From (7) and (9) it is clear that the only essential information contained in E_l is the set of values η_{il} and the index r_l. Note further that in (8), the successive E_l are used with *increasing* l and it follows from (9) that it *is necessary* to know r_l *before* using the η_{il}. On the other hand, in (6), the E_l are used in *decreasing* sequence of l but from (7) it *is not necessary* to know r_l until *after* the η_{il} have been used. The perfect complementarity of the preceding two sentences, together with the fact that $\sum_0^m \eta_{il} a_i$ can obviously be computed starting with $i = m$ as well as with $i = 0$, makes it clear that (6) may be computed using the information in the reverse order of that used in (8).

Let L_l denote the ordered set of 'words' of information

$$L_l = \{r_l; \eta_{0l}, \eta_{1l}, \cdots, \eta_{ml}\}. \tag{10}$$

Then each change of a column of B will produce a new L_{l+1} which may be stored in consecutive order to the previously computed $L_1, L_2, \cdots, L_l$.

On the CPC, by punching two sets of instructions on each card—one being, in form, the reflection, in the vertical center line, of the other (with appropriate adjustments for difference in algorithms (7) and (9))—the transpose use of the inverse may be accomplished by simply turning the cards over using the vertical center line of the card for the axis.

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[2] T. C. Koopmans, Ed., *Activity Analysis of Production and Allocation.* New York, 1951.
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[3] A. Charnes, W. W. Cooper & A. Henderson, *An Introduction to Linear Programming.* New York, 1953.

[4] A. Hoffman, M. Mannos, D. Sokolowsky & N. Wiegmann, "Computational experience in solving linear programs," Soc. Industrial and Applied Math., *Jn.*, v. 1, 1953, p. 17–33.

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On Modified Divided Differences II

[Continued from *MTAC*, v. 8, p. 1–11]

Errors of Type (c). A question that presents itself is the extent to which errors of Type c will mask an isolated error. It will be desirable to approach the problem from a statistical standpoint, and to introduce the simplifying assumptions that the errors of Type (c) behave like round-off errors, subject to the following restrictions:

(a) The errors e_k in the entries u_k all have the same, uniform distribution between $-\frac{1}{2}a + c$ and $\frac{1}{2}a + c$, where a and c are fixed constants. Thus if entries are rounded to a fixed number of decimal places, the assumption is that the rounding will range uniformly between $\pm\frac{1}{2}$ units of the last place; that is, c is zero. Or, if the entries are "chopped"—that is, all digits beyond a fixed decimal place are dropped, without rounding, then the assumption is that the error in the last place ranges between 0 and unity; that is, $c = \frac{1}{2}$. It will be shown that the distribution function is independent of c, provided c is the same for all u_k.

(b) The errors in the tabular entries are independent of one another. The conditions (a) and (b) constitute a useful idealized model.

For the case of equally-spaced arguments the distribution function of the round-off error in differences of the first, second, and third order has been given explicitly by LOWAN & LADERMAN[1], and the method can be used for obtaining the distribution function for differences of all orders. A somewhat more elaborate study has been published by A. VAN WIJNGAARDEN[2]. We shall here follow the method of Lowan and Laderman, based on Fourier transforms. Consider the sum

$$w_n = A_0e_0 + A_1e_1 + \cdots + A_ne_n, \tag{2.16}$$

where the coefficients A_k are constants, and all the values of e_k are subject to the restrictions (a) and (b). Let $f(w, x)dx$ denote the probability element of the distribution; that is, $\int_{-\infty}^{t} f(w,x)dx = F(w,t)$ is the distribution function of w. For the case when $c = 0$ in condition (a), we have

$$f(e_k, x) = \frac{1}{a}, \text{ if } -\frac{1}{2}a \leq x \leq \frac{1}{2}a; f(e_k, x) = 0, \text{ if } |x| > \frac{1}{2}a. \tag{2.17}$$

$$f(Ae_k, x) = \frac{1}{a|A|}, \text{ if } -\frac{1}{2}a|A| < x < \frac{1}{2}a|A|, \tag{2.18}$$

$$f(Ae_k, x) = 0, |x| > \frac{1}{2}aA, \text{ for constant } A.$$

The characteristic function $g(w, t)$ associated with a distribution function $F(w, t)$ is defined by

$$g(w, t) = \int_{-\infty}^{\infty} e^{itx} f(w, x)dx, \tag{2.19}$$

and by the Fourier inversion theorem

$$f(w, x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx} g(w, t)dt. \tag{2.20}$$

