to invert B is n divisions of a vector by p_j. When $v_{ij} \equiv 1$, $A = R$ and $T = B^{-1}$, and hence $\sigma_{ij}^2 = p_j^{-1}(B^{-1})_{ij}[1 - p_j(B^{-1})_{ij}]$, corresponding to the variance of the binomial distribution.

George E. Forsythe
Richard A. Leibler

NBS Institute for Numerical Analysis
Univ. of California, Los Angeles
2901, 18th St.
Washington 9, D. C.

[1] The preparation of this paper was sponsored (in part) by the Office of Naval Research.
[2] A. Kolmogoroff, *Grundbegriffe der Wahrscheinlichkeitsrechnung*, New York, 1946, p. 59. The writers are indebted to T. E. Harris for this reference.
[3] The fact that $\sigma_{ij} = \infty$ does not interfere with the convergence of the average value of N games to $(B^{-1})_{ij}$. However, conventional error estimates in terms of variances no longer apply and, in at least certain matrix inversions where $\sigma_{ij} = \infty$, the accumulated payment after N games cannot be normed so as to be asymptotically normally distributed as $N \to \infty$. See W. Feller, "Über den zentralen Grenzwertsatz der Wahrscheinlichkeitsrechnung," *Mathematische Zeitschrift*, v. 40, 1935, p. 521–559 and v. 42, 1937, p. 301–312, and "Über das Gesetz der grossen Zahlen," Szeged, *Acta Univ.*, *Acta Scient. Math.*, v. 8, 1937, p. 191–201.
[4] It is this case which we learned from von Neumann and Ulam.

Maximum-Interval Tables

Both the article by Herget & Clemence [*MTAC*, v. 1, p. 173–176] and the note by Miller [*MTAC*, v. 1, p. 334] on optimum-interval tables ignore the possibility of a continuously variable interval. It is of some interest to examine the reduction in the number of entries made possible by what might, by analogy, be termed "maximum-interval" tables. Using the principles of optimum-interval tables, with the modifications suggested by Miller, the tabular values of the argument are no longer restricted to terminating decimals so that the interval may be allowed to assume at each point the maximum value consistent with the stated allowable error.

The chief objection to a punched-card table in this form is that all (or nearly all) of the digits in the argument will have to be used in the interpolating factor. In some cases this objection could be overcome by inserting additional cards corresponding to values of the argument terminating in the appropriate number of zeros, or by splitting the whole range into a number of sub-ranges, in each of which the allowable error is varied slightly to make tabular arguments coincide with the end-points; or, of course, by an additional operation of subtraction. Generally, however, the saving in cards is not worth the additional cost of preparation and the resulting complication.

Herget in *The Computation of Orbits* [see *MTAC*, v. 3, p. 418–9] gives an optimum-interval table of $x^{-3/2}$ using quadratic interpolation, with a note "This is the first time such a table has ever been printed for use with a hand calculating machine." A human computer can easily exercise the requisite judgment to use the continuously variable intervals of a maximum-interval table; in the simplest case it will involve nothing more serious than allowing the interpolating factor occasionally to exceed unity in a particular digit. A punched-card machine can only do this at the expense of a separate operation. There may, therefore, be a use for variable interval tables in computation by desk machines.

The chief field of use is likely to be in connection with automatic digital computers, since the number of operations and the number of digits in the multipliers is often less critical than the amount of high-speed storage required. With cubic and quartic interpolation the number of entries required is very small indeed.

Before considering the theory of these tables, it is desirable carefully to examine the significance of "error" as applied to values obtained by interpolation. Corresponding to the function tabulated and the conditions of its use the error may be assessed in four ways: as an absolute or relative error in the tabulated function, or as relative to an absolute or relative error in the argument. Examples of these four cases are:

(a) $\sin x$, to a stated number of decimals; here the error is absolute.

(b) x^m, with a stated percentage error as is always required when multiplying factors are used to extend tables for a limited range of the argument; here the error is relative to the tabulated function.

(c) $\sin x$, with an accuracy corresponding to a stated absolute error (e.g., $0°.0001$ or $1''$) in the argument.

