142[L].—K. Higa, *Table of* $\int_0^\infty u^{-1} \exp \{ -(\lambda u + u^{-2})\} du$. One page typewritten manuscript. Deposited in the UMT File.

The table is for $\lambda = .01, .012(.004).2(.1)1(.5)10$. The values are given to 3S.

L. A. Aroian

Hughes Aircraft Co.
Culver City, California

143[L].—Y. L. Luke. *Tables of an Incomplete Bessel Function.* 13 pages photostat of manuscript tables. Deposited in the UMT File.

The tables refer to the function

$$j_n(\mu,\theta) = \int_0^\theta \exp \{i\mu \cos \phi\} \cos n\phi d\phi.$$

Values are given to 9D for

$$\begin{aligned} n &= 0, 1, 2 \\ \cos \theta &= -.2(.1).9 \\ &= 49\omega/51, \omega = 0(.04).52. \end{aligned}$$

There are also auxiliary tables. The tables are intended to be applied to aerodynamic flutter calculations with Mach number .7.

Y. L. Luke

Midwest Research Institute
Kansas City, Missouri

AUTOMATIC COMPUTING MACHINERY

Edited by the Staff of the Machine Development Laboratory of the National Bureau of Standards. Correspondence regarding the Section should be directed to Dr. E. W. Cannon, 415 South Building, National Bureau of Standards, Washington 25, D. C.

Technical Developments

Fundamental Concepts of the Digital Differential Analyzer Method of Computation

Introduction.—Two fundamentally different approaches have been developed in using machines as aids to calculating. These have come to be known as analog and digital approaches. There have been many definitions given for the two systems but the most common ones differentiate between the use of physical quantities and numbers to perform the required automatic calculations.

In solving problems where addition, subtraction, division and multiplication are clearly indicated by the numerical nature of the problem and the data, a digital machine for computation is appropriate.

When problems have involved calculus methods, as, for instance, in the solution of differential equations, the analog computer has often been used, as the process of integration seems, psychologically at least, to be more aptly handled by analog devices. These devices have been mechanical or electronic integrators. However, the actual process of integration, if one considers the numerical basis for its origin is, in a sense, a numerical additive process. Thus, digital computers "integrate" by successive additions.

By bridging the gap between these two approaches, a new series of instruments for computation is possible.

The method of computation used in a digital differential analyzer resulted from the adoption of a new point of view. Considering the operations involved in the solution of differential equations, it is possible, with this new approach, to obtain many of the advantages of a digital computer and also the essential advantages of an analog differential analyzer. The result is a different type of digital "logic" from that used in the general purpose digital computers.

The advantages gained by the new method in solving ordinary differential equations of any type are:

1. Ease of preparing problems—arising from the use of analog differential analyzer methods instead of numerical methods in the coding process.
2. Increase in computation speed over equivalent general purpose digital computer approaches and equality in speed to some analog methods.
3. Increase in accuracy over analog differential-analyzer procedures.
4. Repeatability and ease of error analysis inherent in the digital method.
5. Small size—In certain embodiments the digital differential analyzer can be much smaller, have fewer tubes and components, weigh and cost less than analog differential analyzers or any of the general purpose digital computers. This is particularly true when the number of integrators needed to solve the equations becomes large.

Review of Analog Differential Analyzer Theory.—There are two different ways of explaining the digital differential analyzer method. The first is a qualitative explanation which follows the analog viewpoint and points out the first advantage. The second is a quantitative numerical explanation which shows the error analysis possibilities and the successive-additions method of integration which actually takes place.

To appreciate the first explanation, it is necessary to review the principles of the analog differential analyzer. In solving an equation such as,

$$\frac{d^2w}{dt^2} - w\frac{dw}{dt} - wt = 0 \tag{1}$$

the analog differential analyzer represents the variables w, t, $\frac{dw}{dt}$ etc., by mechanical rotations of shafts or by variations of voltages in electronic circuits. The rates of shaft rotations or of changes in voltages are always proportional to the rates of change of the variables.

Integration is accomplished by a mechanical wheel and disc integrator or an operational amplifier used as an integrator. Other mathematical operations such as multiplication and addition of variables in the equation are performed either by integrators or other devices, mechanical and electronic.

Integrators and other units are interconnected in such a manner as to produce an analog of the differential equation. The set is "driven" by a single shaft or voltage representing the independent variable (t of Equation 1). The w or dependent variable shaft or voltage varies in accordance with the actual solution to the equation as t varies. For a given set of initial conditions, a solution to Equation 1, $w = f(t)$, is produced as either a graph or a set of tabular values of w as a function of t.

