The differences of the equally-spaced values have been entered. It is required to find the zero of $f(y)$. If it were certain that the three last values of $f(y)$ are correct, and that $\Delta^3 y$ is negligible, this zero could be easily obtained by quadratic inverse interpolation. However, it happens that the computation of $f(y)$ is quite laborious, and it is desirable to check the accuracy of $f(y)$. Let us obtain modified divided differences, using the interval $w = .001$, since in this problem we merely want to have an indication of the magnitude of $\Delta^2 f$ in this region, at an interval of .001 in y. The values follow.

$w = .001$

y	$f(y)$	δ_w	δ_w^2
3.7416573868	+.00824 2550		
		−232201	
3.777	+.00003 5971		80
		−230753	
3.778	−.00019 4782		79
		−230674	
3.779	−.00042 5456		

The fact that the two values of δ_w^2 differ by only one unit is assurance that the values of $f(y)$ are correct, and that quadratic inverse interpolation is adequate. The solution is $y_0 = 3.77715\ 586$, to eight decimal places. [It can be shown that $\lambda = \frac{1}{4}y_0^2$ satisfies the system $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \lambda\right) u = 0$, $u = 0$ on the boundary C, $u > 0$ in interior of region where C is the ellipse $\frac{1}{4}x^2 + y^2 = 1$.]

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[1] L. M. MILNE-THOMSON, *The Calculus of Finite Differences.* London and New York, 1933.

[2] The numbers u_1 and u_2 will be said to have the same order of magnitude if $1/10 < |u_1/u_2| < 10$.

[3] See J. C. P. MILLER, "Checking by Differences I," *MTAC*, v. 4, 1950, p. 3–11.

A Minimum Problem Solved by Mesh Methods

Introduction. In the following, the function $y(x)$, subject to $y(0) = y(1) = 1$, is sought which will minimize the integral

$$I = \int_0^1 y^{-1}(1 + y'^2)^{\frac{1}{2}}\,dx. \tag{1}$$

This problem can be solved by the usual methods of the calculus of variations[1] but the differential equation involved is rather complicated. It is proposed here to solve the problem by mesh or "assumed polynomial" methods.[2] Methods of this sort have been rather extensively applied to the solution of differential equations and boundary value problems, and, recently, also to the determination of characteristic numbers or eigenvalues.[3]

Basic Equations. To solve the above problem by mesh methods it is first assumed that the interval $0 \leq x \leq 1$ is divided into segments each of equal length $h = 1/(2n)$ by the points x_i, $i = 0(1)2n$. It should be noted

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that the solution will be symmetric about $x = \frac{1}{2}$; hence only the portion $0 \leq x \leq \frac{1}{2}$ need be considered together with the condition

$$y'(\tfrac{1}{2}) = 0. \tag{2}$$

We then write, using approximate integration,

$$I \approx 2 \sum_{i=1}^{n} 2hr_i/(y_i + y_{i-1}) \tag{3}$$

where

$$r_i = \{1 + h^{-2}(y_i - y_{i-1})^2\}^{\frac{1}{2}} \tag{4}$$

so that I is now a function of y_j for $j = 0(1)n$. At a minimum we must have

$$\frac{\partial I}{\partial y_k} = 0, \quad k = 1, 2, \cdots, n - 1. \tag{5}$$

If we write

$$s_k = y_k + y_{k-1}, \qquad g_k = (r_k/s_k) + (r_{k-1}/s_{k-1}),$$
$$\rho_k = s_{k+1}r_{k+1}/(s_k r_k), \quad e_k = -h^2 r_k s_k g_k,$$

we can write equation (5) as

$$y_{k+1} = y_k + \rho_k(y_k - y_{k-1}) + e_{k+1}. \tag{6}$$

The process of solution followed here is to assume a curve $y(x)$ and use this to determine the ρ's and e's at mesh points. Then (6) provides a recurrence relation for determining new values of $y_2, y_3, \cdots, y_n$ which we write as $Y_2, Y_3, \cdots, Y_n$ since the mid-point slope condition (2) is not as yet satisfied. Using second order deriving coefficients as given by SOUTHWELL[2] or MILNE[4] (p. 96–98) we calculate Y_n' which usually is not zero. Then, letting $y_i' = Y_i' - Y_n'$ so as to satisfy $y_n' = 0$ we integrate y_i' by use of integrating coefficients[2,4] to get a new set of y's. The above process is then repeated.

