operation would be slowed down further by more intermediate summary punch operations.

It can be shown that if n_0 is the order of the largest matrix which can be inverted given the storage capacity, and $n_0 < n$, then the most efficient method of inverting by partitioning requires inverting of, say, k submatrices, where the first $(k - 1)$ matrices inverted are of order n_0, and the k^{th} inverted matrix is of order $n - (k - 1)n_0$.

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This article is contained in H. M. WAGNER, "Matrix Inversion on an Automatic Calculator by Row and Column Partitioning," P-417, The RAND Corporation, July 14, 1953; P-417 also discusses in detail (1) the inversion by partitioning of non-symmetric matrices; (2) optimal partitioning; (3) inversion of n^{th} order matrices where n exceeds storage limitations.

[1] W. ORCHARD-HAYS, "The Duplex System for IBM's Model II CPC, A Fast Four Address, Double Operation, Floating-Decimal Set-Up," RM-1044, The RAND Corporation, February 23, 1953; H. M. WAGNER, "Coder's Manual for Duplex System" and "Stanford's Revised Duplex System—Wiring Manual," Technical Reports 1 and 4, Stanford Computation Center. These references give set-ups which meet the two operations per card requirement.

[2] The formula and table following do not take into account any saving in cards and time at the beginning of the procedure when r is small enough to invert without any summary punching. E.g., with 19 storage locations, a 5×5 matrix can be inverted without any summary punching in 90 cards at 150 cards/minute. In general, if S is the number of storage locations available, then all n^{th} order symmetric matrices for which $n \leqq N$ where $\frac{N^2 + 3N - 2}{2} = S$ can be inverted without any summary punching.

[3] H. HOTELLING, "Some computational devices," Chapter X in *Statistical Inference in Dynamic Economic Models*, Cowles Commission Monograph 10, T. C. KOOPMANS, ed. New York, 1950, p. 323–328.

Polynomial Approximations to Elementary Functions

The increasing use of high-speed computing machines has revived interest in the approximation of functions of a real variable, particularly by polynomials. An orthodox table of function values at equidistant arguments may require considerable storage space in an electronic machine. In contrast, the coefficients of a polynomial which represents the function to a desired accuracy over a specified range may require very little storage space, and simple instructions will suffice for the evaluation of the polynomial.

If a function is bounded and continuous in a given finite range of argument, a powerful polynomial approximation is usually obtained by truncation of the infinite expansion of the function in Chebyshev polynomials. Many properties of these polynomials, and the means by which the coefficients in the infinite expansions can be derived, are given by LANCZOS,[1] whose notation for the Chebyshev polynomials in the range $0 \leq x \leq 1$ will be used here

$$T_n^*(x) = \cos n\theta, \quad \cos\theta = 2x - 1.$$

In this note, tables are given which ease the calculation of polynomial approximations to some common functions. Coefficients in the infinite

Chebyshev expansions valid in the specified ranges are given to nine decimal places, and formulae are also given for the Chebyshev polynomials.

A polynomial approximation to any of the functions considered is obtained by truncation of the infinite Chebyshev series. The retained coefficients are then rounded-off, and the polynomial rearranged in powers of the independent variable.

As an example, an approximation to $\sin \frac{1}{2}\pi x$ in the range $-1 \le x \le 1$ is

$$x \sum_{n=0}^{5} A_n T_n^*(x^2),$$

where the coefficients are given in the tables. Substituting their values gives

$$\begin{aligned}\sin \tfrac{1}{2}\pi x \doteq x\{&1.27627\ 8962 - 0.28526\ 1569(2x^2 - 1)\\ &+ 0.00911\ 8016(8x^4 - 8x^2 + 1)\\ &- 0.00013\ 6587(32x^6 - 48x^4 + 18x^2 - 1)\\ &+ 0.00000\ 1185(128x^8 - 256x^6 + 160x^4 - 32x^2 + 1)\\ &- 0.00000\ 0007(512x^{10} - 1280x^8 + 1120x^6 - 400x^4 + 50x^2 - 1)\}\\ = &\ 1.57079\ 6326x - 0.64596\ 4102x^3\\ &+ 0.07969\ 2704x^5 - 0.00468\ 1984x^7\\ &+ 0.00016\ 0640x^9 - 0.00000\ 3584x^{11}\end{aligned}$$

In this case, the rapidity of convergence of the Chebyshev series indicates that the truncation error will be small, and most of the error in the above approximation arises from the rounding-off of the coefficients A_n. Thus the maximum error cannot exceed three units in the ninth decimal place.

If an approximation is required to less than nine decimal accuracy, unwanted rounding errors can be avoided by retaining one or two extra decimals in the coefficients.