(d) x^m, with an accuracy corresponding to a stated relative error (say 1 in 10^7) in the argument.

Each of these cases may occur and it is essential that the appropriate one should be adopted if the utmost economy in number of entries is to be achieved. Errors arise from several sources, but all except those due to approximations made in the interpolating formula can be substantially and effectively reduced by the retention of an additional, or guarding, figure; it will be assumed that this is always done in the type of table to be discussed, so that the theoretical errors of approximation may be directly equated with the allowable, or stated, errors.

The TAYLOR series for a function y of x at $x = a + t$ may be replaced, in the range $-\frac{1}{2}h \leqslant t \leqslant +\frac{1}{2}h$, by a series in terms of CHEBYSHEV polynomials:[1]

$$(1)\quad y(a+t) = \alpha_0 + \alpha_1 C_1(4t/h) + \alpha_2 C_2(4t/h) + \cdots + \alpha_p C_p(4t/h) + \cdots$$

where $C_p(\xi) = 2\cos(p \arccos \frac{1}{2}\xi)$ is the Chebyshev polynomial of degree p and where

$$(2)\quad \alpha_p = \frac{(\frac{1}{4}h)^p}{p!}\left\{ y^{(p)} + \frac{(\frac{1}{4}h)^2}{1!(p+1)} y^{(p+1)} + \frac{(\frac{1}{4}h)^4}{2!(p+1)(p+2)} y^{(p+2)} + \cdots \right\}$$

in which $y^{(p)}$ is the p^{th} derivative of y at $x = a$.

It is known that the truncation of (1) after the n^{th} term will provide the most efficient polynomial approximation of degree $(n-1)$ to the function $y(a+t)$ in the range $-\frac{1}{2}h \leqslant t \leqslant +\frac{1}{2}h$. The leading term of the maximum error of the approximation is clearly $2\alpha_n$, which may for the present purpose be simplified to:

$$(3)\qquad \frac{h^n}{n!2^{2n-1}} y^{(n)}.$$

This may now be equated to the allowable error to give the maximum permissible value of h as a function of x. If ϵ_1, ϵ_2, ϵ_3 and ϵ_4 are respectively the maximum absolute and relative errors allowable in y and x, then (3) must be equated with:

(a) ϵ_1 (b) $\epsilon_2 y$ (c) $\epsilon_3 y'$ (d) $\epsilon_4 x y'$

where the signs of the ϵ's are chosen so that h^n is positive.

Let z be an auxiliary variable which takes integer values corresponding to the values of x at which y is to be tabulated. Then, to a very good approximation, h may be identified with dx/dz so that we obtain the following expression for z in terms of x:

$$z = \int^x h^{-1}dx. \tag{4}$$

This will give z as a function of x, and in due course x, y and its derivatives as functions of z from which the table can be prepared.

For a simple power of x,

$$y = x^m \text{ (} m \text{ not a positive integer);}$$

substitution in and integration of (4) with the condition that $z = 0$ at $x = 1$, gives:

$$(5) \qquad \begin{aligned} &\text{(a)} \quad z = \epsilon_1^{-1/n}(n/m)b(m, n)(1 - x^{m/n}) \\ &\text{(b)} \quad z = \epsilon_2^{-1/n}b(m, n) \ln x \\ &\text{(c)} \quad z = (m\epsilon_3)^{-1/n}nb(m, n)(1 - x^{1/n}) \\ &\text{(d)} \quad z = (m\epsilon_4)^{-1/n}b(m, n) \ln x \end{aligned}$$

where

$$\pm\, b(m, n) = [m(m - 1) \cdots (m - n + 1)2^{1-2n}/n!]^{1/n},$$

the sign being taken to make z positive. Cases (b) and (d) are, of course, identical in form. These expressions enable the number of entries for a particular range of x to be determined at once.