It is known that the characteristic function associated with the distribution of the sum of n independent variables is the product of the characteristic functions associated with the distributions of the n individual variables. This gives, for $w = Ae_k$,

$$g(w, t) = \frac{1}{a|A|}\int_{-\frac{1}{2}a|A|}^{\frac{1}{2}a|A|} e^{itx}\, dx = \frac{\sin(\frac{1}{2}aAt)}{\frac{1}{2}aAt}. \tag{2.21}$$

Hence for w_n defined in (2.16)

$$g(w_n, t) = \prod_{k=0}^{n} \frac{\sin(\frac{1}{2}aA_k t)}{\frac{1}{2}aA_k t}. \tag{2.22}$$

Using (2.20), the frequency function for w_n is given by

$$\begin{aligned} f(w_n, x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx} \prod_{k=0}^{n} \frac{\sin(\frac{1}{2}aA_k t)}{\frac{1}{2}aA_k t}\, dt \\ &= \frac{1}{\pi}\int_{0}^{\infty} \frac{\cos(tx)}{t^{n+1}} \prod_{k=0}^{n} \frac{\sin(\frac{1}{2}aA_k t)}{\frac{1}{2}aA_k}\, dt. \end{aligned} \tag{2.23}$$

The probability that w_n takes on a value between b and c, $c > b$, is then given by

$$\int_b^c f(w_n, x)dx.$$

It is to be noted that the integrand in (2.23) is unchanged when A_k is replaced by $-A_k$; it will therefore be convenient to write

$$\begin{cases} \beta_k = a|A_k| \\ G(x, t) = \cos tx \cdot \prod_{k=0}^{n} \dfrac{\sin(\frac{1}{2}\beta_k t)}{\frac{1}{2}\beta_k}. \end{cases} \tag{2.24}$$

Let

$$G^{(k)}(x, t) = d^k G(x, t)/dt^k; \quad G^{(0)}(x, t) = G(x, t).$$

Then

$$G^{(k)}(x, t) = t^{n+1-k} V(x, t), \quad 0 \le k \le n + 1,$$

where $V(x, t)$ is bounded for all t. Moreover $G^{(k)}(x, \infty)$ is bounded. Integrate (2.23) by parts. This gives

$$f(w_n, x) = \frac{1}{\pi}\left[\frac{-G(x, t)}{nt^n}\right]_0^{\infty} + \frac{1}{\pi}\int_0^{\infty} \frac{G^{(1)}(x, t)}{nt^n}\, dt. \tag{2.25}$$

The term within the bracket vanishes at both limits and only the integral remains on the right-hand side of (2.25). Repeat the process of partial integration until we arrive at

$$f(w_n, x) = \frac{1}{\pi}\int_0^{\infty} \frac{1}{n!} \frac{G^{(n)}(x, t)}{t}\, dt. \tag{2.26}$$

By expanding $G(x, t)$ into a sum of sines (or cosines) and performing the differentiations with respect to t, it can be verified that $f(w_n, x) = 0$ if $|x| \ge \frac{a}{2} \sum |A_k|$, as it should. In the case where the numbers A_k are the binomial coefficients $(-1)^k \binom{n}{k}$, w_n represents the error in the nth ordinary difference of u_n, due to the individual errors e_k. In the case of modified divided differences, the numbers A_k represent $M_{n,\,k+n}$ in even central differences or corresponding coefficients in differences of odd order. It should be observed that in all divided differences, whether modified or not,

$$\sum_r M_{k,r} = 0. \tag{2.27}$$

For the coefficients $M_{k,r}$ are independent of the function u_k. In the special case when $u_k = 1$, the differences must vanish; hence (2.27). As a consequence of (2.27) it is clear that if $e_k = c + \delta_k$, where δ_k is uniformly distributed with mean zero, the constant c drops out in the sum represented by w_n. Hence even when entries have been "chopped" it is sufficient to study the distribution functions associated with δ_k.

For purpose of comparison, the known results for ordinary differences of orders up to three are summarized below, along with some of the corresponding distributions for modified divided differences.

