maintenance of automatic digital computing machines are solved, the remaining problems will be relatively simple. This may be the case if attention is confined to standard classical problems; however, if an attempt is made to use these machines fully, one is likely to encounter formidable mathematical difficulties. It is expected that these difficulties will be discussed in the current mathematical journals; but there are also smaller, more technical problems which may cause trouble. It is believed that a discussion of these smaller problems will prove beneficial in avoiding a great many difficulties which are expected to arise when the machines are in actual operation; and we should like to urge interested persons to submit technical notes of this nature for future publication in the Automatic Computing Machinery Section of *MTAC*. These notes could be by-products of or preliminaries to more constructive investigations. It would be a great advantage, for expository purposes, if the authors, even at the expense of a choice of an extravagant example, could exhibit the troubles under discussion on a manual scale.

Solution of Differential Equations by Recurrence Relations

1.1. In general the most satisfactory method for the numerical solution of ordinary differential equations is one of the "extrapolation" methods.[1] These methods have proven very efficient in the hands of a practiced computer. There is little doubt that some of the experience he uses could be codified and adapted for use on automatic digital computing machines. Nevertheless, the use of some direct recursive process is very attractive and worth investigation.

Let us consider the solution by such methods of the equation

$$y'' = -y, \tag{1}$$

with the boundary conditions $y(0) = 0, y'(0) = 1$, by use of the well-known formula [2]

$$h^2y'' = (\delta^2 - \tfrac{1}{12}\delta^4 + \tfrac{1}{90}\delta^6 - \cdots)y. \tag{2}$$

1.2. First let $h = 1$, using only the first term on the right-hand side of (2). If the condition $y'(0) = 1$ is replaced by $y(1) = 1$ the following recurrence relation is obtained

$$y(n+2) = y(n+1) - y(n) \tag{3}$$

with the boundary conditions $y(0) = 0$, $y(1) = 1$. For $n = 0, 1, 2, 3, \cdots$,

$$y(n) = 0, 1, 1, 0, -1, -1, 0, 1, 1, 0, \cdots. \tag{4}$$

Two points are now worth noting. One way of introducing the circular functions analytically is to define $\sin x$ as the solution of (1); in this treatment π is defined as the least positive root of $\sin x = 0$. Observe that 3 has now been obtained as an approximation to π. Secondly, it is seen that the solution (4) is periodic.

1.3. By taking a smaller value of h, a possible improvement can be expected. If we take $h = 0.1$, the recurrence relation is now

$$y(n+2) = 1.99y(n+1) - y(n) \tag{5}$$

with $y(0) = 0$ and $y(1) = \sin 0.1 = 0.09983$. The solution obtained when five decimal places are used is given in column (5) of Table I and may be compared with the corresponding values of $\sin x$, to ten places, given in column (1). It will be seen that for $0 \leq x \leq 16$ the error is always negative and steadily increases in absolute value, being $-35 \cdot 10^{-5}$, when $x = 1.6$.

This solution is apparently not periodic, and we may inquire as to the existence of values of h (other than $h = 1$) for which the solution is periodic, i.e., for which the sequence of its values is periodic. It can be shown that the only values of h are $h = 2 \sin \pi/n$ for $n = 2, 3, \cdots$. When $h = 2 \sin \pi/n$, the period is n and the corresponding approximation to π is $n \sin \pi/n$ which is roughly $\pi[1 - (6n^2)^{-1}]$.

1.4. Let us next consider the possibility of improving the solution by using two terms on the right-hand side of (2). As previous experience has taught us the benefit of taking two further differences into account, it would seem plausible to expect a marked improvement. In fact, however, if we take $h = 0.1$ and work to ten places, using the natural boundary conditions, the solution of the corresponding difference equation $(\delta^2 - \frac{1}{12}\delta^4)y = -0.01y$, i.e.,

$$y(n+4) = 16y(n+3) - 29.88y(n+2) + 16y(n+1) - y(n) \tag{6}$$

rapidly diverges to $+\infty$, as is seen in column (6). The same behavior occurs if 9, 8, 7, or 6 places are used, but if 5 places are used, it will be found [see column (7)] that the solution rapidly tends to $-\infty$.

