NOTES

157.—Analytical Approximations. [See also Note **153.**] The first twelve approximations concern the functions $e^{-x}I_0(x)$ and $e^{-x}I_1(x)$ in which $I_0(x)$ and $I_1(x)$ are the usual Bessel functions of imaginary argument.

(35) To better than .0006 over $(0, \infty)$,

$$e^{-x}I_0(x) \doteq \left(\frac{1 + .297x + .341x^2}{1 + 2.333x + 2.137x^2 + 2.096x^3}\right)^{\frac{1}{2}}.$$

(36) To better than .00005 over $(0, \infty)$,

$$e^{-x}I_0(x) \doteq \left(\frac{1 + .302x + .234x^2 + .114x^3}{1 + 2.2979x + 2.3871x^2 + 1.2032x^3 + .7183x^4}\right)^{\frac{1}{2}}.$$

(37) To better than .000,009 over $(0, 1)$,

$$e^{-x}I_0(x) \doteq (1 + .0302x + .0889x^2)/(1 + 1.0299x + .3728x^2).$$

(38) To better than .0001 over $(0, 2)$,

$$e^{-x}I_0(x) \doteq (1 + .1693x + .0844x^2)/(1 + 1.1665x + .5247x^2).$$

(39) To better than .0007 over $(0, 4)$,

$$e^{-x}I_0(x) \doteq (1 + .4537x + .0955x^2)/(1 + 1.4387x + .8855x^2).$$

(40) To better than .0005 over $(0, \infty)$,

$$e^{-x}I_1(x) \doteq x(3.78 + 9.81x + 3.09x^2 + 6.36x^3)^{-\frac{1}{2}}.$$

(41) To better than .0003 over $(1, \infty)$,

$$e^{-x}I_1(x) \doteq x(4.51 + 9.00x + 3.25x^2 + 6.37x^3)^{-\frac{1}{2}}.$$

(42) To better than .00005 over $(2, \infty)$,

$$e^{-x}I_1(x) \doteq x(10.281 + 3.752x + 4.541x^2 + 6.296x^3)^{-\frac{1}{2}}.$$

(43) To better than .000,006 over $(0, 1)$,

$$e^{-x}I_1(x) \doteq (.49974x - .01695x^2)/(1 + .95935x + .36282x^2).$$

(44) To better than .00007 over $(0, 2)$,

$$e^{-x}I_1(x) \doteq (.4981x + .0066x^2)/(1 + .9805x + .4477x^2).$$

(45) To better than .0003 over $(0, 4)$,

$$e^{-x}I_1(x) \doteq (.4935x + .0268x^2)/(1 + .9667x + .5373x^2).$$

(46) To better than .00008 over $0 \leq x \leq \infty$,

$$e^{-x}I_1(x) \doteq x(15.4 + 74.8x + 67.2x^2 + 235.8x^3 + 43.5x^4 + 59.4x^5 + 39.6x^6)^{-\frac{1}{2}}.$$

(47) Definite Integral: To better than .00055 over $0 \leq x \leq \infty$,

$$N(x) = 30\pi^{-4}\int_0^\infty t^7(e^{t^2} - 1)^{-1}e^{-(x/t)^7}\,dt$$

$$\doteq (1 + .38382x^6 - .55605x^7 + .34791x^8 - .10369x^9 + .01245x^{10})^{-1}.$$

(48) Mach Number in Terms of Pressure Ratio: To better than .0011 over $.3 \leq M \leq 1.0$, the inverse of (a) is (b).

(a) $x = P_S/P_R = [1 + (\gamma - 1)M^2/2]^{-\gamma/(\gamma-1)}, \quad \gamma = 1.4.$

(b) $M \doteq (2.714 - 2.625x)/(1 + 1.650x - 1.955x^2).$

(49) Mach Number in Terms of Pressure Ratio: To better than .0014 over $1 \leq M \leq 3$, the inverse of (c) is (d).

(c) $x = P_S/P_R = [2\gamma M^2/(\gamma + 1) - (\gamma - 1)/(\gamma + 1)]^{1/(\gamma-1)} \times [(\gamma + 1)M^2/2]^{-\gamma/(\gamma-1)}, \quad \gamma = 1.4.$

(d) $M \doteq (8.19 + 29.40x - 24.58x^2)/(1 + 30x).$

(50) Mach Number in Terms of Pressure Ratio: To better than .0021 over $.3 \leq M \leq 3$, the inverse of (a) over $.3 \leq M \leq 1$ conjoined with (c) over $1 \leq M \leq 3$ is given by

$$M \doteq (8.11 + 23.60x - 39.66x^2 + 8.98x^3)/(1 + 28.70x - 15.99x^2 - 5.74x^3).$$

Essentially the same approximation was reported to us by Mr. PHILIP RAPP.

(51) Pearson Cosine Transformation: To better than .0014 over $0 \leq x \leq 1$,

$$r(x) = \cos \pi/(1 + x^{1/2}) \doteq (-1 - 7.47x + 8.47x^2)/(1 + 11.65x + 12.05x^2).$$

$r(x^{-1}) = -r(x)$ can be used to obtain function values over $1 \leq x \leq \infty$.

