1146. G. H. VINEYARD, "Simulation of trajectories of charge particles in magnetic fields," *Jn. Appl. Phys.*, v. 23, 1952, p. 35–39.

The author sets up an equivalence between the motion of a charged particle in an electric and magnetic field and the motion of a sphere rolling on a rotating surface, subject to gravity and viewed from an independently rotating coordinate system. A variety of interesting special cases corresponding to the betatron and magnetron are discussed. In addition an experimental technique involving a rotating glass surface and multi-exposure photography for obtaining the orbits is described. A preliminary setup was made with immediately available components and the author considered the results satisfactory.

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NOTES

163.—ANOTHER UNRECORDED PITISCUS ITEM. In my article on BARTHOLOMÄUS PITISCUS (1561–1613), to whom our word Trigonometry is due (1595) (*MTAC*, v. 3, p. 390–397, 1949), I referred to one of his tables, "unlisted in any of the ordinary bibliographic or historical sources" (p. 396), entitled

> SINUUM/TANGENTIUM/et/SECANTIUM/Canon Manualis/*Accomodatus ad trigo/nometriam/BARTOLOMAEI*/Pitisci/Grünbergensis Silesij./[Decoration]/HEIDELBERGAE,/*Typis Iohan Lincelloti, Acad Typo.*/Imprensis Ionae Rosae./MDCXIII./Signatures A^{12}–H^{12}, I^4 [200 p.].

It contained 5D tables for all six of the trigonometric functions in the quadrant, at interval 1′. Opposite pages record the values for 30 minutes of each function; but both pages are headed "sinus," "tangens," "secans." The angles on the right-hand pages being complementary to those on the left, the values on the right-hand pages are really those of cos, cot, csc for the angles on the left-hand page. Thus 180 pages, A^2 *verso*–H^8 *recto*, are occupied with the tables. "Explicatio numerorum hujus Canonis" occupies pages A^1 *verso* to A^2 *recto*. A^1 *recto* is the title page. H^8 *verso*–H^{12} *recto* are devoted to text explanations; H^{12} *verso* and I^1 *verso* blank; I^1 *recto* comment on the following 230 Errata (sin and cos 46; tan and cot 86; sec and csc 96; 2 others): I^2 *recto*–I^4 *recto*.

The little pages are of size 7.3 × 13.2 cm. A film copy of this table for the Library of Brown University was made from a copy of the original at the University of Illinois.

In the summer of 1953 the bookshop Old Authors Farm, R.R. 1, Harrisburg, Ontario, Canada, offered an extraordinary collection of mint copies of old books—duplicates from the Vatican Library. Among these was a second edition of the Pitiscus volume described above. The displays of the title pages, down through the word HEIDELBERGAE, except for a new decoration, are identical; then follows in the new volume: Typis Joan. Georg. Geyder. Acad. Typ. Imprensis Jonae Rosae./MDCXX./A^{12}–H^{12}. This volume, acquired by Mr. ALBERT E. LOWNES of Providence, R. I., further enhancing the value of his remarkable personal library in the field of history of science, was kindly placed at my disposal for preparing this article.

It will be observed that there are eight fewer pages in the new edition; this was due to elimination of the 230 errata noted in the first edition, and correction of the corresponding places in the earlier pages. I checked the 46 noted errors for sines and cosines and found that while one of those was incorrectly listed, not only were all the others corrected, but also some errors not previously noted. In spot checking of the other functions no error was found in the new edition. Otherwise pages A^2–H^{12} in both editions appear to be textually identical.

The little volume is bound in old vellum, seemingly as fresh as on the day the volume was published 334 years ago—seven years after the death of Pitiscus.

Apart from this copy, the only other one known is in the Vatican Library.

