In this example, the time required per iteration was about 15 minutes for the case $n = 20$. For the larger values of n, the time required was only slightly longer. Since convergence was fairly rapid, a good approximation to the solution $\phi(y)$ could be obtained in a few hours, once the initial A and C decks have been punched.

The method has been applied to about eighteen different cases of the homogeneous equation (1.1). The solutions obtained agree well with those found by Monte Carlo methods and are supported by a limited body of experimental evidence resulting from a study of the one-dimensional movements of fish.

B. A. Griffith
K. W. Smillie

Computation Centre
University of Toronto
Toronto, Ontario

Part of the work described above was sponsored by the Ontario Research Council.

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The Use of Exponential Sums in Step by Step Integration—II

[Continued from *MTAC*, v. 6, p. 63–78]

10. The error analysis for the exponential method is predicated on the convergence of the series representation of the function

$$y/\{(1 + y) \ln (1 + y)\}, \qquad y = e^{-\lambda h} - 1$$

about $y = 0$ (see equation (13)). The radius of convergence of this series is 1. For small h, $y = e^{-\lambda h} - 1$ is small and there will be convergence for all positive h less than the least $h = h_0$ such that

$$|e^{-\lambda h_0} - 1| = 1. \qquad (32)$$

Such an h_0 exists except when λ is real and non-negative, in which case $|y| < 1$ for every positive h.

For rapid convergence and efficient use of the method, it is imperative that h be much smaller than h_0. If $\lambda = \alpha + i\beta$ is not real and non-negative, there is a simple way of computing h_0 based on a table obtained from equation (32).

Equation (32) is equivalent to

$$e^{-\alpha h_0} = 2 \cos \beta h_0. \tag{33}$$

Let $u = \alpha h_0$, $v = \beta h_0$. For $\alpha \neq 0$, let $\beta/\alpha = v/u = R$. If $\alpha = 0$, $\beta h_0 = \pi/3$. This is of interest in itself. We may choose v to be in the first quadrant.

$$u = - \ln (2 \cos v). \tag{34}$$

For values of v between 0 and $\pi/2$, u may be determined, and hence, the ratio R. To use the table, determine R as the ratio β/α, $\beta > 0$. The table then gives the corresponding u and v from which h_0 may be determined either as

$$h_0 = u/\alpha \quad \text{or} \quad h_0 = v/\beta. \tag{35}$$

Table of u and R as functions of v

v	$u = -\ln (2 \cos v)$	$R = v/u$
.0	−.69315	0.00000
.1	−.68814	−.14532
.2	−.67301	−.29717
.3	−.64746	−.46335
.4	−.61092	−.65475
.5	−.56256	−.88879
.6	−.50118	− 1.19717
.7	−.42506	− 1.64682
.8	−.33176	− 2.41141
.9	−.21770	− 4.13404
1.00	−.07752	−12.89978
1.01	−.06177	−16.35005
1.02	−.04567	−22.33275
1.03	−.02921	−35.26553
1.04	−.01236	−84.11687
1.05	.00487	215.61735
1.06	.02251	47.09604
1.07	.04056	26.37861
1.08	.05905	18.28862
1.09	.07799	13.97557
1.1	.09740	11.29353
1.2	.32198	3.72698
1.3	.62549	2.07836
1.4	1.07900	1.29749
1.5	1.95564	.76701
1.51	2.10770	.71642
1.52	2.28721	.66456
1.53	2.50629	.61046
1.54	2.78737	.55249
1.55	3.17990	.48744
1.56	3.83542	.40673
1.57	6.44235	.24370

11. The calculation of the coefficients, $a_0, \cdots, a_{n-1}$, of the open method of integration may run into considerable difficulty because of the small determinant involved. In this section practical methods for computing these coefficients without great losses of accuracy are given.

The coefficients, $a_0, a_1, \cdots, a_{n-1}$, for the open method of integration are determined by equations (4). By means of the quantities $x_j = e^{-\nu_j h}$,

$y_j = x_j - 1$ and the function

$$g(y_j) = \frac{y_j}{(1 + y_j)\ln(1 + y_j)} \tag{36}$$

described in § 8, (4) may be written

$$\sum_{r=0}^{n-1} a_r x_j^r = g(y_j). \tag{37}$$

One can then evaluate the a_j's by Cramer's rule. Thus, if α is the Vandermonde determinant of the x_j's and α_t the result of replacing the $t + 1$ column by $g(y_j)$,

$$a_t = \alpha_t/\alpha. \tag{38}$$

This can be evaluated directly. However, to obtain control over the accuracy required for these coefficients, it is desirable to arrange the computation so that the loss in accuracy in computing these coefficients is a minimum. The following is believed to be a practical method for this purpose.