(52) Pearson Cosine Transformation: To better than .00017 over $0 \leq x \leq 1$,

$$r(x) \doteq \frac{-1 - 4.828\eta + 7.866\eta^2 - 2.038\eta^3}{1 + 5.560\eta - 4.985\eta^2 + .385\eta^3}, \quad \eta = \frac{x}{.16 + .84x}.$$

(53) Natural Addition Logarithm: To better than .00026 over $0 \leq x \leq \infty$, $\ln(1 + e^{-x}) \doteq (1 + .3581x + .1151x^2 + .0094x^3 + .0052x^4)^{-2} \ln 2.$

(54) Natural Addition Logarithm: To better than .000,045 over $0 \leq x \leq \infty$,

$$\ln(1 + e^{-x}) \doteq (1 + .36123x + .10204x^2 + .02411x^3 - .00055x^4 + .00069x^5)^{-2} \ln 2.$$

(55) Natural Addition Logarithm: To better than .000,008 over $0 \leq x \leq \infty$, use constants .360571, .105546, .018760, .002654, −.000100, .000066 in next form of sequence.

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158.—COUNTS OF TWIN PRIMES LESS THAN 100000. The writer has constructed a table making the counting of twin primes relatively easy. The counts differ from those of previous observers to such an extent that it seems desirable to publish the results.

The previous tables are identified merely by giving the author's names. The exact references are given in LEHMER's *Guide.*[1]

Table I shows the various counts of twin primes for each of the 10 myriads. Following GLAISHER 1 and 3 are counted as pair of twin primes.

TABLE I

Myriad	SUTTON	STÄCKEL	POLETTI	GLAISHER & SEXTON
1	206	206	203	206
2	137	137	135	137
3	125	125	125	125
4	124	124	124	124
5	113	114	113	114
6	106	106	105	106
7	93	94	94	94
8	102	102	102	102
9	108	109	110	109
10	108	108	105	108
Totals	1222	1225	1216	1225

The results of GLAISHER agree exactly with those of the author. POLETTI gives a total of 1217 but the figures he gives add in fact to 1216.

The total 1225 is in agreement with HARDY & LITTLEWOOD.

Table II shows the distribution of these twin primes as to their final digits.

TABLE II

	Digital Endings			
Myriad	(1, 3)	(7, 9)	(9, 1)	*Total*
1	68	64	72	204
2	49	46	42	137
3	42	45	38	125
4	46	39	39	124
5	36	40	38	114
6	39	32	35	106
7	24	36	34	94
8	34	33	35	102
9	33	35	41	109
10	31	44	33	108
Totals	402	414	407	1223
POLETTI	401	411	403	1215

The reference here is to POLETTI 2, Table XVII.

The author has made a count of double prime pairs like 191, 193, 197, 199. The number n_r of these in the r-th myriad is as follows:

r:	1	2	3	4	5	6	7	8	9	10
n_r:	11	7	3	2	1	2	3	3	3	2

There are in all 37 such double pairs. The set 1, 3, 7, 9 is not counted.

Primes of the forms $6n - 1$ and $6n + 1$ have also been counted and are compared with Poletti in Table III.

TABLE III

Myriad	$6n - 1$ Poletti	$6n - 1$ Sexton	$6n + 1$ Poletti	$6n + 1$ Sexton
1	616	616	614	611
2	519	520	513	513
3	498	497	486	486
4	479	479	479	479
5	463	463	467	467
6	466	466	458	458
7	449	449	429	429
8	447	447	455	455
9	437	436	442	440
10	432	433	444	446
Totals	4806	4806	4787	4784

The total 4784 agrees with the count given in CUNNINGHAM 28.

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[1] D. H. LEHMER, *Guide to Tables in the Theory of Numbers.* Nat. Res. Council, *Bull.* no. 105. Washington, 1941.

159.—A NEW APPROXIMATION TO THE RECIPROCAL OF π. In 1949 I decided to extend to about 1120D my previously published[1] approximation to π^{-1}. To attain a completely reliable result it was decided to evaluate the reciprocal of π by two methods: first, the summation of a suitable highly convergent series; and second, machine division of unity by an approximation to π that LEVI B. SMITH and I had computed to about 1120D[2] in extension of an 808D value we had previously published[3] in collaboration with D. F. FERGUSON.

Smith computed by desk machine the sum of the following series due to RAMANUJAN:[4]

$$\frac{4}{\pi} = \frac{1123}{882} - \frac{22583}{882^3}\frac{1}{2}\frac{1\cdot 3}{4^2} + \frac{44043}{882^5}\frac{1\cdot 3}{2\cdot 4}\frac{1\cdot 3\cdot 5\cdot 7}{4^2\cdot 8^2} - \cdots$$

Each term, omitting the coefficients 1123, 22583, $\cdots$, in arithmetic progression, was evaluated to 1130D and then these coefficients were introduced in the calculation to complete the summation. Two independent congruential checks were applied at intervals of 100D at every stage of this calculation.