For example, in the seven-decimal table of reciprocals considered by Herget & Clemence the value of ϵ_1 is taken as $5 \cdot 10^{-7}$. A more realistic arrangement would arise from specifying maximum relative errors; a comparable table should therefore be based on $\epsilon_2 = \epsilon_4 = 5 \cdot 10^{-13/2}$, the geometric mean of the relative errors at beginning and end. Case (c), of a stated maximum absolute error in x, may well arise in practical computation; ϵ_3 can be taken as $5 \cdot 10^{-7}$. The corresponding expressions for z are:

$$\begin{aligned} &\text{(a)} \quad z = \tfrac{1}{4}n(4 \cdot 10^6)^{1/n}(1 - x^{-1/n}) \\ &\text{(b), (d)} \quad z = \tfrac{1}{4}(4 \cdot 10^{11/2})^{1/n} \ln x \\ &\text{(c)} \quad z = \tfrac{1}{4}n(4 \cdot 10^6)^{1/n}(x^{1/n} - 1) \end{aligned}$$

The number of entries required in a table to cover the range $x = 1$ to 10, with $n = 2, 3, 4$ (corresponding to linear, quadratic and cubic interpolation), are:

	$n = 2$	$n = 3$	$n = 4$
(a)	685	65	21
(b), (d)	648	64	21
(c)	2163	139	36

The reduction to 685, for case (a), as compared with the 1368 of Herget & Clemence and the 924 of Miller is substantial. There is here little difference between the requirements of cases (a), (b), and (d), though (b) is certainly the more realistic table. The very considerable reduction effected by the inclusion of a quadratic term would suggest that it might well be worth while introducing the additional step into the interpolation.

As a second example, consider a table of $x^{-3/2}$ with a maximum relative error of 10^{-7} which corresponds approximately to that of Herget's table. The number of entries (case (b)) required are:

$n = 2$ (linear) 3526.
$n = 3$ (quadratic) 204, as compared with Herget's 316.
$n = 4$ (cubic) 50.

The coefficients of the interpolating polynomial of degree $(n - 1)$ are obtained by expanding the individual terms of the truncated series (1) and rearranging as a power series in t. This series is then appropriate to the range $-\frac{1}{2}h \leqslant t \leqslant \frac{1}{2}h$. In order to facilitate computation it is desirable still further to transform the series as a power series in s where

$$s = (a + t) - x_0$$

and x_0 is a value of x terminating with several zero digits. It is only necessary to retain sufficient figures in the coefficients of powers of s to cover the range appropriate to t, provided they are rigorously consistent with the coefficients in the t-series; they should be calculated one at a time starting with that of s^{n-1}, which is of course the same as that of t^{n-1}.

It will be noted that for $y = x^m$, y in case (a) and x in case (c) are both polynomials in z of degree $n - 1$. In cases (b) and (d) both x and y are of exponential form; since the coefficient of $\ln x$ in (5) does not change rapidly for small changes in m, it would be possible to use the same series of values of x (or of y) for a number of fractional powers.

In his note Miller referred to the previously accepted convention of not modifying function values to reduce the error due to neglect of second differences. Actually such modification (for a slightly different reason) is in use in both the British and American surface and air almanacs; but in many cases there are sound reasons why the tabular values should be more accurate than the interpolates. It is possible to reduce considerably the error due to neglect of second differences without modifying the function values: in principle, the first difference is modified so that the error at the end of the range is numerically equal, with opposite sign, to the maximum error in the range. Specifically $\frac{1}{4}(3 - 2\sqrt{2}) = 0.043$ of the double second difference is subtracted from the first difference; the error is then -0.043 of the double second difference at the end of the interval and $+0.043$ at a point 0.414 along the interval. This error compares with 0.0625 without modification and 0.03125 with modification of the function. The device has obvious disadvantages, but it has some valuable applications—mainly in navigational tables.

D. H. Sadler

Royal Greenwich Observatory
Herstmonceux Castle, Sussex

[1] This development is taken, with only slight change of notation, from Miller's paper "Two numerical applications of Chebyshev polynomials," R. Soc. Edinburgh *Proc.* Sec. A, v. 62, 1946, p. 204–210. It is, of course, simple to obtain the optimum expansions for small values of n by direct methods.