Let

$$g(y_j) = \sum_{k=0}^{n-1} A_k y_j^k + h_j^{(n)}. \tag{39}$$

We will manipulate the determinant α_t by its columns only and we will indicate it by a row without subscripts, i.e.,

$$\alpha_t = |1, \cdots, x^{t-1}, g(y), x^{t+1}, \cdots, x^{n-1}|. \tag{40}$$

As in the discussion of $\epsilon(\lambda h)$ following equation (24), one may replace x_j by y_j in the first t columns.

$$\alpha_t = \left|1, \cdots, y^{t-1}, \sum_{k=0}^{n-1} A_k y^k + h^{(n)}, x^{t+1}, \cdots, x^{n-1}\right|.$$

Now $x = 1 + y$ and by an obvious reduction one obtains

$$\alpha_t = \left|1, \cdots, y^{t-1}, \sum_{k=t}^{n-1} A_k y^k + h^{(n)}, y^{t+1} + (t+1)y^t, \cdots, y^{n-1} + (n-1)y^{n-2} + \cdots + \binom{n-1}{t} y^t\right|.$$

By an elementary argument, the latter columns in this determinant can be reduced so that

$$\begin{aligned}\alpha_t &= \left|1, \cdots, y^{t-1}, \sum_{k=t}^{n-1} A_k y^k + h^{(n)}, y^{t+1} + (t+1)y^t, \cdots, y^{t+k} + (-1)^{k-1}\binom{t+k}{k} y^t, \cdots\right| \\ &= \left|1, \cdots, y^{t-1}, D_t y^t + h^{(n)}, y^{t+1} + (t+1)y^t, \cdots, y^{t+k} + (-1)^{k-1}\binom{t+k}{k} y^t, \cdots\right|,\end{aligned}$$

where

$$D_t = A_t - (t+1)A_{t+1} + \binom{t+2}{2}A_{t+2} + \cdots + (-1)^{n-t-1}\binom{n-1}{t}A_{n-1}. \tag{41}$$

Thus,

$$\alpha_t = D_t\alpha + \left|1, \cdots, y^{t-1}, h^{(n)}, y^{t+1} + (t+1)y^t, \cdots, y^{t+k} + (-1)^{k-1}\binom{t+k}{k}y^t, \cdots\right|.$$

Let

$$\begin{aligned}\beta_t &= \left|1, \cdots, y^{t-1}, h^{(n)}, y^{t+1} + (t+1)y^t, \cdots, y^{t+k} + (-1)^{k-1}\binom{t+k}{k}y^t, \cdots\right| \\ &= \sum_j (-1)^{t+j+1}h_j^{(n)}[\,|1, \cdots, y^{t-1}, y^{t+1}, \cdots, y^{n-1}|_{\text{no } j \text{ row}} \\ &\quad + (t+1)\,|1, \cdots, y^{t-1}, y^t, y^{t+2}, \cdots, y^{n-1}|_{\text{no } j \text{ row}} \\ &\quad + \binom{t+2}{2}|1, \cdots, y^{t-1}, y^t, y^{t+1}, y^{t+3}, \cdots, y^{n-1}|_{\text{no } j \text{ row}} + \cdots \\ &\quad + \binom{n-1}{n-t-1}|1, \cdots, y^{n-2}|_{\text{no } j \text{ row}}].\end{aligned}$$

Now, using the notation of § 7, we have

$$\begin{aligned}\beta_t &= \sum_j (-1)^{t+j+1}h_j^{(n)}\left(\sum_{k=0}^{n-t-1}\binom{t+k}{t}V_{0,\,t+k}(y_1, \cdots, y_{j-1}, y_{j+1}, \cdots, y_n)\right) \\ &= \sum_j (-1)^{t+j+1}h_j^{(n)}\left(\sum_{k=0}^{n-t-1}\binom{t+k}{t}(-1)^{t+k}V_0\,(\text{no } y_j) \times (-1)^{t+k}p_{(n-1)-t-k}S_{0,\,n-1}\right) \\ &= \sum_j (-1)^{t+j+1}h_j^{(n)}\left(\sum_{k=0}^{n-t-1}\binom{t+k}{t}V_0\,(\text{no } y_j)p_{(n-1)-t-k}\,(\text{no } y_j)\right) \qquad (42) \\ &= \sum_j (-1)^{t+j+1}h_j^{(n)}\,V_0\,(\text{no } y_j)\left(\sum_{k=1}^{n-t}\binom{n-k}{t}p_{k-1}\,(\text{no } y_j)\right) \\ &= \alpha(-1)^{t+n+1}\Big[\sum_j h_j^{(n)}(\pi_{i\neq j}(y_j - y_i))^{-1}\Big]\left(\sum_{k=1}^{n-t}\binom{n-k}{t}p_{k-1}\,(\text{no } y_j)\right) \\ &= \alpha(-1)^{t+n+1}\Big(\sum_j K_j^{(n)}E_{j,\,t}\Big).\end{aligned}$$

