Conclusions. It can be seen from the table that the process converges with fair rapidity. Further calculations by the author indicate that the results obtained after three trials contain an error of less than .001 for $y(\frac{1}{2})$. If the usual methods of the calculus of variations were employed the resulting non-linear differential equation would presumably have to be solved by finite difference methods anyway and it does not appear that this would be as easy a computation to carry through as the above.

In writing (3), first order divided differences have been used, these being the simplest and at the same time adequate. Higher order expressions for the derivatives may be employed but will in general result in more complicated recurrence relations. The iterative procedure for the solution apparently must be devised anew for each different class of problems.

G. C. Best

3717 Crestline
Fort Worth, Texas

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[2] R. V. Southwell, *Relaxation Methods in Theoretical Physics*. Oxford, 1946.
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A Modification of the Aitken-Neville Linear Iterative Procedures for Polynomial Interpolation

A. C. Aitken[1] has described a method of interpolation which is equivalent to the use of Lagrange's polynomial formula but consists principally of the repeated computation of an attractively simple algorithm, well suited to desk calculators. The method does not require uniform spacing in the values of the argument at the points at which the values of the required function are given, though uniformity permits of a convenient check of some aspects of the calculation. It can therefore be used for both direct and inverse interpolation, and it is particularly valuable for the latter. The procedure depends on the following property, which is also discussed at greater length by E. H. Neville[2] and by W. E. Milne:[3]

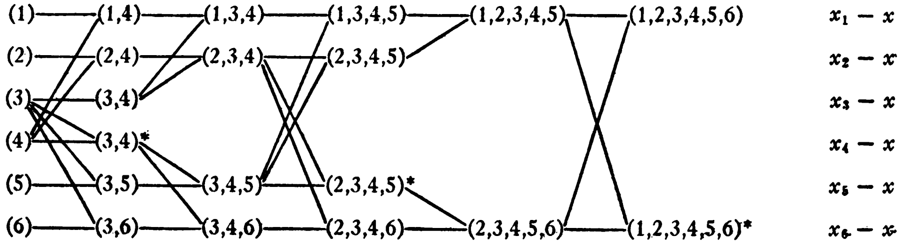
On the basis of the known values of a function u at n values of the argument X—that is, at a point P and $n - 1$ other points, denoted collectively by Q—we may have obtained a polynomial interpolate u_{PQ} of degree $n - 1$ for the value of the function at some other point $X = x$. We may have obtained also another interpolate u_{QR} of degree $n - 1$, for the same value x, on the basis of the $n - 1$ points Q and a further point R. Then the polynomial interpolate of degree n for $X = x$, based on all $n + 1$ points P, Q, R, is

$$u_{PQR} = \begin{vmatrix} u_{PQ} & x_P - x \\ u_{QR} & x_R - x \end{vmatrix} \Big/ (x_R - x_P).$$

In this formula, x_P and x_R are the values of the argument at the points P and R respectively. It is easily computed on most desk calculators, especially as the divisor $x_R - x_P$ can be found (or checked), without extra work, as the net total of the two multipliers $x_R - x$ and $-(x_P - x)$ which are used in evaluating the determinant. Moreover, the linearity of the formula permits u_{PQ} and u_{QR} to be adjusted equally by any suitable amount before interpolation, thus minimizing the number of digits to be used at each stage. We use the term 'interpolation' loosely here to include extrapolation, which strictly occurs when $x_P - x$ and $x_R - x$ have the same sign. Aitken[1] calls this elementary process 'linear iteration by proportional parts,' and has also (according to WOMERSLEY[4]) called the derived value of u_{PQR} a 'linear cross-mean' of u_{PQ} and u_{QR}.

This iterative linear process or algorithm can therefore be used to increase the order of approximation of an interpolated value for a function, starting with a simple linear interpolation between two known values ($n = 1$) and progressively incorporating others. The final result, when the same known values have all been used, will necessarily be the same (apart from rounding-off variations) regardless of the order in which they are incorporated. The algorithm will be computed $\frac{1}{2}n(n - 1)$ times to incorporate n points, but with a suitable schedule considerable economies and simplifications can be effected. Aitken[1] proposes a schedule in which the work sheet for the computations can be laid out in such a way that the values required for each determinant always appear at the corners of rectangles. Neville[2] proposes a schedule which lacks this advantage but which usually gives a more rapidly convergent process. Neville's schedule permits of the addition of further known points, if desired, after the completion of the calculation of an interpolate, with greater freedom than Aitken's does, owing to the fundamental asymmetry of the latter. As Neville[2] and KINCAID[5] have demonstrated, Neville's method is also readily adapted to utilize the known values of differential coefficients at points at which the function is known.

The convenience of Aitken's schedule as regards the determinants, and much of the convenience of Neville's as regards the latitude in the addition of further points, are possessed by a third schedule which is also usually a little more rapidly convergent than Neville's in the early stages. With a condensed notation, using $(i, j, k, \cdots)$ to stand for the interpolate based on the points $X = x_i, x_j, x_k, \cdots$, the new procedure follows the schedule shown diagrammatically below for the case $n = 6$. The schedule for smaller values of n is the appropriate fragment of this one, and the procedure generalizes immediately to greater n. The cross products involving each line are taken with the items nearest the axis of symmetry of the schedule, and alternately on the further and nearer sides of this axis for successive columns, as is indicated by the lines in the diagram. In this diagram, the value of each interpolate $(i, j, k, \cdots)$ is written on the same line as one of the values on which it is directly based. Certain entries, marked by asterisks, are duplicated so that the values used in the determinant always occur at the corners of a rectangle. These entries can be computed separately as a partial check without interrupting the rhythm of the process. Incidentally, the geometrical interpretation given by Aitken[1] (p. 73–74) for his schedule can be adapted to represent the new schedule, and also to represent Neville's.