The accompanying table shows results for three trials using $h = 1/8$ and starting with $y \equiv 1$. An approximate value of $\frac{1}{2}I$ was found for each assumed curve $y(x)$. These successive values, i.e. .5, .48148, .48144, indicate a minimum being reached.

	k	y_k	$y_k - y_{k-1}$	r_k	s_k	ρ_k	g_k	e_k	Y_k	Y_k'	y_k
	0	1.00000	—	—	—	—	—	—	1.00000	+.0625	1.00000
	1	1.00000	.00000	1.00000	2.00000	1.00000	—	—	1.00000	−.0625	1.05469
Run 1	2	1.00000	.00000	1.00000	2.00000	1.00000	.50000	−.015625	.984375	−.1875	1.09375
	3	1.00000	.00000	1.00000	2.00000	1.00000	.50000	−.015625	.953125	−.3125	1.11719
	4	1.00000	.00000	1.00000	2.00000	1.00000	.50000	−.015625	.906250	−.4375	1.12500
	0	1.00000	—	—	—	—	—	—	1.00000	.50504	1.00000
	1	1.05469	.05469	1.09152	2.05469	1.003638	—	—	1.05469	.37000	1.05362
Run 2	2	1.09375	.03906	1.04769	2.14844	.99937	.48553	−.01708	1.09250	.24124	1.09071
	3	1.11719	.02344	1.01743	2.21094	.99870	.43512	−.01529	1.11500	.12268	1.11241
	4	1.12500	.00781	1.00195	2.24219	—	.40744	−.01430	1.12347	.00804	1.11953
	0	1.00000	—	—	—	—	—	—	1.00000	.49656	1.00000
	1	1.05362	.05362	1.08812	2.05362	1.00096	—	—	1.05362	.36136	1.05358
Run 3	2	1.09071	.03709	1.04309	2.14433	.99971	.48486	−.01695	1.09034	.23280	1.09064
	3	1.11241	.02170	1.01496	2.20312	.99976	.43596	−.01523	1.11182	.11448	1.11234
	4	1.11953	.00712	1.00162	2.23194	—	.41018	−.01433	1.11896	−.00024	1.11947

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Conclusions. It can be seen from the table that the process converges with fair rapidity. Further calculations by the author indicate that the results obtained after three trials contain an error of less than .001 for $y(\frac{1}{2})$. If the usual methods of the calculus of variations were employed the resulting non-linear differential equation would presumably have to be solved by finite difference methods anyway and it does not appear that this would be as easy a computation to carry through as the above.

In writing (3), first order divided differences have been used, these being the simplest and at the same time adequate. Higher order expressions for the derivatives may be employed but will in general result in more complicated recurrence relations. The iterative procedure for the solution apparently must be devised anew for each different class of problems.

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[1] R. WEINSTOCK, *Calculus of Variations*. New York, 1952.
[2] R. V. SOUTHWELL, *Relaxation Methods in Theoretical Physics*. Oxford, 1946.
[3] L. COLLATZ, *Numerisches Behandlung von Differentialgleichungen*. Berlin, 1951.
[4] W. E. MILNE, *Numerical Calculus*. Princeton, 1949.

A Modification of the Aitken-Neville Linear Iterative Procedures for Polynomial Interpolation

A. C. AITKEN[1] has described a method of interpolation which is equivalent to the use of Lagrange's polynomial formula but consists principally of the repeated computation of an attractively simple algorithm, well suited to desk calculators. The method does not require uniform spacing in the values of the argument at the points at which the values of the required function are given, though uniformity permits of a convenient check of some aspects of the calculation. It can therefore be used for both direct and inverse interpolation, and it is particularly valuable for the latter. The procedure depends on the following property, which is also discussed at greater length by E. H. NEVILLE[2] and by W. E. MILNE:[3]

On the basis of the known values of a function u at n values of the argument X—that is, at a point P and $n - 1$ other points, denoted collectively by Q—we may have obtained a polynomial interpolate u_{PQ} of degree $n - 1$ for the value of the function at some other point $X = x$. We may have obtained also another interpolate u_{QR} of degree $n - 1$, for the same value x, on the basis of the $n - 1$ points Q and a further point R. Then the polynomial interpolate of degree n for $X = x$, based on all $n + 1$ points P, Q, R, is

$$u_{PQR} = \begin{vmatrix} u_{PQ} & x_P - x \\ u_{QR} & x_R - x \end{vmatrix} \Big/ (x_R - x_P).$$

In this formula, x_P and x_R are the values of the argument at the points P and R respectively. It is easily computed on most desk calculators, espe-

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