Thus, one can calculate a_t by the formula

$$a_t = \alpha_t/\alpha = D_t + (-1)^{n+t+1}\sum_j K_j^{(n)}E_{j,\,t},$$

where D_t is given by (41)

$$K_j^{(n)} = h_j^{(n)}/\Pi_{i\neq j}(y_j - y_i) \tag{43}$$

$$E_{j,\,t} = \sum_{k=1}^{n-t}\binom{n-k}{t}p_{k-1}\,(\text{no } y_j) = \sum_{k=0}^{n-t-1}\binom{n-k-1}{t}p_k\,(\text{no } y_j). \tag{44}$$

It is of interest to note that D_t is the coefficient corresponding to the case in which each ν_i is zero.

The critical stage in the computation of a_t is in computing $h_j^{(n)}$ for $K_j^{(n)}$, because of the cancellation of many significant digits.

$$h_j^{(n)} = g(y_j) - \sum_{k=0}^{n-1} A_k y_j^k.$$

It is probably best to use $u_j = h\nu_j$ (h is the time interval) as the basis for this computation, since the ν_j are chosen precisely. Thus,

$$y_j = e^{-u_j} - 1, \qquad g(y_j) = (e^{u_j} - 1)/u_j.$$

The A_k's are most readily computed by the inductive formula

$$(45) \quad A_0 = 1, \qquad A_n = (-1)^n \sum_{k=1}^{n} |A_{n-k}|/k(k+1) \qquad n = 1, 2, \cdots.$$

This formula is obtained from the fact that $g(y) = A_0 + A_1 y + A_2 y^2 + \cdots$ is the reciprocal of $(y + 1) \log (1 + y)/y$ whose series can be readily written out. The loss of accuracy can be judged from the computation of $h_1^{(6)}$ in § 12 below. Thus, for nine places after the decimal point:

$$g(y_1) = .98457\,9722 + .11180\,8598\,i$$

$$\sum_{k=0}^{5} A_k y_1^k = .98462\,3553 + .11180\,4513\,i$$

$$h_1^{(6)} = -.00004\,3831 + .00000\,4081\,i.$$

The following computational procedure is suggested for computing the a_t.

1) The ν_j, h and n are chosen and the $u_j = h\nu_j$ calculated.

2) The D_t are obtained.

3) From the u_j, $y_j = e^{-u_j} - 1$ and $g(y_j) = (e^{u_j} - 1)/u_j$ are obtained and used in turn to compute $h_j^{(n)}$.

4) The various differences $y_i - y_j$ are computed and from these the products $\Pi_{i \neq j}(y_i - y_j)$ are formed. These can be used to compute $K_j^{(n)}$ in conjunction with the result of 3.

5) The p_k (no y_j) are obtained by forming the polynomial

$$y^{n-1} - p_1 y^{n-2} + p_2 y^{n-3} + \cdots + (-1)^{n-1} p_{n-1}.$$

The complex y_j are combined in conjugate pairs to yield the quadratic factors for this polynomial except, of course, for the conjugate of the y_j omitted. The p_k are then used to obtain the $E_{j,t}$.

6) a_t is then obtained from the above formula.

The process 3) is readily seen to be the only one critical as far as accuracy is concerned and overall estimates for the accuracy of a_t can be obtained based on this.