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164.—PERTURBATION OF LAGRANGIAN COEFFICIENTS. Suppose it is desired to interpolate a function $g(x)$ polynomially to a certain value when the independent variable x is a function, $f(t)$, of a parameter, t, where the values g are known at equally spaced values of t. Under certain circumstances we find it advantageous to carry out a first order perturbation of the Lagrangian coefficients for equally spaced values. Suppose $f(t)$ is a function with a non-vanishing derivative and with a continuous third derivative. Also, suppose we want to interpolate $g(x)$ to the value $x = f(a)$, given $f(t)$ for $t = a + (i - b)h$, $i = 0, \cdots, m$ where b is a positive number less than m and h is a positive number. Then, from Taylor's series

$$(1)\quad f(a + (i - b)h) = f(a) + (i - b)hf'(a) + \tfrac{1}{2}(i - b)^2h^2f''(a) + R_i$$

where R_i is the appropriate remainder. For the purpose of interpolation $f(a)$ has no effect on the coefficients. Since the values $(i - b)hf'(a)$, $i = 0, \cdots, m$ are equally spaced, the Lagrangian coefficients for interpolating to x are given by

$$(2)\qquad L_j = \prod_{i\neq j} \frac{(x - x_i)}{(x_j - x_i)} \quad j = 0, 1, \cdots, m.$$

Now, considering the x's as variables we find the logarithmic derivative of L_j to be

$$(3)\qquad \frac{dL_j}{L_j} = \sum_{i\neq j} \frac{dx - dx_i}{x - x_i} - \sum_{i\neq j} \frac{dx_j - dx_i}{x_j - x_i}.$$

Then letting

$$x_i = (i - b)hf'(a), \quad dx_i = \tfrac{1}{2}(i - b)^2h^2f''(a)$$

$$x = 0, \qquad dx = 0$$

$$\frac{dL_j}{L_j} = \frac{hf''(a)}{2f'(a)}\left[\sum_{i\neq j} (i - b) - \sum_{i\neq j} (i + j - 2b)\right]$$

$$= \frac{hf''(a)}{2f'(a)}\left[m(b - j)\right] \quad j = 0, \cdots, m.$$

Hence the perturbed values of the Lagrangian coefficients are given by

$$(4) \qquad L_j + dL_j = \prod_{i \neq j} \left(\frac{b - i}{j - i} \right) \left[1 + \frac{hf''(a)}{2f'(a)} m(b - j) \right].$$

The sum of the perturbed coefficients in (4) equals 1 since

$$\sum_{j=0}^{m} \left[(b - j) \prod_{i \neq j} \left(\frac{b - i}{j - i} \right) \right] = 0.$$

While an error estimate might be calculated, in any application it is simple to calculate actual coefficients to compare with the first order approximations.

For example, let $f(t) = t^3$, $t_i = a + (i - 1.5)h$ to $i = 0, 1, 2, 3$. Then $f''(a) = 6a, f'(a) = 3a^2$ and for interpolation to $x = a^3$ from 4

$$L_0 + dL_0 = -\frac{1}{16}\left(1 + \frac{9h}{2a}\right)$$

$$L_1 + dL_1 = \frac{9}{16}\left(1 + \frac{3h}{2a}\right)$$

$$L_2 + dL_2 = \frac{9}{16}\left(1 - \frac{3h}{2a}\right)$$

$$L_3 + dL_3 = -\frac{1}{16}\left(1 - \frac{9h}{2a}\right).$$

Clearly these coefficients should be used only if $\frac{h}{a}$ is small. Formulas of this type when appropriate have been used at the Los Alamos Scientific Laboratory in a calculation on the IBM 701 to save appreciable storage space compared to that required for direct calculation of the correct coefficients. Of course, $f''(a)/f'(a)$ may not be simple to calculate in many applications.

In case it is inconvenient to calculate $f'(a)$ and $f''(a)$ we may interpolate to $x = f(a)$ using the following adjusted coefficients

$$(5) \quad L_j + dL_j = \prod_{i \neq j} \left(\frac{p + b - i}{j - i} \right) \left[1 + \frac{hf''(c)}{2f'(c)} m(p + b - j) \right] \qquad j = 0, \cdots, m,$$

where c is a convenient value of t, usually taken to be one of the equally spaced values near a and $p = (a - c)/h$. When $a = c$, (5) reduces to (4). The sum of the adjusted coefficients in (5) is 1.