For instance, suppose that an approximation to $\ln(1 + x)$ is required in the range $0 \le x \le 1$, accurate to three decimal places. Examination of the coefficients A_n in the series

$$\ln(1 + x) = \sum_{n=0}^{\infty} A_n T_n^*(x)$$

given in the tables, shows that $|A_n| < 0.0005$ when $n > 3$, and that

$$\sum_{n=4}^{\infty} |A_n| = 0.000503\cdots.$$

Hence

$$\begin{aligned}\ln(1 + x) &\doteq 0.37645 + 0.34315(2x - 1) - 0.02944(8x^2 - 8x + 1)\\ &\qquad + 0.00337(32x^3 - 48x^2 + 18x - 1)\\ &= 0.00049 + 0.98248x - 0.39728x^2 + 0.10784x^3\end{aligned}$$

with a maximum error not exceeding 0.00053.

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[1] *Tables of the Chebyshev Polynomials $S_n(x)$ and $C_n(x)$*. NBS Applied Math. series 9, 1952.

Tables of Chebyshev coefficients

$\sin \frac{1}{2}\pi x$

n	A_n
0	1.27627 8962
1	−0.28526 1569
2	0.00911 8016
3	−0.00013 6587
4	0.00000 1185
5	−0.00000 0007

$$\sin \tfrac{1}{2}\pi x = x \sum_{n=0}^{\infty} A_n T_n^*(x^2),$$

$$-1 \leq x \leq 1.$$

$\cos \frac{1}{2}\pi x$

n	A_n
0	0.47200 1216
1	−0.49940 3258
2	0.02799 2080
3	−0.00059 6695
4	0.00000 6704
5	−0.00000 0047

$$\cos \tfrac{1}{2}\pi x = \sum_{n=0}^{\infty} A_n T_n^*(x^2),$$

$$-1 \leq x \leq 1.$$

arctan x

n	A_n
0	0.88137 3587
1	−0.10589 2925
2	0.01113 5843
3	−0.00138 1195
4	0.00018 5743
5	−0.00002 6215
6	0.00000 3821
7	−0.00000 0570
8	0.00000 0086
9	−0.00000 0013
10	0.00000 0002

$$\arctan x = x \sum_{n=0}^{\infty} A_n T_n^*(x^2),$$

$$-1 \leq x \leq 1.$$

for $|x| > 1$, use

$$\arctan x = \tfrac{1}{2}\pi - \arctan (1/x)$$

arcsin x, arccos x

n	A_n
0	1.05123 1959
1	0.05494 6487
2	0.00408 0631
3	0.00040 7890
4	0.00004 6985
5	0.00000 5881
6	0.00000 0777
7	0.00000 0107
8	0.00000 0015
9	0.00000 0002

$$\arcsin x = x \sum_{n=0}^{\infty} A_n T_n^*(2x^2),$$

$$-\tfrac{1}{2}\sqrt{2} \leq x \leq \tfrac{1}{2}\sqrt{2}.$$

$$\arccos x = \tfrac{1}{2}\pi - x \sum_{n=0}^{\infty} A_n T_n^*(2x^2),$$

$$0 \leq x \leq \tfrac{1}{2}\sqrt{2}.$$

For $\frac{1}{2}\sqrt{2} \leq x \leq 1$, use
$\arcsin x = \arccos (1 - x^2)^{\frac{1}{2}}$,
$\arccos x = \arcsin (1 - x^2)^{\frac{1}{2}}$.

e^x

n	A_n
0	1.75338 7654
1	0.85039 1654
2	0.10520 8694
3	0.00872 2105
4	0.00054 3437

e^{-x}

n	A_n
0	0.64503 5270
1	−0.31284 1606
2	0.03870 4116
3	−0.00320 8683
4	0.00019 9919

5	0.00002 7115
6	0.00000 1128
7	0.00000 0040
8	0.00000 0001

$$e^x = \sum_{n=0}^{\infty} A_n T_n^*(x), \quad 0 \leq x \leq 1.$$

5	−0.00000 9975
6	0.00000 0415
7	−0.00000 0015

$$e^{-x} = \sum_{n=0}^{\infty} A_n T_n^*(x), \quad 0 \leq x \leq 1.$$

$\log(1 + x)$

n	A_n
0	0.37645 2813
1	0.34314 5750
2	−0.02943 7252
3	0.00336 7089
4	−0.00043 3276
5	0.00005 9471
6	−0.00000 8503
7	0.00000 1250
8	−0.00000 0188
9	0.00000 0029
10	−0.00000 0004
11	0.00000 0001

$$\log(1 + x) = \sum_{n=0}^{\infty} A_n T_n^*(x), \quad 0 \leq x \leq 1.$$

$\Gamma(1 + x)$

n	A_n
0	0.94178 5598
1	0.00441 5381
2	0.05685 0437
3	−0.00421 9835
4	0.00132 6808
5	−0.00018 9303
6	0.00003 6069
7	−0.00000 6057
8	0.00000 1056
9	−0.00000 0181
10	0.00000 0031
11	−0.00000 0005
12	0.00000 0001

$$\Gamma(1 + x) = \sum_{0=n}^{\infty} A_n T_n^*(x), \quad 0 \leq x \leq 1.$$

$J_0(x)$

n	A_n
0	0.03154 0613
1	−0.21461 6183
2	0.00433 6620
3	−0.26620 3654
4	0.30612 5520
5	−0.13638 8770
6	0.03434 7540
7	−0.00569 8082
8	0.00067 7504
9	−0.00006 0947
10	0.00000 4309
11	−0.00000 0246
12	0.00000 0012