Let $a_0^0, \cdots, a_{n-1}^0$ be the result of the above computation. When one substitutes back in (4) or (37) one may find a discrepancy ϵ_j in each equation, i.e.,

$$\sum_{r=0}^{n-1} a_r^{(0)} x_j^r = g(y_j) - \epsilon_j.$$

Using (37) one obtains

$$\sum_{r=0}^{n-1} (a_r - a_r^0)x_j^r = \epsilon_j.$$

Thus if $a_r = a_r^0 + \Delta a_r$, Δa_r satisfies an equation

$$\sum_{r=0}^{n-1} \Delta a_r x_j^r = \epsilon_j.$$

Now the previous argument can be used to show that

$$\Delta a_t = (-1)^{n+t+1} \sum_j L_j E_{j,t}, \tag{46}$$

where $L_j = \epsilon_j / \Pi_{i \neq j}(y_j - y_i)$ and the $E_{j,t}$ are the quantities previously computed. Thus, the computation can be readily continued as far as the accuracy to which y_j and $g(y_j)$ will justify.

12. A report on the aspects of numerical solutions of differential equations pertinent to analogue computing installations is being prepared by Project Cyclone. As part of the report, a system of differential equations representative of the types encountered is solved by various numerical procedures. The exponential method gave the most accurate solution, although it was used with a very large interval width. The details of the problem as solved by the exponential method are considered in this section.

The problem is the following:

$$\begin{aligned}
\dot{V} &= 9.295 \cos\alpha - 32.2 \sin\gamma - .00056022\, V^2[.129 + .051632(.965 + 5.1\,\alpha)^2] \\
V\dot{\gamma} &= 9.295 \sin\alpha - 32.2 \cos\gamma + .00056022\, V^2(.965 + 5.1\,\alpha) \\
\dot{y} &= -.00009421\, V^2(.215y + .44\alpha - .026) \\
\dot{\theta} &= y \\
\alpha &= \theta - \gamma.
\end{aligned}$$

The initial conditions of the system are:

$$V(0) = 200 \qquad y(0) = -.0204$$
$$\gamma(0) = 0 \qquad \theta(0) = .0525$$

Solutions of the system of equations had been obtained by other procedures before the exponential method was used. Based upon these solutions, characteristic matrices were calculated at $t = 0$ and 5.7 seconds. The characteristic values were

t	λ_1, λ_2	λ_3, λ_4
0	$-.72140\,2212 \pm 1.28266\,534i$	$-.01586\,3538 \pm .19708\,3422i$
5.7	$-.86065\,6195 \pm 1.43671\,649i$	$-.01938\,6195 \pm .17749\,5026i$

In the exponential method $n = 4$ and $\nu_1, \nu_2 = -.80 \pm i1.36$ and $\nu_3, \nu_4 = -.018 \pm .19i$. These values are rounded averages obtained from λ_1, λ_2 and λ_3, λ_4, respectively, of the above table. It is clear from the table of values that the chosen quantities are not very different from the characteristic values of the $t = 0$ matrix that would normally be chosen.

Based on the characteristic values, the short period of the solution is approximately 4 seconds. An h value was chosen by a rule generally applied to cubic polynomial approximation methods, namely, h is not to exceed

$T/12$. Thus, h was originally chosen as .3. The natural limitation on the size of h, discussed in § 10, shows that h must be chosen less than .9. This was obtained by evaluating a maximum h for the characteristic values of the $t = 0$ and $t = 5.7$ second matrices.

t	α	β	max h
0	−.721	1.28	1.041
5.7	−.861	1.44	.934
0	−.016	.197	5.562
5.7	−.019	.177	6.276

Having chosen an interval width, it is necessary to estimate the error of the solution. A bound on the error can be obtained by an evaluation of the step error formula, equation (10), using the relationships obtained for $\epsilon(\lambda_j h)$ and $e(t, \sigma)$ in § 8, 9. An estimate of the total error in the solution was obtained by a recomputation at a half interval width, $h = .15$. Since the step error is proportional to h^5, the truncation error of the second procedure

