card. Approximately eight cards per minute can be calculated, with eight digit $N$'s and square roots to 4D.

The answer from the previous card is taken as $x_0$. The chief problem in the wiring is to obtain a starting value for the first card, since the 602A performs all the programming wired, in dummy form, before the first card is read. If the cards are in descending order on $N$, the first starting value can be taken as $N$, provided that the dummy programming can be skipped. This latter can be accomplished by wiring to "read" from an early program through the normal side of a selector which is then latched for the remainder of the run.

Diagrams of the setups used at the Numerical Analysis Laboratory will be published shortly.

J. Hollingsworth
F. Gruenberger

Numerical Analysis Laboratory
University of Wisconsin
Madison, Wisconsin

**151. A Numerical Study of a Conjecture of Kummer.**—The generalization to the cubic case of the well-known (quadratic) Gauss sum was first investigated by Kummer.[1] He showed that the expression

$$x_p = 1 + 2 \sum_{\nu=1}^{(p-1)/2} \cos (2\pi\nu^3/p) \tag{1}$$

for all $p \equiv 1 \pmod 3$ satisfies the cubic equation

$$f(x) = x^3 - 3px - pA = 0 \tag{2}$$

where $A$ is uniquely determined by the requirements

$$4p = A^2 + 27B^2, \qquad A \equiv 1 \pmod 3). \tag{3}$$

Equation (2) clearly has three real roots for each $p$. Kummer classified the primes $p \equiv 1 \pmod 3$ according to whether the Kummer sum is the largest, middle or smallest root of equation (2). He conjectured that the asymptotic frequencies for these classes of $p$ are (in the order above) $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{6}$. To check this surmise he calculated the first 45 of the $x_p$ and found the densities to be .5333, .3111, .1556.

This problem was brought to the attention of the authors by E. Artin who suggested the desirability of further testing the conjecture since its truth would have important consequences in algebraic number theory.

Accordingly the primes $p \equiv 1 \pmod 3$ from 7 through 9,973 were tested. We give below a summary of the resulting densities. In this tabulation we have arbitrarily divided the primes into six groups of 100 each, designated by I, $\cdots$, VI and a final group of eleven primes, VII.

Number of primes $p \equiv 1 \pmod 3$ such that

| Group | $x_p$ is the largest root | $x_p$ is the middle root | $x_p$ is the smallest root |
|---|---|---|---|
| I | 54 | 28 | 18 |
| II | 41 | 38 | 21 |
| III | 46 | 33 | 21 |
| IV | 39 | 32 | 29 |
| V | 43 | 29 | 28 |
| VI | 44 | 38 | 18 |
| VII | 5 | 3 | 3 |
| Total | 272 | 201 | 138 |
| Density | .4452 | .3290 | .2258 |

These results would seem to indicate a significant departure from the conjectured densities and a trend toward randomness.

The method of calculation was this: Each root of (2) lies in one of the intervals $(-2p^{\frac{1}{2}}, -p^{\frac{1}{2}})$, $(-p^{\frac{1}{2}}, +p^{\frac{1}{2}})$, $(p^{\frac{1}{2}}, 2p^{\frac{1}{2}})$ as may be seen directly from the form of (2) with the help of (3). For each relevant $p$ the expression (1) for $x_p$ was evaluated, its sign was determined and its square compared to $p$. This determined in which of the three intervals just described the $x_p$ lies. To check that $x_p$ was indeed a solution of (2), (3) and to determine the precision of the evaluation the expression

$$f(x_p)/f'(x_p) \tag{4}$$

was then calculated. This latter check was performed by first transforming the $x_p$ into decimal form for tabulation and then retransforming these results back into binary form before evaluating the expression (4). In this manner both the calculation proper and the conversion to decimal form of the results were checked.

The trigonometric expressions appearing in (1) were evaluated by power series. Each angle was reduced mod $2\pi$ and then mod $\pi$ until it lay between $-\pi/2$ and $+\pi/2$. Then the cosine of $\frac{1}{4}$ of this angle was calculated keeping five terms in the series expansion. The "double-angle" formula for cosines was then used twice to obtain the desired cosine.

The calculation involved about 15 million multiplications counting the checking mentioned above. The values of $p$ were introduced in blocks of 200. The entire calculation was carried out twice to ensure reliability. The authors are indebted to Mrs. ATLE SELBERG who programmed and coded the calculation.

J. VON NEUMANN
H. H. GOLDSTINE

Institute for Advanced Study
Princeton, New Jersey

[1] E. E. Kummer, "De residuis cubicis disquisitiones nonnullae analyticae," *Jn. f. d. reine u. angew. Math.*, v. 32, 1846, p. 341–365.

## CORRIGENDA

v. 6, p. 262, *insert* Emch, G. F. 247.
v. 6, p. 265, under Myers *insert* 54.
v. 6, p. 268, under Yowell *insert* 254.
v. 7, p. 31, l. 11 of MTE 218; *for* $-\log p$ *read* $\log p$.