The integrator is the key to the machine's operation, all other units being straightforward in comparison. It may be regarded as a "black box" with two inputs and one output shown schematically in Figure 1.

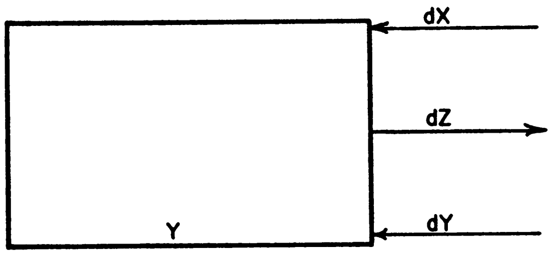


FIG. 1.

The inputs[1] dx and dy are the rates of change of some x and y variables in an equation as represented physically by shafts rotating or voltages changing. The differential notation is used because the same dt is inherently used throughout the machine. The inputs and outputs are related by the integrator Equation 2.

$$dz = KYdx, \tag{2}$$

where $Y = S\,dy$. The constant K is determined by the physical properties of the integrator.

In the mechanical integrator the dy input causes a worm gear to move a small disc across the surface of a large wheel (see Figure 2) such that its distance from the center of the wheel is Y. The dx input turns the wheel and friction causes the disc to rotate at a speed dz.

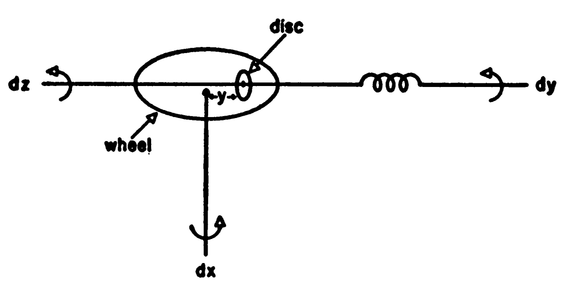


FIG. 2.

In the electronic integrator the dy input is a varying voltage, the dx input is always time and the z voltage is produced by using the integrating characteristics of capacitors in connection with a feedback amplifier to produce linearity. It should be noted that to interconnect integrators it is necessary that the inputs and outputs all be of the same form.

Qualitative Explanation of the Digital Integrator.—The digital integrator is the heart of the new type of computer, the digital differential analyzer, and may be visualized as a black box with the same schematic (Figure 1) and the same equation relating its inputs and output (Equation 2). The dx, dy, and dz variables are represented by pulse rates, i.e., the rates of occurrence of streams of electronic pulses entering or leaving the integrator. As stated before the equation relating these pulse rates is still Equation 2, and the inputs and output are of the same form.

Without describing the nature of such an integrator, it can be seen that the two properties above will allow these black boxes to be intercoupled, in the same manner as were the analog differential analyzer integrators, to

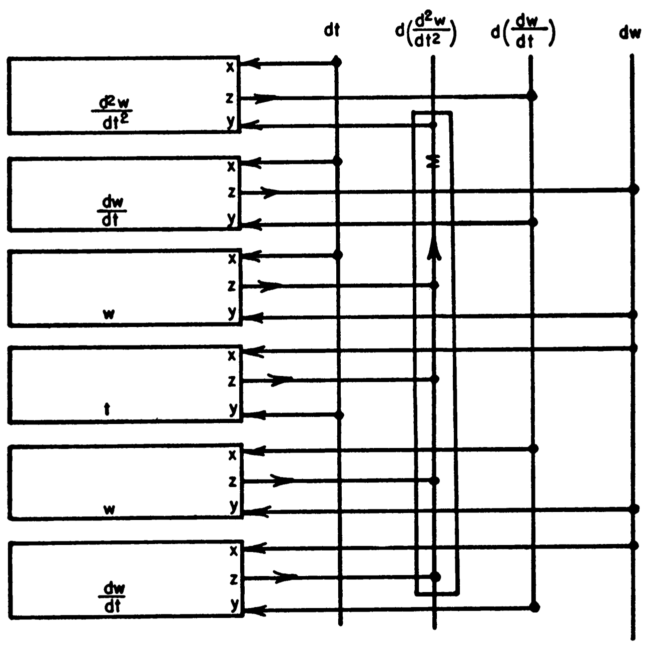


FIG. 3.

solve ordinary differential equations. The same techniques of intercoupling integrators to perform various operations such as multiplication, scaling function generation, division, etc., can be used. A set of digital integrators intercoupled by wires carrying pulse streams can be "driven" by a pulse source representing the independent variable. A solution is produced as a set of tabular values and a graph can be produced. The same schematic or connection diagram can be used for both the digital and the analog differential analyzers. Only the K in Equation 2 changes. Such a schematic is illustrated in Figure 3 for the solution of Equation 1.