TABLE A

t	V $h = .3$	V $h = .15$	$\Delta\ 10^3$	γ $h = .3$	γ $h = .15$	$\Delta\ 10^7$
0	200.000	200.000		.00000000	.00000000	
		200.700			−.00298124	
.3	201.407	201.407		−.00576910	−.00576886	
		202.121			−.00833398	
.6	202.840	202.840		−.0106580	−.0106575	
		203.562			−.0127295	
.9	204.286	204.286		−.0145486	−.0145079	
		205.010			−.0161168	
1.2	205.734	205.734	0	−.0174438	−.0174457	19
		206.457			−.0185476	
1.5	207.178	207.178	0	−.0194343	−.0194384	41
		207.897			−.0201350	
1.8	208.612	208.612	0	−.0206523	−.0206549	26
		209.325			−.0210150	
2.1	210.033	210.033	0	−.0212278	−.0212311	33
		210.738			−.0213174	
2.4	211.438	211.438	0	−.0212849	−.0212861	12
		212.134			−.0211476	
2.7	212.824	212.824	0	−.0209110	−.0209099	−11
		213.510			−.0205790	
3.0	214.190	214.190	0	−.0201594	−.0201593	− 1
		214.864			−.0196533	
3.3	215.532	215.532	0	−.0190644	−.0190624	−20
		216.193			−.0183866	
3.6	216.847	216.847	0	−.0176267	−.0176256	−11
		217.494			−.0167781	
3.9	218.132	218.132	0	−.0158419	−.0158429	10
		218.762			−.0148186	
4.2	219.383	219.383	0	−.0137044	−.0137039	− 5
		219.995			−.0124979	
4.5	220.596	220.597	−1	−.0111973	−.0112000	17
		221.188			−.00980993	
4.8	221.768	221.769	−1	−.00832525	−.00832802	26
		222.338			−.00675493	
5.1	222.896	222.896	0	−.00509068	−.00509185	12
		223.442			−.00334027	
5.4	223.975	223.975	0	−.00149742	−.00150204	46
		224.496			.000420690	
5.7	225.004	225.004	0	.00242675	.00242555	12
		225.499			.00451008	
6.0	225.980	225.980	0	.00667289	.00667159	13

TABLE B

t	y $h = .3$	y $h = .15$	$\Delta\, 10^7$	θ $h = .3$	θ $h = .15$	$\Delta\, 10^7$
0	−.0204000	−.0204000		.0525000	.0525000	
		−.0165274			.0497366	
.3	−.0131725	−.0131726		.0475157	.0475158	
		−.0103520			.0457580	
.6	−.00805566	−.00805626		.0443837	.0443841	
		−.00625330			.0433168	
.9	−.00489427	−.00489545		.0424856	.0424860	
		−.00392390			.0418290	
1.2	−.00328144	−.00327280	−86	.0412954	.0412928	26
		−.00287426			.0408345	
1.5	−.00267128	−.00266115	−101	.0404182	.0404212	−30
		−.00257054			.0400300	
1.8	−.00254925	−.00254868	−6	.0396415	.0396467	−52
		−.00253762			.0392653	
2.1	−.00250967	−.00250611	−36	.0388805	.0388865	−60
		−.00242001			.0385161	
2.4	−.00225020	−.00225703	68	.0381541	.0381643	−102
		−.00200326			.0378435	
2.7	−.00164177	−.00165228	105	.0375630	.0375681	−51
		−.00120409			.0373527	
3.0	−.000656351	−.000664002	76	.0372078	.0372115	−37
		−.0000412040			.0371577	
3.3	.000665564	.000652188	134	.0372004	.0372027	−23
		.00140242			.0373562	
3.6	.00219999	.00219509	49	.0376306	.0376256	50
		.00301588			.0380042	
3.9	.00385298	.00385141	16	.0385338	.0385311	27
		.00468956			.0391718	
4.2	.00552401	.00552000	40	.0399427	.0399376	51
		.00633420			.0408270	
4.5	.00711795	.00712560	−76	.0418444	.0418368	76
		.00788956			.0429633	
4.8	.00862167	.00862310	−14	.0442031	.0442021	10
		.00932485			.0455486	
5.1	.00999301	.00999468	−17	.0470037	.0469980	57
		.0106334			.0485455	
5.4	.0112334	.0112428	−94	.0501893	.0501865	28
		.0118249			.0519169	
5.7	.0123873	.0123822	51	.0537306	.0537328	−22
		.0129170			.0556305	
6.0	.0134275	.0134319	−44	.0576129	.0576069	60

may be considered negligible in comparison with that of the first computation. The difference in values of the solution at $h = .3$ and $h = .15$ was very small. These values are given in Table A. The maximum errors given as a percentage of the absolute maximum of the variables V and γ are .000 and .022, respectively.

By the use of § 10 to calculate the coefficients of the integration formulas, the following values were obtained:

h	a_0	a_1	a_2	a_3
.3	2.09056 049	−1.92175 461	1.07630 795	−.23931 8630
.15	2.19751 098	−2.19425 286	1.29578 949	−.29906 1942

The corresponding results for the variables y and θ are given in Table B. The percentage errors in these cases are .066 for y and .020 for θ.

The initial four values to start the procedure were calculated in the $h = .15$ case using the Runge-Kutta method. In the $h = .3$ case these values were calculated using an unpublished method of R. F. CLIPPINGER.

