very briefly the computer used. No reference is made to the large amount of literature on the paper.

As far as computing technique goes the main point of interest is an electronic circuit used to represent the boundary conductance which is not constant but rather a function of temperature. This nonlinear condition has heretofore on other computers been represented in discrete finite steps. The author shows in figure 8 a circuit for representation of such boundary conductance. The check of the computations with actual experiments is rather unsatisfactory possibly because of poor assumption of physical constants of the system; the correction of assumptions determines in analog computers the validity of the result. The author, in designing the computing circuit, disregards a number of influences (for example the thermal resistance of the ice).

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1015. F. Weiner, "Further remarks on intermittent heating for aircraft ice protection," ASME *Trans.*, v. 73, 1951, p. 1131–1137.

The paper deals with the deicing of a propeller of ovoid cross section. The butt end of this ovoid will be called A, the extension, B. Heat is generated by electric heaters only in the region A and is conducted to the region B. A cross section through the propeller in the region A shows (proceeding inwards) the steel blade (extending all the way to B), a layer of nylon, the heater, another layer of nylon, and finally sponge rubber. Disregarding the heat flow along the propeller length, the problem is two dimensional.

The greatest problem in solving two dimensional problems on analog computers, one on which little information is available, is that of how to section the body in which heat flow occurs. The authors disregard the thermal resistance across the thickness of the steel, and along the length of the nylon and heater layers. The sponge rubber is represented by a number of parallel sections, ending in a fictitious center with zero volume (Fig. 3 of the paper). This design is not discussed or analyzed; thus, the most crucial problem, from a computational view-point, is not dealt with in the paper. Regarding deicing, the paper shows the desirability of using high rates of energy production, which results in a lower total heat consumption than heating at a lower rate.

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NOTES

143.—Analytical Approximations. [Editorial Note: In Note **139**, *MTAC*, v. 6, p. 251–253, Cecil Hastings has described the Rand *Collection of Illustrative Approximations*. The interest that these approximations have aroused during the past year is considerable. It is hoped to publish as Notes from time to time additional examples of such approximations contributed by our readers. To encourage this hope, Mr. Hastings is submitting a dozen new examples, prepared with the assistance of Mr. JAMES P. WONG and Mrs. DAVID K. HAYWARD. These differ from the RAND Collection in form, especially since they do not give illustrative error curves. For convenience in future references we are numbering these approximations consecutively.]

(1) Square Root: To better than 1 part in 12 over $.1 \leq x \leq 10$,
$$x^{\frac{1}{2}} \doteq (1 + 4x)/(4 + x).$$
(2) Pearson Cosine Transformation: To .003 over $0 \leq x \leq 1$,
$r(x) = \cos (\pi/(1 + x^{\frac{1}{2}})) \doteq (-1 - 4x + 5x^2)/(1 + 8x + 6x^2)$.
$r(x^{-1}) = -r(x)$ can be used to obtain function values over $1 \leq x \leq \infty$.
(3) Common Logarithmic Function: To better than .005 over $.1 \leq x \leq 1$, $\log x \doteq -.076 + .281\,x - .238/(x + .15)$.
This approximation is the result of a request for a very simple formula to use in the reduction of certain data.
(4) Common Logarithmic Function: To better than .000,004 over $1 \leq x \leq 10$,
$\log x \doteq \frac{1}{2} + .86857y + .29059y^3 + .15783y^5 + .20269y^7$,
where $y = (x - \sqrt{10})/(x + \sqrt{10})$.
(5) Common Logarithmic Function: To better than .000,000,015 over $1 \leq x \leq 10$,
$\log x \doteq \frac{1}{2} + .8685888y + .2895497y^3 + .1731159y^5 + .1314381y^7 + .0547562y^9 + .1832415y^{11}$,
where $y = (x - \sqrt{10})/(x + \sqrt{10})$.
(6) Inverse Tangent: To better than .005 over $-1 \leq x \leq 1$,
$$\arctan x \doteq x/(1 + .28x^2).$$

This approximation is the result of a request for a very simple formula to use in the reduction of certain data.