Two distinct advantages of the digital over the analog integrator in addition to increased accuracy should be noted at this point. It will be observed that Figure 3 contains no adders. The terms

$$w\,dt,\ t\,dw,\ \left(\frac{dw}{dt}\right) dw, \text{ and } w\,d\left(\frac{dw}{dt}\right)$$

created by the lower four integrators would in the analog machines have to be added in extra units, known as adders, to form $d\left(w\frac{dw}{dt} + w\,t\right) = d\left(\frac{d^2w}{dt^2}\right)$ to be fed back into the "Y" input of the upper integrator. The addition is indicated on the diagram by a box with $\sum$ sign.

Since the outputs of these integrators in the digital case are pulse streams, they may be mixed together directly and sent into the same input, provided, of course, that the pulses do not coincide in time. In several embodiments of the digital differential analyzer this is the case. If time coincidence does occur, it is only necessary to delay one pulse with respect to another.

The other advantage is the superior ability of the digital integrator to receive the output of another integrator at its "dx" input. This greatly facilitates multiplication, division and the solution of non-linear equations without the use of special devices.

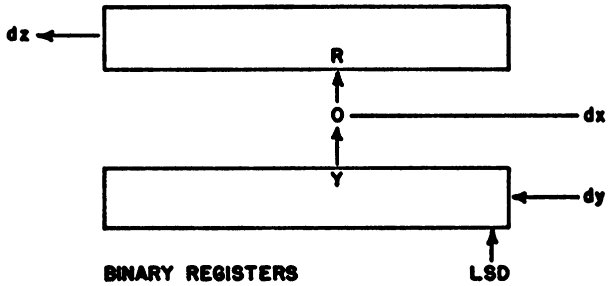


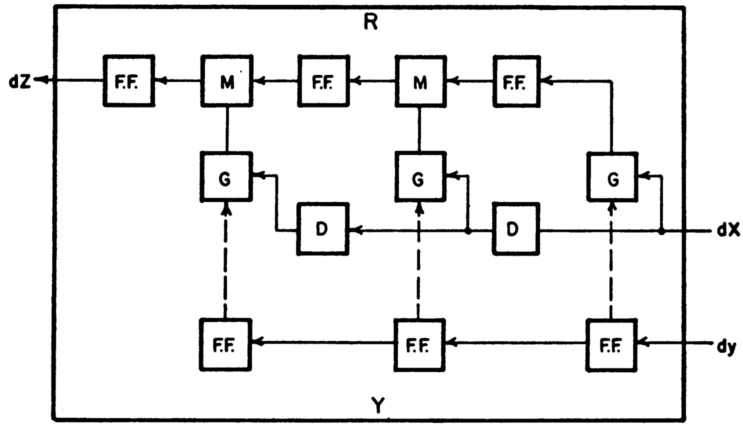
FIG. 4.

Embodiments of the Digital Integrator.—The "contents" of the black box digital integrator may take many physical forms and still have the external properties described. All of the forms so far devised have one common property. Two numbers appear within the box and may be designated as a coupled pair. These two numbers, always labelled Y and R, may appear in any one of several physical forms. Two specific examples are: Numbers stored in vacuum tube registers made up of two-stable-state devices, and numbers appearing in pulse form on a cathode ray tube screen. The numbers may be of any length and in any number base system. For convenience they will usually be represented in this paper schematically as appearing in two registers as binary numbers. (See Figure 4.) Some other methods of storing the Y and R numbers are: relays, mechanical registers as on desk calculators, magnetic tapes or drums, and mercury or other delay lines. Each of the storage media must be capable of changing the numbers digitally by the receipt of information at the inputs to the box.

In the integrator diagram in Figure 4, the Y register acts as a counter when receiving dy pulses and in a sense integrates the dy pulse rate to produce the number Y. The dx pulses are treated as instructions to transfer in an additive manner the number Y into the R register without removing Y from the Y register. If the R register contained some previous R before the transfer, it contains $R + Y$ after the transfer.

The R register will of course overflow after a certain number of transfers. Each time it does so, a pulse is transmitted from the integrator as a dz pulse. The Y and R registers have the same length and capacity and, if binary registers are used, the capacity is 2^N, where N is the number of binary stages in each register.