13. The interest of the authors in the use of exponential sums for integrating differential equations arose from their experience in integrating a system of fourteen first order differential equations in order to provide a large scale test problem for a continuous computer. For references to this work see *MTAC*, v. 6, p. 78, ftn. 3.

The system is stable and for each of the fourteen unknown dependent variables, z_i, one can express the solution, regarded as a function of t, as the sum of two terms, one of which varies very slowly. The other term varies much more quickly and appears to be a combination of exponentials. The complex frequencies that appear in this latter combination of exponentials agree very closely with those obtained from a linearized version of the problem as described in Chapter 2 of part II of the above mentioned work. The error analysis in this report was based on the assumption that the functions to be integrated could be expressed in terms of these frequencies and the results seem to justify this assumption. (Cf. loc. cit., p. 121–123.)

These frequencies are listed for various time values. The values are taken from tables on pages 72 and 73 of the report. λ_6 and λ_9 are identically zero. The complex frequencies are listed in Table 2, the real frequencies in Table 3. The Table of § 10 may be used to compute the maximum h. In

TABLE 1

$v_1, v_2 = -.35 \pm 5.667i$	$u_1, u_2 = -.014 \pm .22668i$
$v_3, v_4 = -.234 \pm 1.064i$	$u_3, u_4 = -.00936 \pm .04256i$
$v_5 = -2.9$ $\quad v_6 = 0$	$u_5 = -.116$ $\quad u_6 = 0$

TABLE 2

t	λ_1, λ_2	λ_3, λ_4
2.30	$-.34965 \pm 5.66490i$	$-.22970 \pm 1.06540i$
2.70	$-.34664 \pm 5.62710i$	$-.22886 \pm 1.05708i$
12.500	$-.28158 \pm 5.07139i$	$-.21667 \pm .95000i$
30.020	$-.19358 \pm 4.31071i$	$-.18938 \pm .80357i$
50.020	$-.11056 \pm 3.40846i$	$-.14739 \pm .62710i$

t	$\lambda_{10}, \lambda_{11}$	$\lambda_{12}, \lambda_{13}$
2.30	$-.35142 \pm 5.66870i$	$-.23858 \pm 1.06253i$
2.70	$-.35160 \pm 5.62501i$	$-.23895 \pm 1.05327i$
12.500	$-.27892 \pm 5.06239i$	$-.22938 \pm .94652i$
30.020	$-.19116 \pm 4.30492i$	$-.20374 \pm .79798i$
50.020	$-.10608 \pm 3.40693i$	$-.16836 \pm .62433i$

TABLE 3

t	λ_5	λ_{14}	λ_7	λ_8
2.30	−2.9009	−2.8940	−.062634	+.0027338
2.70	−2.8980	−2.8949	−.055760	+.0027604
12.50	−2.9165	−2.9124	−.042666	+.0029659
30.020	−2.9496	−2.9485	−.040503	+.0035029
50.020	−2.9761	−2.9761	−.044609	+.0042092

TABLE 4

t	$D_t^{(6)}$	a_t
0	2.97013 88889	2.90460 07614
1	−5.50208 33333	−5.19498 93728
2	6.93194 44444	6.35668 29407
3	−5.06805 55556	−4.52952 65434
4	1.99791 66667	1.74595 45079
5	− .32986 11111	− .28272 22943

this case the smallest h_0 obtainable from the table is associated with

$$\lambda_{10} = -.35 \pm 5.669i.$$

This yields $h_0 = .178$. (The value of h_0 corresponding to λ_1, λ_2 or λ_{11} is essentially the same. The values for $\lambda_3, \lambda_4 = -.23 \pm 1.065i$ is .85 and for the real $\lambda_5 = -2.9$ is .239.) The experience with the problem seems, to indicate that even $h = .12$ is too large when polynomials are used but it is conceivable that, with a proper choice of the ν_i even $h = .08$ might be effectively used with the exponential method. For comparison with the polynomial results let $h = .04$.