(7) Descending Exponential Function: To better than .000,000,11 over $0 \leq x \leq \infty$,
$e^{-x} \doteq (1 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5)^{-8}$,
where $a_1 = .125{,}000{,}204$, $a_2 = .007{,}811{,}604$, $a_3 = .000{,}326{,}627$, $a_4 = .000{,}009{,}652$ and $a_5 = .000{,}000{,}351$.
(8) Incomplete Gamma Function Type Integral: To better than .000,000,1 over $0 \leq x \leq 1$,
$$F(x) = \int_1^\infty e^{-t}t^{-x-1}dt$$
$$\doteq [.219384+x(.024717+.000803x)]/[1+x(.558651+.090584x)].$$
$F(x) + (x + 1)F(x + 1) = e^{-1}$ can be used to obtain function values outside the range indicated.
(9) Exponential Integral of Negative Argument: To better than .000,000,1 over $10 \leq x \leq \infty$,
$$xe^x \int_x^\infty t^{-1}e^{-t}dt$$
$$\doteq [1.15198 + x(4.03640 + x)]/[4.19160 + x(5.03637 + x)].$$

(10) Segmental Area Function: To better than .0012 over $-1 \le x \le 1$,
$$A(x) = \int_{-x}^{x} (1 - t^2)^{\frac{1}{2}}\, dt \doteq 2.0083x - .4160x^3 + .1604x^5 - .1808x^7.$$
In terms of elementary functions, $A(x) = \arcsin x + x(1 - x^2)^{\frac{1}{2}}$.

(11) Segmental Area Function: To better than .00016 over $-1 \le x \le 1$,
$$A(x) = \int_{-x}^{x} (1 - t^2)^{\frac{1}{2}}\, dt$$
$$\doteq (1.99916x - 2.39484x^3 + .58673x^5)/(1 - 1.03472x^2 + .15634x^4).$$

(12) Segmental Area Function: To better than .000,016 over $-1 \le x \le 1$,
$$A(x) = \int_{-x}^{x} (1 - t^2)^{\frac{1}{2}}\, dt$$
$$\doteq x(1.999872 + 4.143151\eta - 3.153670\eta^2 - 1.430807\eta^3)/$$
$$(1 + 2.901498\eta - 1.811287\eta^2 - 1.098016\eta^3),$$
where $\eta = x^2/(5 - 4x^2)$.

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144.—ZEROS OF THE DERIVATIVE OF BESSEL FUNCTIONS OF FRACTIONAL ORDER. The NBS Computation Laboratory[1] has published extensive tables of Bessel functions of fractional order, $J_\nu(x)$, $\pm\nu = \frac{3}{4}, \frac{2}{3}, \frac{1}{3}, \frac{1}{4}$, and zeros of $J_\nu(x)$ have been tabulated by ABRAMOWITZ[2] and by the Computation Laboratory when the latter was known as the Mathematical Tables Project.[3] (The zeros in the last two sources are also given in the first, p. 384–385.) This present note gives the first seven or eight positive zeros of the first derivative of those Bessel functions. Following standard notation, $j'_{\nu,s}$ will be used to denote the s-th positive root of $J_\nu'(x) = 0$. The zeros $j'_{\nu,s}$ were obtained from the published values[1] of $J_\nu(x)$, and this table includes all such zeros within the range of tabulation of $J_\nu(x)$ itself (i.e., not exceeding 25).

These zeros were computed to the maximum accuracy obtainable from the NBS tables. All entries are given here to seven decimal places; although the seventh decimal place is not absolutely guaranteed, it has a high probability of being correct. The zeros were computed using the 5-point case of two different formulas for inverse interpolation for the derivative, given in SALZER[4] on p. 214 and p. 215. They were also checked by (a) calculating an approximate expression for the error in $j'_{\nu,s}$, (b) recomputing the last $j'_{\nu,s}$ given here by the asymptotic formulas for each ν (see below), and (c) computing the seventh divided difference of $j'_{\nu,s}$ as a function of ν, using a formula in Salzer.[5] (This divided difference check was not fully applicable to the first few zeros.)