First modified divided difference:

$$w_1 = \frac{w}{x_1 - x_0}(e_1 - e_0) = \frac{e_1 - e_0}{c_1};$$

$$f(w_1, t) = \begin{cases} (c_1/a)^2 \left[\dfrac{a}{c_1} - |t| \right], & -\dfrac{a}{c_1} \le t \le \dfrac{a}{c_1} \\ 0 \text{ elsewhere.} \end{cases} \tag{2.28}$$

If $c_1 = 1$, we have the frequency function for the first ordinary difference.

Second modified divided difference:

$$w_2 = M_{0,-1}e_{-1} + M_{0,0}e_0 + M_{0,1}e_1 = -\frac{2e_0}{c_1 c_0} + \frac{2e_1}{c_1(c_0 + c_1)} + \frac{2e_{-1}}{c_0(c_0 + c_1)}.$$

Assume $c_0 \le c_1$

$$f(w_2, t) = \begin{cases} \dfrac{2a^2 c_0 c_1 - \left[c_0 c_1 \left(\dfrac{c_0 + c_1}{2} \right) \right]^2 t^2}{4a^3}, & \text{if } 0 \le |t| \le \dfrac{2a}{c_1(c_0 + c_1)} \\ \dfrac{c_0}{4a^2} \left[-|t| c_0 c_1 (c_0 + c_1) + a c_0 + 2a c_1 \right], & \text{if } \dfrac{2a}{c_1(c_0 + c_1)} \le |t| \le \dfrac{2a}{c_0(c_1 + c_0)} \\ \left(\dfrac{c_0 + c_1}{2} \right)^2 \left[\dfrac{(t c_0 c_1)^2 - 4|t| c_0 c_1 a + 4a^2}{4a^3} \right], & \text{if } \dfrac{2a}{c_0(c_0 + c_1)} \le |t| \le \dfrac{2a}{c_0 c_1} \\ 0 \text{ elsewhere.} \end{cases} \tag{2.29}$$

Thus $f(w_2, x)$ consists of five curves, symmetric about the y-axis, two of those curves being linear. If $c_0 = c_1 = 1$, each line shrinks to a point, and $f(w_2, t)$ consists of three parabolas.

Third ordinary difference: The following table is taken from LOWAN & LADERMAN[1]:

$$
(2.30) \qquad f(w_3, t) = \begin{cases} \dfrac{1}{a^4}\left[\dfrac{|t|^3}{27} - \dfrac{at^2}{9} + \dfrac{8a^3}{27}\right], & 0 \le |t| \le a \\ \dfrac{1}{a^2}\left[-\dfrac{|t|}{9} + \dfrac{a}{3}\right], & a \le |t| \le 2a \\ \dfrac{1}{a^4}\left[\dfrac{|t|^3}{54} - \dfrac{at^2}{9} + \dfrac{a^2|t|}{9} + \dfrac{5a^3}{27}\right], & 2a \le |t| \le 3a \\ \dfrac{1}{a^4}\left[-\dfrac{|t|^3}{54} + \dfrac{2at^2}{9} - \dfrac{8a^2|t|}{9} + \dfrac{32a^3}{27}\right], & 3a \le t \le 4a \\ 0 \text{ elsewhere.} \end{cases}
$$

There are seven curves in $f(w_3, t)$, two of them being straight lines. Let us consider the frequency function for the third modified divided difference, and write for brevity

$$
w_3 = \sum_{k=0}^{3} A_k e_k; \quad B_k = a|A_k|.
$$

Making use of (2.27), we can put $A_3 = -(A_0 + A_1 + A_2)$. If the arguments x_k form an increasing sequence, the *sign* of A_k is independent of the magnitude of the intervals $(x_k - x_{k-1})$; hence the sign will be the same as in the ordinary third difference, and it is permissible to write

$$
|A_3| = |A_2| - |A_1| + |A_0|.
$$

$$
G(x, t) = \frac{\cos xt[\sin(\frac{1}{2}B_0 t)\sin(\frac{1}{2}B_1 t)\sin(\frac{1}{2}B_2 t)\sin(\frac{1}{2}(B_2 - B_1 + B_0)t]}{[B_0 B_1 B_2(B_2 - B_1 + B_0)]/16}.
$$