The process of finding the a_t's is illustrated by considering the use of 6 coefficients for the "Lark problem" and a time interval of $h = .04$. The following values of ν_j and u_j were chosen.
This will be referred to as the $n = 6$ case. In Table 4 the D_t and the a_t appear so that the amount of change of coefficients needed is indicated. Table 5 contains the y_j, $g(y_j)$, $\sum_{r=0}^{5} A_r y_j^r$, $h_j^{(6)}$, $\Pi_{i\neq j}(y_j - y_i)$. The values of

TABLE 5

j	y_j	$g(y_j)$
1	−.01184 4296 − .22791 2241i	.98457 97220 + .11180 85936i
2	−.01184 4296 + .22791 2241i	.98457 97220 − .11180 85936i
3	.00848 9886 − .04294 7263i	.99503 48169 + .02114 44856i
4	.00848 9886 + .94294 7263i	.99503 48169 − .02114 44856i
5	.12299 5872	.94417 91121
6	0	1.

j	$\sum_{r=0}^{5} A_r y_j^r$	$h_j^{(6)}$
1	.98462 35536 + .11180 45132i	−.00004 38316 + .00000 40804i
2	.98462 35536 − .11180 45132i	−.00004 38316 − .00000 40804i
3	.99503 48145 + .02114 44916i	.00000 00024 − .00000 00060i
4	.99503 48145 − .02114 44916i	.00000 00024 + .00000 00060i
5	.94417 81327	.00000 09794
6	1.	0

j	$\prod_{i=1, i\neq j}^{6} (y_j - y_i)$
1	−.00096 98667 − .00099 90172i
2	−.00096 98667 + .00099 90172i
3	.00002 01775 + .00001 15376i
4	.00002 01775 − .00001 15376i
5	.00012 89991
6	−.00001 22777

the $p_{k,j}$ and $E_{t,j}$ are given in Tables 6 and 7. The values of $K^{(6)}$ are given in Table 8. The a_t's are good to at least 7 figures after the decimal point.

The case in which one is permitted only four coefficients may also be considered. In this case, one may choose ν_1 and ν_2 to be $-.292 \pm 3.3655i$ corresponding to an average of the previously used ν. The error formula shows that for λ_1 and λ_2, the truncation error is reduced by 28 percent over the polynomial case. This relatively slight improvement is due to the need of approximating 6 complex numbers by 4 in a certain way. If one were

TABLE 6

$p_{k,j}$

l \ j	1	2
0	1.	1.
1	.12813 1348 + .22791 2241i	.12813 1348 − .22791 2241i
2	.00234 70745 + .03190 21627i	.00234 70745 − .03190 21627i
3	.00018 82909 + .00091 27857i	.00018 82909 − .00091 27857i
4	−.00000 27920 + .00005 37251i	−.00000 27920 − .00005 37251i
5	0	0

l \ j	3	4
0	1.	1.
1	.10779 7166 + .04294 7263i	.10779 7166 − .04294 7263i
2	.05001 37854 + .00426 49759i	.05001 37854 − .00426 49759i
3	.00682 36045 + .00211 17460i	.00682 36045 − .00211 17460i
4	.00005 43875 + .00027 51267i	.00005 43875 − .00027 51267i
5	0	0

l \ j	5	6
0	1.	1.
1	−.00670 8820	.11628 7052
2	.05359 85956	.05277 34384
3	.00083 89789	.00743 13849
4	.00009 98219	.00020 30128
5	0	.00001 22777

TABLE 7

$E_{t,j}$

j \ t	0	1
1	1.13066 39214 + .26078 09145i	5.51994 04053 + 1.00923 47486i
2	1.13066 39214 − .26078 09145i	5.51994 04053 − 1.00923 47486i
3	1.16468 89434 + .04959 91116i	5.59493 16167 + .189082 5984i
4	1.16468 89434 − .04959 91116i	5.59493 16167 − .18909 25984i
5	1.04782 85764	5.13573 82865
6	1.17670 71658	5.63853 43058

j \ t	2	3
1	10.77601 76025 + 1.46409 27198i	10.51487 24665 + .94355 11267i
2	10.77601 76025 − 1.46409 27198i	10.51487 24665 − .94355 11267i
3	10.80364 79567 + .27259 02517i	10.48120 24494 + .17605 40279i
4	10.80364 79567 − .27259 02517i	10.48120 24494 − .17605 40279i
5	10.12138 18457	10.02676 33156
6	10.86347 40121	10.51792 16464

j \ t	4	5
1	5.12813 1348 + .22791 2241i	1
2	5.12813 1348 − .22791 2241i	1
3	5.10779 7166 + .04294 7263i	1
4	5.10779 7166 − .04294 7263i	1
5	4.99329 1180	1
6	5.11628 7052	1

TABLE 8

j	K_j
1	.01982 50786 − .02462 81210i
2	.01982 50786 + .02462 81210i
3	−.00005 18207 − .00027 53833i
4	−.00005 18207 + .00027 53833i
5	.00759 23010
6	0