For zeros > 25, the following asymptotic formulas, whose coefficients were calculated from the general expression for $j'_{\nu,s}$ in WATSON,[6] will give at least seven decimal accuracy:

$$\pm\nu = \tfrac{3}{4},\quad j'_{\nu,s} = y - .65625000y^{-1} - .54931641y^{-3}$$

$$\text{where } y = \pi s - \begin{cases} .39269908 & \nu = -\tfrac{3}{4} \\ 1.17809724 & \nu = \tfrac{3}{4} \end{cases}$$

$$\pm\nu = \tfrac{2}{3},\quad j'_{\nu,s} = y - .59722222y^{-1} - .41380530y^{-3}$$

$$\text{where } y = \pi s - \begin{cases} .26179939 & \nu = -\tfrac{2}{3} \\ 1.30899694 & \nu = \tfrac{2}{3} \end{cases}$$

$$\pm\nu = \tfrac{1}{3},\quad j'_{\nu,s} = y - .43055556y^{-1} - .07507073y^{-3}$$

$$\text{where } y = \pi s + \begin{cases} .26179939 & \nu = -\tfrac{1}{3} \\ (-1.83259571) & \nu = \tfrac{1}{3} \end{cases}$$

$$\pm\nu = \tfrac{1}{4},\quad j'_{\nu,s} = y - .40625000y^{-1} - .03108724y^{-3}$$

$$\text{where } y = \pi s + \begin{cases} .39269908 & \nu = -\tfrac{1}{4} \\ (-1.96349540) & \nu = \tfrac{1}{4} \end{cases}$$

To obtain $j'_{\nu,s}$ for any other ν that is less than one in absolute value, the present table may be used in conjunction with a special table of interpolation coefficients,[1] p. 393–413. The user is cautioned that for interpolation as well as for forming divided differences, the values of $j'_{\nu,s}$ for $\nu < 0$ are not continued into $j'_{\nu,s}$ for $\nu > 0$, but into $j_{\nu,s+1}$.

Mrs. Ruth E. Capuano and Miss Mary M. Dunlap assisted in the computations.

Herbert E. Salzer

NBSCL

Table of $j'_{\nu,s}$

s	$\nu = -\frac{3}{4}$	$\nu = -\frac{2}{3}$	$\nu = -\frac{1}{3}$	$\nu = -\frac{1}{4}$	s
1	2.47861 49	2.65267 49	3.27468 22	3.41838 81	1
2	5.77630 68	5.92026 00	6.47892 00	6.61491 38	2
3	8.95866 64	9.09725 75	9.64204 42	9.77606 19	3
4	12.11945 77	12.25581 13	12.79457 06	12.92770 62	4
5	15.27226 12	15.40738 64	15.94278 34	16.07542 28	5
6	18.42121 33	18.55556 21	19.08881 57	19.22113 82	6
7	21.56801 08	21.70182 42	22.23359 29	22.36569 56	7
8	24.71348 00	24.84690 21			8

s	$\nu = \frac{3}{4}$	$\nu = \frac{2}{3}$	$\nu = \frac{1}{3}$	$\nu = \frac{1}{4}$	s
1	1.51433 70	1.40121 80	0.90999 85	0.76906 15	1
2	4.97223 54	4.85063 49	4.35291 38	4.22515 79	2
3	8.16610 90	8.04140 90	7.53529 41	7.40675 25	3
4	11.33027 35	11.20403 00	10.69360 09	10.56453 27	4
5	14.48452 05	14.35735 04	13.84430 89	13.71489 82	5
6	17.63422 27	17.50643 40	16.99164 33	16.86199 56	6
7	20.78145 96	20.65322 86	20.13718 52	20.00736 48	7
8	23.92720 84	23.79864 53	23.28166 09	23.15170 93	8