1.5. If use is made of the equation obtained by neglecting all but the first two terms on the right-hand side of (2) and substituting in (1), we find

$$(\delta^2 - \tfrac{1}{12}\delta^4)y = -h^2 y. \tag{8}$$

If the term $\delta^4 y$ on its left is replaced by its approximate value

$$-h^2\delta^2 y \sim \delta^2(\delta^2 y), \tag{9}$$

a three-term relation is obtained

$$(12 + h^2)\delta^2 y = -12h^2 y. \tag{10}$$

Using $h = 0.1$, the following equation replaces (6)

$$y(n+2) = 1.99000\,83333y(n+1) - y(n). \tag{11}$$

TABLE I

x	(1)	(5)	(6)	(7)	(11)
0	0.00000 00000	0.00000	0.00000 00000	0.00000	0.00000 00000
0.1	0.09983 34166	0.09983	0.09983 34166	0.09983	0.09983 34166
0.2	0.19866 93308	0.19866	0.19866 93308	0.19867	0.19866 93310
0.3	0.29552 02067	0.29550	0.29552 02067	0.29552	0.29552 02077
0.4	0.38941 83423	0.38939	0.38941 83685	0.38934	0.38941 83450
0.5	0.47942 55386	0.47939	0.47942 59960	0.47819	0.47942 55440
0.6	0.56464 24734	0.56460	0.56464 90616	0.54721	0.56464 24828
0.7	0.64421 76872	0.64416	0.64430 99144	0.40096	0.64421 77021
0.8	0.71735 60909	0.71728	0.71864 22373	−2.67357	0.71735 61128
0.9	0.78332 69096	0.78323	0.80125 45441		0.78332 69403
1.0	0.84147 09848	0.84135	1.09135 22239		0.84147 10261
1.1	0.89120 73601	0.89106	4.37411 56871		0.89120 74139
1.2	0.93203 90860	0.93186			0.93203 91543
1.3	0.96355 81854	0.96334			0.96355 82701
1.4	0.98544 97300	0.98519			0.98544 98328
1.5	0.99749 49866	0.99719			0.99749 51092
1.6	0.99957 36030	0.99922			0.99957 37469

The solution of this equation, using the natural boundary conditions, is shown in column (11) of the table. The error is positive and steadily increases to the value 1439×10^{-10} at $x = 1.6$. The device used here is well known and is the basis of the NUMEROV-MILNE method for the solution of second order differential equations.

1.6. The reason for the surprising results mentioned in **1.4** is clear; they are essentially due to the large coefficient 16 of $y(n + 3)$ in (6). Assuming that $y(0)$, $y(1)$, $y(2)$, and $y(3)$ are correct, apart from rounding off errors, we may expect an error in $y(4)$ due either to the rounding off of the given values or to the truncation error caused by neglect of the terms involving the sixth and higher differences in (2) or to both these causes. These errors can easily be estimated. The first error cannot exceed $(16 + 29.88 + 16 + 1)\cdot\frac{1}{2}\cdot 10^{-r}$ which is about $3\cdot 10^{-r+1}$ when one is working to r decimal places. The second error may be estimated as

$$- 12\cdot\tfrac{1}{90}\delta^6 y - 1.\dot{3}\times 10^{-7} y$$

which is about $2.\dot{6}\times 10^{-8}$, when $x = 0.2$. When 10 decimal places are used, the truncation error is the dominant one. Examination of the rounding off errors caused by carrying out $\sin x$ (for $x = 0.1$, 0.2, and 0.3) to 9, 8, 7, or 6 places shows that the resultant error in $y(4)$ is positive, but if 5 places are used, it is negative and greater in magnitude than the truncation error.