TABLE 9

t	$D_t^{(4)}$	a_t
0	2.29166 66667	2.23915 28992
1	−2.45833 33333	−2.30866 86852
2	1.54166 66667	1.39918 51376
3	−0.37500 00000	−0.32966 93515

TABLE 10

j	y_j	$g(y_j)$
1	.00259 45860 − .13579 05660i	.99119 12907 + .06668 75407i
2	.00259 45860 + .13579 05660i	.99119 12907 − .06668 75407i
3	.12299 58721	.94417 91095
4	0	1.00000 00000

j	$\sum_{r=0}^{3} A_r y_j^r$	$h_j^{(4)}$
1	.99107 63783 + .06666 37665i	.00011 49124 + .00002 37742i
2	.99107 63783 − .06666 37665i	.00011 49124 − .00002 37742i
3	.94410 76360	.00007 14735
4	1.00000 00000	.00000 00000

j	$\prod_{\substack{k=1 \\ k \neq j}}^{k=4} (y_j - y_k)$
1	.00434 44938 + .00509 25453i
2	.00434 44938 − .00509 25453i
3	.00405 09364
4	−.00226 87585

TABLE 11

$p_{l,j}$

l \ j	1	2
0	1.00000 00000	1.00000 00000
1	0.12559 04581 + .13579 05660i	.12559 04581 − .13579 05660i
2	0.00031 91234 + .01670 16791i	.00031 91234 − .01670 16791i
3	0.00000 00000	0.00000 00000

l \ j	3	4
0	1.00000 00000	1.00000 00000
1	0.00518 91720	.12818 50441
2	0.01844 58097	.01908 40564
3	0.00000 00000	.00226 87585

TABLE 12

$E_{t,j}$

j \ t	0	1
1	1.12590 95815 + .15249 22451i	3.25150 00396 + .28828 28111i
2	1.12590 95815 − .15249 22451i	3.25150 00396 − .28828 28111i
3	1.02363 49817	3.02882 41537
4	1.14953 78590	3.27545 41446

j \ t	2	3
1	3.12559 04581 + .13579 05660i	1
2	3.12559 04581 − .13579 05660i	1
3	3.00518 91720	1
4	3.12818 50441	1

TABLE 13

j	K_j
1	.01384 34770 − .01075 48348i
2	.01384 34770 + .01075 48348i ·
3	.01764 36945
4	.00000 00000

permitted 6 coefficients, initially the truncation error practically vanishes. Even after a change of 10 seconds in the independent variable, the truncation error has been reduced by 76 percent relative to the polynomial case.

Tables 9 to 13 for $n = 4$ are analogous to Tables 4 to 8, respectively, for the $n = 6$ case.

P. BROCK
F. J. MURRAY

Reeves Instrument Corp.
Columbia University

The preparation of this paper was assisted by the Office of Naval Research.

RECENT MATHEMATICAL TABLES

992[C].—G. W. SPENCELEY, R. M. SPENCELEY & E. R. EPPERSON, *Smithsonian Logarithmic Tables to Base e and Base 10.* Smithsonian Misc. Collections, v. 118, Washington, 1952, xii + 402 p., 14.6 × 22.9 cm. Price $4.50.

This volume gives both common and natural logarithms of numbers of the forms

$$n, \qquad 1 + n \cdot 10^{-4}, \qquad 1 + n \cdot 10^{-8},$$

where $n = 1(1)10^4$.

The values are given to 23D. In the case of common logarithms the characteristics are omitted. P. xii contains 23D values of the natural logarithms of 10^k, $k = 1(1)10$ as well as log e.

The table is intended to be used with a calculating machine to find logarithms and antilogarithms to accuracy not exceeding 23D by the well-known factorization method. It was computed on desk calculators with the assistance of 4 students. The natural logarithms of integers were built up from the table of WOLFRAM.[1] All common logarithms were found by multiplication by log e. The work was carried to 28D and then rounded to 23D.

These tables prove that it is still possible to produce a hand-set volume of over a million digits from a very small computing organization. The existence of large scale computing units that could calculate the present table in three days does not seem to daunt the authors.

The FMR *Index* lists only one table comparable with the present one: the four-figure radix table of STEINHAUSER[2] to 21D which "contains many errors." The need for the present table is certainly not as great as it was in 1880 with the advent of the modern automatic desk calculator. Having both natural and common logarithms is something of a luxury in the face of present day printing costs. The hand computer who has a need for many