By expanding the above into a sum of sines and cosines, it can be verified that

$$
(2.31) \quad G^{(3)}(x, t) = C \sum_{k=1}^{8} \{b_k(x + D_k)^3 \sin[(x + D_k)t] + b_k(x - D_k)^3 \sin[(x - D_k)t]\}.
$$

In (2.31) C is some constant, $b_k = \pm 1$ if $D_k \neq 0$, $b_k = \frac{1}{2}$, if $D_k = 0$, and D_k assumes the following values:

$$
(2.32) \quad D_1 = 0, D_2 = B_2 - B_1, D_3 = B_0, D_4 = (B_2 - B_1 + B_0), D_5 = (B_1 - B_0), D_6 = B_1, D_7 = B_2, D_8 = B_2 + B_0.
$$

Let us consider the special case when

$$
D_1 \le D_2 \le D_3 \le D_4 \le D_5 \le D_6 \le D_7 \le D_8 \le B_0 + B_1 + B_2,
$$

and let x be positive. Since

$$\frac{1}{\pi}\int_0^\infty (\sin bt/t)dt = \tfrac{1}{2} \text{ if } b > 0, \quad -\tfrac{1}{2} \text{ if } b < 0, \text{ and } 0 \text{ if } b = 0,$$

the terms of (2.31) involving $(x + D_k)^3$ will contribute, after integration, terms that have the same sign for all positive values of x between 0 and $(B_0 + B_1 + B_2)$. On the other hand, the terms involving $(x - D_k)^3$ will vary in sign, depending on whether x is greater than or less than D_k. Clearly x can fall into any one of the eight regions separated by the inequalities of (2.32), and in the most general case when no two D_k are equal, the set of terms of (2.31) will comprise, after integration, cubic polynomials (or polynomials of lower degree) with coefficients that will be different in the several regions. Since $f(w_3, x)$ is an even function of x, the curve comprising the eight arcs will be reflected through the y-axis, with a common central arc. There will therefore be 15 arcs to the frequency function for the most general values of D_k. These shrink to seven arcs in the case of ordinary differences. The labor of computing the exact distribution seems to be prohibitive, and alternative approximations will be considered.

It is known that for large enough n, $F(w_n, t)$ tends to approach the normal distribution with mean 0 and standard deviation $\sigma(w_n)$, where

$$\sigma(w_n) = \left(\sum_{k=0}^{n} A_k^2\right)^{\frac{1}{2}} \sigma(e_k) = a\left(\sum_{k=0}^{n} A_k^2/12\right)^{\frac{1}{2}}.$$

In the above, $\sigma(e_k)$ is the standard deviation of e_k. It will be instructive to examine the probabilities that w_3, associated with the third ordinary difference, will fall into certain intervals, and to compare them with the probabilities obtained from the corresponding normal distribution. For w_3, $\sigma(w_3) = 1.29099a$. The following schedule lists some calculations:

Range of w_3	Theoretical Probability Based on (2.30)	Probability Based on Normal Distribution
$2\ \sigma \leq \lvert w_3\rvert \leq 4a$	.03687	.04350
$2.4\sigma < \lvert w_3\rvert \leq 4a$	.00612	.01440
$2.5\sigma \leq \lvert w_3\rvert \leq 4a$	.00331	.01236

It is to be observed that in the third ordinary difference, the normal distribution exaggerates the area of the "tail"-end of the distribution. However, the discrepancy between the two distributions is no worse than by a factor of 2.4 for the first two ranges of the schedule, and agreement is expected to be closer in higher differences.

Assuming that the probability of w_1 being numerically greater than 2.4σ is approximately the same in higher differences, it might be reasonable to tolerate a discrepancy of 2.4σ. Such a discrepancy is expected to occur once in 163 listings of the third difference, according to the exact distribution, and if we use the normal distribution as a guide, it may occur once in 69 entries in differences of high order. It is costly to examine too many doubtful entries; often it is much more advantageous to calculate two added decimal places in the entries, so that discrepancies with even higher probability can be passed. Much depends on the problem at hand.