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165.—AN INVESTIGATION INTO THE REAL ROOTS OF CERTAIN POLYNOMIALS. In connection with work on the application of statistical theory to the determination of the number and location of roots of high order polynomial equations, it was required to investigate the real roots of the equations:

$$(1) \qquad x^8 \pm x^7 \pm x^6 \pm x^5 \pm x^4 \pm x^3 \pm x^2 \pm x \pm 1 = 0$$

where all possible combinations of sign are taken.

It can be shown simply, that all roots of these equations lie in the range:

$$\tfrac{1}{2} < |x| < 2$$

and the computational problem therefore reduces to that of evaluating each of the 256 different equations obtained by changing the signs of (1), over the range $\frac{1}{2} \leq x \leq 2$, and at a sufficiently fine interval to ensure that no roots are missed.

Preliminary calculations indicated that the functions are quite well behaved, and that tabulation at intervals of $\frac{1}{16}$ in x would be adequate, and it was thus necessary to evaluate each equation at 25 points.

The polynomials were written in the form:

$$x(\cdots x(x(x \pm 1) \pm 1) \cdots) \pm 1$$

and were evaluated by successive addition of the coefficients and multiplication by x.

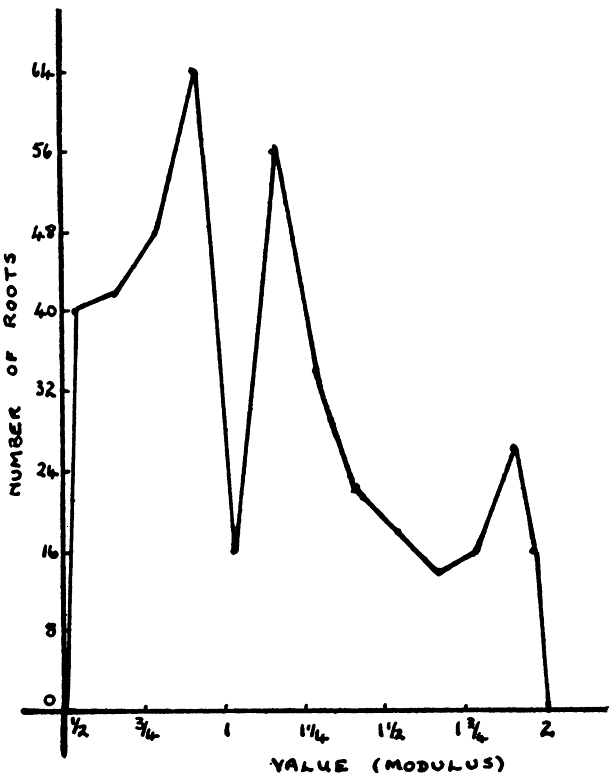

FIG. 1.

The calculations were performed on the HEC 1 and the time for calculation and printing of each polynomial was 50 seconds. For convenience of reference, the sign combination of each polynomial was printed as well.

An analysis of the results of this tabulation shows that of the 256 equations considered:

58 have no real roots
190 have two real roots
8 have four real roots.

The distribution of the roots by numerical value is shown in Fig. 1. The detailed results of this tabulation are available for reference at the Computation Laboratory, Birkbeck College.

The author wishes to thank Mr. R. L. Michaelson of the British Tabulating Machine Company for providing computing facilities for this calculation.

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166.—Numerical Study of Signature Rank of Cubic Cyclotomic Units. In the search for algebraic fields (particularly beyond the quadratic) where unique factorization fails, little work seems feasible on high speed computers for the reason that the straightforward evaluation of forms must be controlled by group-theoretic data-processing.[1] A rare opportunity, however, is afforded by the cubic field $R(\alpha)$ generated by the Gaussian sum

$$\alpha = \sum_{x=0}^{p-1} \exp 2\pi i x^3/p$$

where $p = 6n + 1$ is a prime. Here the so-called *cyclotomic unit*[2] is given by the simple formula (for three conjugates $0 \leq i \leq 2$):

$$\Theta_i = -\prod_{t=0}^{n-1} \sin\left(\tfrac{1}{2}\pi g^{3t+i}/p\right) \csc\left(\tfrac{1}{2}\pi g^{3t+i+1}/p\right)$$

where g is an odd primitive root modulo p (or an even one augmented by p). Now while the decimal accuracy of such a formula would be lost very rapidly the Θ_i have the property that the possible non-unique factorization in $R(\alpha)$ can be of a certain frequent type (namely *even* class number) only when all three Θ_i are positive.[2] This condition is easy to determine by machine.