Concurrently I used the ENIAC approximation[5] to π as the divisor in an independent evaluation of π^{-1}. Smith's final result was compared with mine in October 1952 and complete agreement through 1123D was observed, thereby verifying to a comparable degree of accuracy the ENIAC determination of π.

Subsequently I continued the desk machine reciprocation of π to 1408D. As an exercise on the UNIVAC the complete product of this approximation to π^{-1} and the ENIAC 2040D work sheet value of π was computed on 20 April 1953 by two independent routines in an average time of about 8 minutes. The accuracy of my desk machine calculation was confirmed and the extension of π^{-1} to 2037D was at once deduced on the UNIVAC. The final result of these investigations is reproduced below.

π^{-1} = 0.31830	98861	83790	67153	77675	26745	02872	40689	19291	48091
28974	95334	68811	77935	95268	45307	01802	27605	53250	61719
12145	68545	35159	16073	78582	36922	29157	30575	59348	21463
39967	84584	79933	87481	81551	46155	49279	38506	15377	43478
57924	34795	32338	67247	80483	44725	80236	64760	22844	53995
11431	88092	37801	73805	34791	22409	78821	87387	56881	71057
44619	98928	86800	49734	46954	78919	22179	66461	93566	14981
23339	72925	60939	88973	04375	76314	95731	33928	48207	79917
48278	69721	99677	36198	39992	48857	51170	34235	77168	62235
03753	43210	93095	07397	60194	78920	72951	86675	36118	60498
89932	70610	65431	35510	06440	64955	56327	94332	04589	34962
39196	33168	12120	33606	07199	62678	23974	99766	55733	08870
55951	01400	32481	35512	87776	99142	62176	02443	98752	29536
27555	29475	78126	61360	92915	95696	35226	24854	62813	99215
50049	00059	55197	14178	11380	55935	70263	05042	00326	35492
04184	96232	12481	12291	24062	92968	17849	69183	82870	42315
08151	12401	74305	32136	04434	31828	15149	49165	44519	54925
70799	75031	06587	81627	96354	48187	16509	59414	66574	38081
39995	18153	15415	69869	40787	17965	61743	46851	28073	37902
33250	91411	88665	52625	37300	05224	54359	42306	42251	99008
77335	89007	52511	21672	63423	39051	95162	56449	88324	66686
29021	22470	73757	12622	72733	84334	28413	94939	20258	50115
66721	06239	21718	90196	79113	43741	99094	93020	86324	76310
35161	67888	59599	41999	01050	87751	32258	89176	66136	92101
57058	30302	82080	97859	77012	77632	15523	93986	14682	07799
91573	83781	19618	74755	44123	75086	44543	78602	73251	05224
77560	77507	77622	13628	13530	86816	56557	05386	68535	99112
14158	07721	20705	47799	24902	51991	49855	25940	47188	19116
86023	29659	28237	11554	24811	50889	89140	43579	53958	48189
80654	58954	04332	99207	13063	63070	88007	68137	97494	35383
17752	63819	33013	92880	95539	41375	36731	35562	09559	59090
07067	91516	60376	36773	75875	53224	96299	06119	93116	04381
67197	50207	02542	58086	46316	09974	39373	75551	89313	26924
42068	40888	17109	95700	75854	77388	58707	32387	55658	57471
87568	69406	46047	42916	75847	11423	72726	83858	92036	63645
83928	33001	75661	58662	70699	55819	94917	29858	05349	01219
78737	81891	76610	06740	61076	10946	24643	16188	63953	52064
56626	28379	61949	96448	76670	34871	39796	95002	07900	13677
60079	57344	71992	16048	00547	80217	49909	70957	58471	36522
27989	78065	37994	85416	69922	29841	65780	75535	69486	07100
91369	12167	34295	86169	13446	65407	09707	85		

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[1] H. S. Uhler, "Log π and other basic constants," Nat. Acad. Sci., *Proc.*, v. 24, 1938, p. 23–30.

[2] J. W. Wrench, Jr. & L. B. Smith, "Values of the terms of the Gregory series for arccot 5 and arccot 239 to 1120 and 1150 decimal places, respectively," *MTAC*, v. 4, 1950, p. 160–161.

[3] J. W. Wrench, Jr. & L. B. Smith, "A new approximation to π," *MTAC*, v. 2, 1947, p. 245–248; v. 3, 1948, p. 18–19.

[4] Srinivasa Ramanujan, "Modular equations and approximations to π," *Quart. Jn. Math.*, v. 45, 1914, p. 350–372, *Collected papers of Srinivasa Ramanujan*. Cambridge, 1927, p. 23–39.

[5] George W. Reitwiesner, "An ENIAC determination of π and e to more than 2000 decimal places," *MTAC*, v. 4, 1950, p. 11–15.

QUERY

43. Incomplete Hankel Function. Has the Incomplete Hankel function

$$f(x, y) = \int_x^\infty dt \exp(-iyt)/(t^2 - 1)^{\frac{1}{2}}$$

been tabulated? If so, where can such a table be obtained?

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CORRIGENDUM

V. 7, p. 275, l. —2, *for* Mag *read* Mac.