The following schedule lists the value of 2.4σ in differences of orders n up to 10. Corresponding values of $.4\sum|A_k|a$ are also included; for some values of n, $.3\sum|A_k|a$ is also tabulated. A_k is the binomial coefficient $\binom{n}{k}$.

n	2.4σ	$.4\sum\lvert A_k\rvert a$	n	2.4σ	$.4\sum\lvert A_k\rvert a$	$.3\sum\lvert A_k\rvert a$
2	$1.697a$	$1.6a$	7	$40.588a$	$51.2a$	$38.4a$
3	$3.098a$	$3.2a$	8	$78.598a$	$102.4a$	$76.8a$
4	$5.797a$	$6.4a$	9	$152.766a$	$204.8a$	$153.6a$
5	$10.998a$	$12.8a$	10	$297.797a$	$409.6a$	$307.2a$
6	$21.060a$	$25.6a$				

It should be observed that the simpler function $.4\sum|A_k|a$ is close in magnitude to 2.4σ if $n \leq 6$ and can be used in its place as a basis for a reasonable tolerance. In differences of higher order $.3\sum|A_k|a$ is close to 2.4σ.

To what extent can the above schedule be used as a basis for establishing a reasonable tolerance for modified divided differences? One way is to compute a few values of $M_{k,\,k+r}$ as a basis for estimating σ. That may be laborious. Some qualitative estimates can be obtained from the form of (1.12). Let A_k denote the coefficient of u_k in the ordinary difference of order $2n$. Then it is clear that

$$M_{0,k}/A_k = \prod_{\substack{j=-n\\ k\neq j}}^{n}\left(\frac{k-j}{p_k-p_j}\right) = \prod_{\substack{j=-n\\ k\neq j}}^{n}\left(\frac{|k-j|}{c_s+c_{s-1}+\ldots+c_{t-1}}\right),$$

where s is the greater of k and j and t is the smaller. Thus in a region where the values of c_k are consistently greater than unity, the standard deviation is *lower* than that in the ordinary difference, and the tolerance should be more stringent than for ordinary differences. On the other hand, if the c_k are all *smaller* than unity in a region, then σ is larger. Wherever the standard deviation for ordinary differences can be used as a basis, the schedule of 2.4σ offers one means of judging the extent to which an isolated error will be masked by smaller errors in neighboring entries.

Secondary Effects in Round-offs. When ordinary differences are considered, all the round-off errors occur in the entires u_k, and the process of taking differences introduces no other errors. In forming divided differences, however, (modified or not) multiplications and divisions are involved, and the result is rounded off to a fixed number of decimals. Hence the cumulative effect of such roundings must be considered, and we must specify the order in which various operations are to be performed. Let $g(k, r)$ denote for brevity the rth modified divided difference associated with t_k. Then $g(k, r)$ is formed as follows:

$$g(k, r) = y(k, r)[g(k+1, r-1) - g(k, r-1)],$$

where

$$y(k, r) = rw/[t_{k+\frac{1}{2}r} - t_{k-\frac{1}{2}r}], \quad \text{if } r \text{ is even}$$

$$y(k, r) = rw/[t_{k+\frac{1}{2}(r+1)} - t_{k-\frac{1}{2}(r-1)}], \quad \text{if } r \text{ is odd.}$$

The differences are useful for interpolation or error detection only in the case where successive differences fall off in magnitude. Let $y(k, r)$ be computed to the maximum attainable accuracy (depending on the computing machine), and then multiplied by $g(k + 1, r - 1) - g(k, r - 1)$, which in successive differences is expected to have fewer significant figures than $y(k, r)$ in the useful case. Thus the error due to carrying an inexact $y(k, r)$ is expected to be of a lower order of magnitude than the error in $g(k, r)$ and we shall neglect its consideration. The principal new error is therefore the rounding of $g(k, r)$ to a fixed number of decimals. Thus in each successive difference there is introduced a new rounding ρ_j, assumed to satisfy conditions (a) and (b). If w_n is the error function in the nth difference with exact operations in obtaining $g(k, r)$ then the total error due to all types of roundings is

$$V_n = w_n + w_{n-1} + w_{n-2} + \ldots + w_1 + \rho_n,$$

where ρ_j replaces e_j in w_{n-j}, $j > 0$.