$J_1(x)$

n	A_n
0	0.06942 43523
1	−0.11557 79057
2	0.12167 94099
3	−0.11488 40465
4	0.05779 05331
5	−0.01692 38801
6	0.00323 50252
7	−0.00043 70609
8	0.00004 40991
9	−0.00000 34583
10	0.00000 02172
11	−0.00000 00112
12	0.00000 00005

$$J_0(x) = \sum_{n=0}^{\infty} A_n T_n^*\left(\frac{x^2}{100}\right), \qquad J_1(x) = x\sum_{n=0}^{\infty} A_n T_n^*\left(\frac{x^2}{100}\right),$$
$$-10 \le x \le 10. \qquad\qquad -10 \le x \le 10.$$

n	$T_n^*(x)$
0	1
1	$2x - 1$
2	$8x^2 - 8x + 1$
3	$32x^3 - 48x^2 + 18x - 1$
4	$128x^4 - 256x^3 + 160x^2 - 32x + 1$
5	$512x^5 - 1280x^4 + 1120x^3 - 400x^2 + 50x - 1$
6	$2048x^6 - 6144x^5 + 6912x^4 - 3584x^3 + 840x^2 - 72x + 1$
7	$8192x^7 - 28672x^6 + 39424x^5 - 26880x^4 + 9408x^3 - 1568x^2 + 98x - 1$
8	$32768x^8 - 1\ 31072x^7 + 2\ 12992x^6 - 1\ 80224x^5 + 84480x^4 - 21504x^3 + 2688x^2 - 128x + 1$
9	$1\ 31072x^9 - 5\ 89824x^8 + 11\ 05920x^7 - 11\ 18208x^6 + 6\ 58944x^5 - 2\ 28096x^4 + 44352x^3 - 4320x^2 + 162x - 1$
10	$5\ 24288x^{10} - 26\ 21440x^9 + 55\ 70560x^8 - 65\ 53600x^7 + 46\ 59200x^6 - 20\ 50048x^5 + 5\ 49120x^4 - 84480x^3 + 6600x^2 - 200x + 1$
11	$20\ 97152x^{11} - 115\ 34336x^{10} + 273\ 94048x^9 - 367\ 65696x^8 + 306\ 38080x^7 - 164\ 00384x^6 + 56\ 37632x^5 - 12\ 08064x^4 + 1\ 51008x^3 - 9680x^2 + 242x - 1$
12	$83\ 88608x^{12} - 503\ 31648x^{11} + 1321\ 20576x^{10} - 1992\ 29440x^9 + 1905\ 13152x^8 - 1203\ 24096x^7 + 506\ 92096x^6 - 140\ 57472x^5 + 24\ 71040x^4 - 2\ 56256x^3 + 13728x^2 - 288x + 1.$

RECENT MATHEMATICAL TABLES

1202[A,P].—M. L. Clinnick, (a) *Gear Ratios No.* 43. (b) *Gear Ratios No.* 59. Privately printed, 3211 School Street, Oakland 2, California, 1953. Each book has 84 unnumbered pages 8.9 × 11.4 cm. and 8.26 × 14.0 cm. respectively, photo-offset from typescript. Price $1.00 each.

These pocket-sized tables are designed for use in selecting appropriate sprocket gears in motorcycle racing events. They are triple entry tables giving 2D values of

$$R = kr/(ec)$$

for $c = 10(1)23$, $e = 15(1)25$, $r = 46(1)75$, $k = 43$, 59. Unrealistic values of $R \ge 15$ are omitted. In the intended application r, e and c are the numbers of teeth in the rear, engine, and countershaft sprockets respectively. The clutch sprocket is assumed to have 43 teeth or 59 corresponding to certain popular British and American makes of motorcycle. Instructions are given in (b).

D. H. L.

1203[A,B,P].—J. K. Lynch, *Kilocycle-Radian Frequency Conversion Tables.* Commonwealth of Australia, Postmaster-general's Dept., Research Laboratories, *Report* No. 3726. Melbourne, 1953, 24 p. 20.2 × 25.4 cm.

This table gives 6S values of

$$\omega = 2000\pi F$$