[1] NBSCL, *Tables of Bessel Functions of Fractional Order.* V. 1, New York, Columbia University Press, 1948.
[2] M. Abramowitz, "Zeros of certain Bessel functions of fractional order," *MTAC*, v. 1, p. 353–354.
[3] Mathematical Tables Project, National Bureau of Standards, "More zeros of certain Bessel functions of fractional order," *MTAC*, v. 2, p. 118–119.
[4] H. E. Salzer, "Formulas for finding the argument for which a function has a given derivative," *MTAC*, v. 5, p. 213–215.
[5] H. E. Salzer, "The checking of functions tabulated at certain fractional points," *MTAC*, v. 2, p. 318–319.
[6] G. N. Watson, *A Treatise on the Theory of Bessel Functions.* Cambridge University Press, 1944, p. 507.

145.—An Example in the Use of the Differential Analyser. In a recent article Sprague[1] has discussed, as an example, a differential analyser setup for solving the equation

$$w'' - ww' - wt = 0,$$

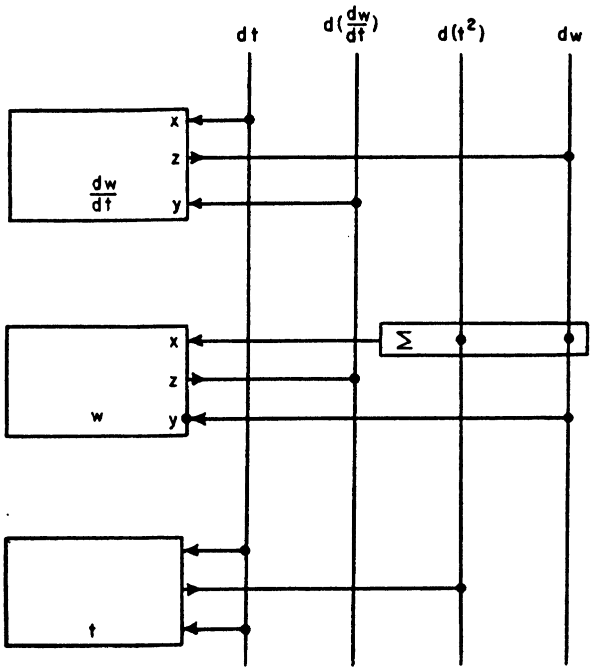

Figure 1

primes denoting differentiation with respect to t. There are several simpler ways of setting up this equation. One is to take the once-integrated form

$$w' = \int_0^t wd(w + \tfrac{1}{2}t^2) + w'(0)$$

and so place it on the analyser in accordance with Figure 1. This uses three integrators in lieu of the six shown in Figure 3 of Sprague's paper.

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[Editorial note: Mr. Sprague informs us that an additional integrator will be required in case the differential analyzer is of digital type, thus the above method would require four integrators.]

[1] R. E. Sprague, "Fundamental concepts of the digital differential analyzer method of computation," *MTAC*, v. 6, p. 41–49.

146.—Two New Mersenne Primes. The program described in Notes **131**(c) and **138** [*MTAC*, v. 6, p. 61, 204] has been continued. Two more Mersenne primes, $2^{2203} - 1$ and $2^{2281} - 1$, were discovered by the SWAC on October 7 and 9, 1952. The time required for either of the tests is one hour. This makes 17 Mersenne primes, and a corresponding number of perfect numbers, now known. They are $2^n - 1$ for $n = 2, 3, 5, 7, 13, 17, 19, 31, 61, 89,$ 107, 127, 521, 607, 1279, 2203, and 2281.

D. H. L.

CORRIGENDA

v. 6, p. 129, l. 22 *for* $r_i(1 - \rho_{i-1})$ *read* $r_i(1 + \rho_{i+1})$
v. 6, p. 132, l. − 6 and − 18 *for* Thompson *read* Thomson
v. 6, p. 152, l. −5 *for* 8 *read* 9
v. 6, p. 187, l. 12 *for* QVAC *read* QUAC
v. 6, p. 189, l. − 8 *for* Connoly *read* Connolly