By qualitative analysis of the relations between the variables, it may be seen that Equation 2, $dz = KY\,dx$, does hold for the integrator, where $K = 1/2^N$ and Y is regarded as an integer, provided that Y remains constant. In other words if $Y = 1$ the output rate dz will be $1/2^N\,dx$, since it requires



Transfer method.

FIG. 5.

2^N additions of Y to R to cause an overflow. If $Y = 2^N$, or the register is filled to capacity, an R overflow or dz pulse will occur for every dx pulse. In this case, $dz = dx$. In general, dz is certainly proportional to dx for a constant Y and is proportional to $Y/2^N$ for constant dx.

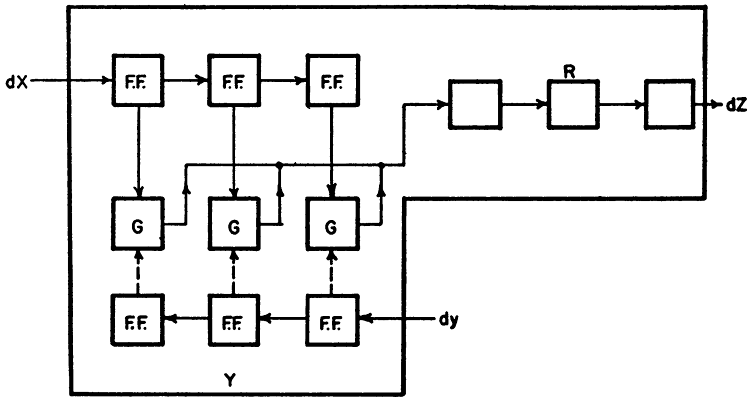
Two fundamentally different ways exist (as well as combinations of the two) to cause the transfer of Y to R to take place. In the one described above, called the transfer method, a single pulse at the dx input caused the entire Y number to be added into R. In the second system a large number of dx pulses are required to transfer Y to R. The Y number may change during this transfer process so that the number of stages required in the register for the same accuracy is larger. This method is called the sieve method.

Figures 5 and 6 illustrate electronic methods of causing the two types of transfer. There are, of course, many other electronic ways of effecting the transfer.

In Figure 5 the additive transfer of Y into R by a single dx pulse is accomplished by transmitting the dx pulse into successive gates and delays. The gates are controlled (dashed lines) by the Y register flip-flops, or two-state devices. If a flip-flop is in its "1" state (binary digit one at that digit position), its gate is "open" and the dx pulse passes up to one of the R register flip-flops causing it to trigger. If the Y flip-flop is in its "0" state (binary digit zero), the gate is closed and the pulse does not pass through.

If an R flip-flop triggers from "1" to "0" it transmits a "carry" pulse to the next flip-flop to the left. The dx pulse in the meantime is being delayed through D, and if it passes the next gate it will arrive at the pulse mixer M non-coincident with the carry pulse from the R flip-flop. Any carries from the left R flip-flop represent overflow pulses and are transmitted to the dz output.

The sieve method of Figure 6 requires 2^N dx pulses to transfer Y into R. A third register is used to distribute the dx pulses through the gates controlled by the Y flip-flops in such a manner that when the outputs from each gate are mixed they are non-coincident and can be accumulated in R. The entire operation resembles the action of sieving the dx pulses through the gates.



Sieve method.

FIG. 6.

Carry pulses are taken from the third register flip-flops at different times to feed to the gates and to trigger the next flip-flop to the right. This means that each set of pulses reaching the gates as the operation proceeds from left to right is anticoincident with all preceding sets and equal to half the adjacent left hand set. The effect of 2^N (2^3 in case shown) dx pulses for constant Y will be to transfer Y into R.

When other storage methods are used, the electronic operations change. For instance with magnetic drum storage, the two numbers are stored in two parallel channels, and the digits of Y and R appear at magnetic read heads one digit at a time in serial fashion. The addition of Y to R is then that of time-serial binary addition. The same thing would be true of any serial or delay type of number storage.

Quantitative Explanation of the Digital Integrator.—The second or quantitative explanation of the digital differential analyzer method will be covered rigorously in another paper. Briefly, the successive addition process of multiplying an ordinate of a curve, $Y \equiv f(x)$ (see Figure 7), by a Δx

increment and adding the resulting products to get the area under a curve is really being carried out in an integrator.