It is this initial error which determines the ultimate behavior of $y(n)$. The error in $y(5)$ is roughly 16 times that occurring in $y(4)$, as the errors in $y(3)$, $y(2)$, $y(1)$ are negligible compared with that in $y(4)$; the error in $y(6)$ is roughly 16 times that occurring in $y(5)$, for similar reasons; and so on. This exponential increase in the error serves as the explanation of the observed divergences. It is important to note that the trouble has been caused not by an accumulation of rounding off errors but rather by a single error (caused either by rounding off or truncation) and an unfortunate choice of formula.

Let us examine this more precisely. The solution of the difference equation (6) is of the form

$$y(n) = A_1\alpha_1^n + A_2\alpha_2^n + A_3\alpha_3^n + A_4\alpha_4^n$$

where α_1, α_2, α_3, α_4 are the roots of the equation

$$x^4 - 16x^3 + 29.88x^2 - 16x + 1 = 0.$$

This equation has two real roots α_1, α_2 and two complex roots α_3, α_4. Since it is a reciprocal equation with real coefficients, the following relations must hold

$$\alpha_1\alpha_2 = 1, \quad \alpha_3\alpha_4 = 1, \quad \alpha_3 = \bar{\alpha}_4.$$

The solution is therefore of the form

$$y(n) = A\alpha^n + B\alpha^{-n} + C\cos n\theta + D\sin n\theta,$$

where $\alpha = \alpha_1$, and $\theta = \arg \alpha_3$. The following approximate values are found

$$\alpha = 13.94, \qquad \theta = 0.1000.$$

It is clear that, if $A \neq 0$, then the term $A\alpha^n$ will dominate the others in the solution.

It will be found that the ratio of successive errors in column (6) approaches 13.94 very rapidly.

1.7. The solution of the difference equation (10) is

$$y(n) = A\cos n\theta + B\sin n\theta$$

where, for general h, θ is defined by

$$\cos\theta = 1 - [6h^2/(12 + h^2)].$$

We are interested in the pure sine solution. A small error will introduce a component $A \cos n\theta$ which will remain small. Significant results are to be expected in this case although there will, in general, be an error caused by the accumulation of the rounding off error or by the truncation of the formula (2).

Some idea of the magnitude of the first kind of error can be obtained by consideration of the difference equation arising from (10) with $h = 0$,

$$y(n + 2) = 2y(n + 1) - y(n).$$

There is no rounding off here, since the coefficients are integral. For this equation, we have

$$y(n) = (-n + 1)y(0) + ny(1) + (n - 1)y(2) + (n - 2)y(3) + \cdots + 2y(n - 1).$$

If we denote by $e(n)$ the error committed in rounding off the right-hand side of the last equation, then the total error in $y(n)$ due to rounding off is

$$(-n + 1)e(0) + ne(1) + (n - 1)e(2) + (n - 2)e(3) + \cdots + 2e(n - 1) + e(n).$$

Assuming that the $e(r)$'s, where $r = 0, 1, 2, 3, \cdots n$, are independent and have a rectangular distribution, then, for large values of n, the distribution of the total error in $y(n)$ is approximately normal with zero mean and variance

$$\sigma^2 = \tfrac{1}{12}\{(n-1)^2 + n^2 + (n-1)^2 + (n-2)^2 + (n-3)^2 + \cdots + 2^2 + 1^2\} \simeq n^3/36.$$

The probable total error is thus, in units of the last decimal,

$$0.6745\sigma \simeq 0.6745n^{3/2}/6, \text{ i.e., about } 0.1n^{3/2}.$$

The maximum error of this kind cannot exceed

$$\tfrac{1}{2}\{(n-1) + n + (n-1) + (n-2) + (n-3) + \cdots + 2 + 1\} \simeq \tfrac{1}{4}n^2$$

units of the last decimal.