It is necessary to have an estimate of V_n, as compared with w_n. Let us consider the case where the standard deviation of w_n is used as an estimate, and let

$$\phi_n = a^2 \sum A_k^2/12; \quad \sigma_n = \phi_n^{\frac{1}{2}}.$$

Then the standard deviation of V_n is

$$\sigma_{V_n} = \left(\frac{1}{12} a^2 + \sum_{k=1}^{n} \phi_k \right)^{\frac{1}{2}}.$$

In the special case where the A_k are binomial coefficients (i.e., for the ordinary difference) it can be verified that

$$\sigma_{V_n} = d_n \sigma_n$$

where $d_2 = 1.155$, $d_3 = d_4 = 1.183$, and for n ranging between 4 and 10, d_n decreases steadily up to 1.165 for $n = 10$. Hence if a tolerance of $2.4\sigma_n$ is allowed for ordinary differences, a tolerance of 2.8 σ_n should be reasonable for divided differences. If storage space permits, it is of course possible to carry two more decimals in the divided differences than in the function, so as to lessen the rounding error. If that is done, the added tolerance is not necessary.

The writer acknowledges gratefully the help given by Dr. J. LADERMAN, who read the manuscript and offered many valuable suggestions.

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[1] ARNOLD N. LOWAN & JACK LADERMAN, "On the distribution of errors in nth tabular differences," *Annals of Math. Statistics*, v. 10, 1939, p. 360–364. One minor detail of the paper is faulty, but the results are correct. In evaluating integrals involving products of sines or cosines, it is stated, for example, that

$$\int_{-\infty}^{\infty} dt[\sin at]/t^{2n+1} = [(-1)^n a^{2n}/(2n)!] \int_0^{\infty} dt \sin at/t.$$

This of course is not true, since the left-hand integral does not exist if n is different from zero. However, it turns out that the contribution from the region around the origin in a sum of such integrals vanishes, and the result is right.

[2] A. VAN WIJNGAARDEN, Afrondingsfouten MR3, Tevens ZW-(1950)-.001. Math. Centrum Rekenfdeling, Amsterdam (in Dutch). See also A. M. OSTROWSKI. *Two Explicit Formulae for the Distribution Function of the sums of n Uniformly Distributed Independent Variables.* Archiv d. Math., v. 3, 1952, p. 3–11.

RECENT MATHEMATICAL TABLES

1165[B,F].—H. S. UHLER, "On the 16th and 17th Perfect Numbers," *Scripta Math.*, v. 19, 1953, p. 128–131.

This note contains exact values of $2^{n-1}(2^n - 1)$ for $n = 2203$ and 2281, numbers of 1327 and 1373 decimal digits. These are the 16th and 17th perfect numbers. Exact values are given also of 2^n for $n = 560, 2202, 2280$ and those digits of 2^{4405} and 2^{4561} which are not identical with the corresponding digits of the perfect numbers mentioned above.

The author has informed the reviewer of the fact that the 1023rd digit was printed incorrectly: *for* 32633 *read* 32638. This substitution occurred between page proof and printing and would have gone undetected by any author but one having Uhler's indefatigable perspicacity.

D. H. L.

1166[C].—NBSCL, *Tables of* 10^x (*Antilogarithms to the Base* 10). NBS Applied Math. Series, No. 27, U. S. Gov. Printing Office, Washington, 1953, viii + 543 p., 19.3 × 26.0 cm. Price $3.50.

The main table in the work is Table I, a 500 page table of 10^x for $x = 0(.00001)1$. These 100000 values are given to 10D. The arrangement is in four columns of 50 pairs $(x, 10^x)$ each so that consecutive entries lie one under the other making linear interpolation easy. All eleven digits of 10^x are given in each entry. No differences are given. Linear interpolation gives 9D accuracy. The effect of the second difference on the 10th decimal may be read from a chart on p. vi. This amounts to at most 7 units in the 10th place.

Table II is a 15D radix table of 10^x. Specifically it gives 10^y where

$$y = n \cdot 10^{-p}, \quad n = 1(1)999, \quad p = 3(3)15.$$

From this table 14 figure antilogarithms can be found by multiplying five entries together. The table can also be made to serve as a table of common logarithms to 15D.

Table II is similar to that of DEPREZ[1] which gives 13D antilogarithms of $x = m \cdot 10^{-r}$ for

$$m = 1(1)999, \quad r = 7(3)13$$

in connection with a basic table for

$$x = 0(.0001)1.$$

Table II will be found very useful in connection with any ordinary radix type logarithm table for very precise work.

Table I is based on DODSON'S[2] rare table of 1742. The entire Dodson table was transferred to punched cards and differenced on a tabulator. After correcting errors the table was checked by summing sets of 50 consecutive entries (as a geometric progression). Finally the printed page proof