The new schedule can be regarded as a modification of Aitken's to introduce properties of symmetry which are often—though not always—desirable in an interpolation formula. Both for ease of computation and for accuracy of interpolation of a given order, it is normally most satisfactory when the numbers of given points on each side of the required point are either equal or differ by unity. That is to say, the required point should lie in the interval enclosed by the two given points which lie nearest to the axis of symmetry of the general pattern. Then the linear cross-mean procedure used at each stage is equivalent to an interpolation (for the columns of interpolates of odd degree) or an extrapolation (for the columns of interpolates of even degree) on a straight line fitted as a chord to a curve representing the previous column of values, and this concept is useful in judging the closeness of approximation in the new method. Moreover, the latter is designed to make the fullest use of the points nearest the required value from the earliest stages onwards, and to use them for interpolation rather than for extrapolation. Both of these features normally increase the rate of convergence, and thus facilitate both the computations and the detection of errors.

In the six-point example used by Milne[3] to illustrate Aitken's and Neville's methods, the range of variation of the results in the earlier columns is considerably reduced under the new schedule. We shall not, however, use this as an example here, since the given values of the function are at uniformly spaced values of the argument, and the NBS *Tables of Lagrangian Interpolation Coefficients*[6] enable the final interpolate to be found in two minutes easily with a calculator. The time taken to compute the fifteen linear cross-means under Aitken's or Neville's or the new schedule cannot practicably be reduced much below ten minutes (including writing down the necessary digits of the intermediate interpolates) with a fast electrical desk calculator.

Whittaker & Robinson[7] demonstrated a method of inverse interpolation for finding the positive root of the equation $u^7 + 28u^4 - 480 = 0$, given the values of $x = u^7 + 28u^4 - 480$ at the five points $u = 1.90(.01)1.94$. Aitken[1] (p. 71) and Neville[2] (p. 94–97) used this example to illustrate their schedules of linear iteration by proportional parts. Working to 10D for u, with 8D for x, the new schedule may be applied to solve this problem as follows:

u_i					$x - x_i$
	$u = 1.9$	22	88	41	
1.90	2271 69289	88 48558	41731	527	−25.71402 610
1.91	2279 04030	88 45462	41643		−14.62541 674
1.92	2286 32562				− 3.30746 392
1.93	2286 32562*				+ 8.24394 354
1.94	2283 41050	88 36413	41643*		+20.03298 301

Thus the estimated root is $u = 1.92288\ 41527$, which differs by six units in the last place from the correct 10D value, of which the last two digits are 33. The complete inverse interpolation process takes about 12 to 15 minutes on the desk calculator referred to above. Further values of x_i, say for $u_i = 1.95$ first, followed by $u_i = 1.89, 1.96, 1.88, 1.97$, etc., could be added if desired, and the existing results incorporated completely into the extended schedule up to the point at which the number of decimal places carried from the start ceases to justify it. In this example not more than two values can advantageously be added.

M. C. K. Tweedie

Statistical Laboratory
Virginia Agricultural Experiment Station
Virginia Polytechnic Institute
Blacksburg, Virginia

[1] A. C. Aitken, "On interpolation by iteration of proportional parts, without the use of differences." Edinburgh Math. Soc., *Proc.*, ser. 2, v. 3, 1932, p. 56–76.

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[3] W. E. Milne, *Numerical Calculus*. Princeton University Press, 1949, chap. III.

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[6] NBSMTP, *Tables of Lagrangian Interpolation Coefficients*. New York, Columbia University Press, 1944.

[7] E. T. Whittaker & G. Robinson, *The Calculus of Observations*. Fourth ed., London, 1948, chap. IV.

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1144[F].—J. Touchard, "On prime numbers and perfect numbers," *Scripta Math.*, v. 19, 1953, p. 35–39.

This note contains a small table of the function

$$C_p(n) = \sum_{k=1}^{n-1} k^p \sigma(k)\sigma(n-k)$$

for $p = 0(1)3$ and $n = 2(1)11$. Here $\sigma(k)$ denotes the sum of the divisors of k. These coefficients occur in the expansion of certain elliptic functions. More specifically,

$$\sum_{n=1}^{\infty} C_p(n)x^n = \Phi_{p-1,\,p}\Phi_{0,\,1}$$

where

$$\Phi_{r,\,s} = \sum_{n,m=1}^{\infty} m^r n^s x^{mn}.$$

D. H. L.

1145[I].—E. L. Kaplan, "Numerical integration near a singularity," *Jn. Math. Phys.*, v. 31, 1952, p. 1–28.

The types of singularities considered in this paper are of the forms

$$x^{\frac{1}{2}}A(x), \quad x^{-\frac{1}{2}}A(x), \quad x^n\{A(x)\log x + B(x)\}, \quad n = 0, 1,$$