Some idea of the magnitude of the truncation error is obtained by noticing that, on the assumption that no rounding off errors are committed, the solution obtained is

$$y(n) = \sin n\theta = \sin nh[1 + (h^4/480) + O(h^6)]$$

instead of $y(n) = \sin nh$.

The main source of error in column (11) is due to truncation while that in column (5) is due to rounding off.

1.8. The solution of a differential equation of the form

$$y'' = -k^2 y,$$

where k is a constant, can be discussed in exactly the same way as we have dealt with the case where $k = 1$. Similar considerations will apply to

$$y'' + I(x)y = 0,$$

in the oscillatory regions, i.e., where $I(x)$ is positive. In the exponential regions, where $I(x)$ is negative, there will be solutions which diverge and solutions which converge. Contamination of a divergent solution with a small component of a convergent will, in general, cause no serious trouble, but the reverse effect must be avoided, e.g., by working backwards so that the recurrence relations are used in the form

$$y(n) = a_1 y(n+1) + a_2 y(n+2) + a_3 y(n+3) + \cdots + a_r y(n+r).$$

The behavior of the solutions in the transition case near (simple) zeros of $I(x)$ will, in general, be on the pattern of those of

$$y'' = xy$$

which are the Airy Integrals, $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$.[3]

1.9. Recently L. Fox and E. T. Goodwin [4] suggested the following type of method for the practical solution of ordinary differential equations with one-point boundary conditions: Use a recurrence relation of the form of (5) or (11) to obtain an approximation to the solution. Difference this solution and use the differences in the untruncated formula to correct the solution at each stage of the recursion. Difference again and correct again. Repeat until satisfactory solutions are obtained. The process appears likely to be of considerable use, but when using it, as was done in this instance, care must be taken in the choice of the recurrence relation used.

1.10. Similar phenomena for partial differential equations have been investigated by L. COLLATZ,[5] who indicates some conditions under which recurrence relations can be useful in the solution of such problems.

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[1] See, e.g., L. J. COMRIE, *Chamber's Six-figure Mathematical Tables*, London, v. 2, 1949, p. 545–549 and *Interpolation and Allied Tables*, H. M. Stationery Office, London, 1947, p. 942–943.

[2] See, e.g., W. E. MILNE, *Numerical Calculus*, Princeton Univ. Press, 1949, p. 192.

[3] See, e.g., J. C. P. Miller, *The Airy Integral*, Brit. Assn. Math. Tables, Part vol. B, Cambridge Univ. Press, 1946.

[4] L. Fox & E. T. GOODWIN, "Some new methods for the numerical integration of ordinary differential equations," Cambridge Phil. Soc., *Proc.*, v. 45, 1949, p. 373–388.

[5] L. COLLATZ, Über das Differenzenverfahren bei Anfangswertproblemen partieller Differentialgleichungen," *ZAMM*, v. 16, 1936, p. 239–247.

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6. HERMAN H. GOLDSTINE and JOHN VON NEUMANN, *Planning and Coding of Problems for an Electronic Computing Instrument*, Institute for Advanced Study, Princeton, N. J., part II, v. 2, 1948, 68 pages, 7 figs. 21.6 × 27.9 cm.

The second volume on the planning and coding of problems for an electronic computer illustrates the preparation of several problems for the proposed Institute for Advanced Study electronic computing machine. The many steps, from the initial statement of the problem and its mathematical foundation through the final coding of the problem in machine language, are given in complete detail.

The first volume of this report [see *MTAC*, v. 3, p. 54] presented the fundamental information and explanations necessary to the understanding of the publication now under review, i.e., the instruction code of the IAS computer, the preparation of flow diagrams, and several examples of the coding of some basic arithmetic operations. The problems encountered in the second volume are numerical integration, interpolation schemes, sorting, and collating.

Automatically-sequenced computing machines are most efficiently utilized when the problem to be computed can be reduced to an iterative process. The report gives rigorous treatment to the reduction of problems to such form, which lends itself to easy translation into machine language. There