If the R register were of unlimited length and each dx pulse were assumed to have a value of 1, then the successive additions of Y to R would produce a number similar to the sum of the area of the rectangles under the curve of Figure 7 (assuming the Y values to be correct). Since the R register is broken off and dz pulses transmitted, it can be seen the sum of these dz pulses will be in error from the rectangular areas by the remainder R in the R register. The total error consists of this roundoff error plus the truncation error difference between the true curve and the rectangles. Automatic corrections of various electronic types can be and have been made for both of these errors. In the future mathematical paper it will be demonstrated that the total truncation and roundoff error for many equations will not exceed the two least significant binary digits of a Y register.

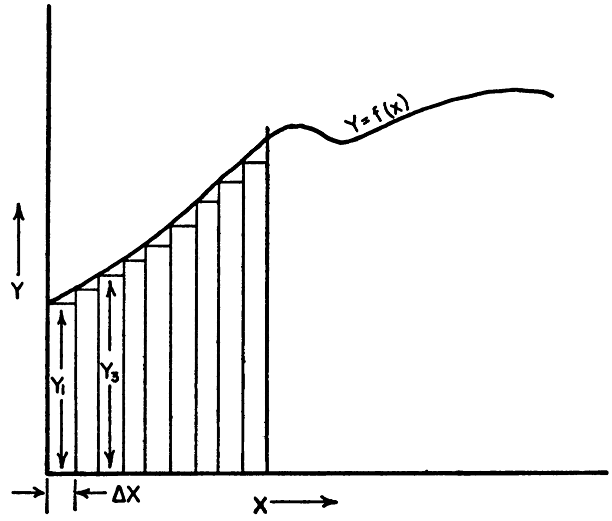


FIG. 7.

Further Advantages of the Digital Integrator.—In the foregoing discussion "black boxes" comprising digital integrators which have two fundamental properties were described. Ordinarily, to obtain 50 "black boxes" it would be necessary to use 50 times the equipment required for one box. This is certainly true in case of the mechanical and electronic integrators. However, where the integrators consist merely of paired numbers operating on each other in accordance with the methods already described, it becomes possible to time share operational circuits among all of the number pairs if they are stored in a serial or delay type of memory and the integrators are strung out in a line timewise.

One set of electronic circuits can operate on all integrators in sequence, or rather on all paired numbers in sequence. An integrator, as such, does not really exist when such a system is used. In a scheme like this it is easily seen that no coincidence problems exist, since no two integrator outputs occur simultaneously. It will also be seen that the amount of equipment does not increase linearly with the number of integrators and that beyond a certain point the digital differential analyzer is smaller and involves less components than the analog.machine.

The problems of handling the signs of the variables and their derivatives and of the scales or scale factors for a problem will also be covered in detail in a future paper. They are similiar to analog sign and scale problems except that special provisions must be made for handling signs, and scaling takes place digitally.

The writer wishes to express thanks to the following men who furnished the ideas for much of the material this article covered: D. E. Eckdahl, H. H. Sarkissian, I. S. Reed, C. Isborn, W. Dobbins, F. G. Steele, B. T. Wilson, J. Donan, J. Matlago, and A. E. Wolfe.

R. E. Sprague

Computer Research Corporation
Torrence, California

[1] V. Bush and others use the x, y, and $z = y\,dx$ notation for inputs and outputs.

Bibliography Z–XVIII

1. Anon., "Computing machines," *Mechanical Engineering*, v. 73, Apr. 1951, p. 325–327.

The MADDIDA (Magnetic Drum Digital Differential Analyzer), a new small electronic computer built by Northrup Aircraft, Inc., is briefly described, including some of the main specifications and the uses for this type of computer. The machine is capable of solving many types of differential equations or sets of such equations. The machine is being manufactured for general use in science and industry, at a relatively low cost. It is a 29 binary digit machine and adds binary digits at a rate of 100,000 per second. A big advantage of the machine is that differential equations can be solved without reducing them first to difference equations.

A survey of the Federal Computer Program is also included in the article and a list of many of the analog and digital computers either being used or being constructed at various institutions in the United States.

Donald Larson

NBSCL

2. Anon., "High-speed analog to digital converter," *Review Sci. Instr.*, v. 22, July 1951, p. 544.

Expository article.

3. S. Gill, "The diagnosis of mistakes in programmes on the EDSAC," R. Soc., London, *Proc.*, v. 206A, 1951, p. 538–554.

This paper discusses in detail two methods which have been used to diagnose errors in programming on the EDSAC. The "Blocking Order" routine prints the contents of any given location whenever the blocking order is obeyed, and then returns the control to the main routine immediately after the blocking order. This is useful in investigating arithmetical failures.