We accordingly form three tallies A_0, A_1, A_2 for each p as follows: We increment the A_i tally $(0 \leq i \leq 2)$ when the least positive residues of g^{3t+i+1} and g^{3t} modulo $2p$, $0 \leq t \leq n - 1$, lie in different halves of the interval $(0, 2p)$ subdivided at p. Such a procedure tallies the possible negative sign of $\sin(\tfrac{1}{2}\pi g^{3t+i+1}/p) \csc(\tfrac{1}{2}\pi g^{3t}/p)$; hence the three Θ_i are seen to be all positive exactly when the A_0, A_1, and A_2 have (final) values that are odd, even, and odd respectively.

Since there are many primitive roots g for each p we find that the A_i depend on the g chosen, except for the fact that the occurrence of three positive Θ_i must be independent[2] of g. Assuming that the powers of g lay in half-intervals "at random" we should expect the A_i to be each approximately $p/12$, while assuming "random parities," (which is more precarious), we should expect the Θ_i to be all positive approximately $\frac{1}{4}$ of the time; (since $\Theta_0\Theta_1\Theta_2 = +1$, or, equivalently, A_2 is always odd).

The computation was performed on the MIDAC starting Jan. 27, 1954, for the first 207 values of p (<3000). The values[3] of p and g in the form of binary-coded decimals were placed as alternate words on high-speed photoelectric input tape. They were read in by pairs (taking approximately .15 sec. for each pair). The conversion to binary form was internal. For the main induction loop the starting point was each g^{3t}, $0 \leq t \leq n - 1$, from which were formed g^{3t+i+1} for $0 \leq i \leq 2$ (always modulo $2p$). The three tally increment decisions (for A_i) were made as described above (taking .04 sec. for each circuit of the loop). In addition, every time the MIDAC chose not to increment an A_i tally it incremented a common D tally. After the loop had been traversed n times two checks on accuracy were made. First the sum of the three A_i and the D tallies was verified to be $(p-1)/2$ (the number of increment decisions), and secondly the last power, $g^{(p-1)/2}$ (reduced modulo $2p$), was verified to be $2p - 1$. The MIDAC was made to examine the three final A_i tallies for the occurrence of alternate parities and indicate such occurrences or non-occurrences in the form of a -0 or 0 respectively in a temporary storage space. The output for each 0 consisted of

$$p \; g \; A_0 \; A_1 \; A_2 \; -0 \text{ (or } 0)$$

where the first five items were converted to decimal internally and shifted for a short-word (4 character) print-out while the sixth item was the signed zero indicating presence or absence of alternate parities. (Print-out time for each p was approximately 5 seconds.)

The total running time was about one hour of which 20 minutes was input-output.

The 21 values of $p(<3000)$ for which the A_i have alternate parity (or for which the Θ_i are all positive) are reproduced below with j their order in the list of primes (of the form $6n + 1$).

j	p	j	p	j	p
18	163	71	853	143	1879
27	277	77	937	145	1951
33	349	81	1009	156	2131
37	397	107	1399	169	2311
48	547	129	1699	191	2689
52	607	135	1777	199	2797
60	709	137	1789	200	2803

It will be seen that the frequency of these p is close to $\frac{1}{10}$, which is not in agreement with "randomness" of parities (as postulated above). The magnitudes of the A_i did come out, however, to agree rather well with the randomness assumption made earlier, and they are omitted here. A spot check[4] of $p = 163, 277, 349, 2803$ by hand calculations revealed that unique factorization does fail in the field $R(\alpha)$, and that the class number is 4 for each of these cases.

The author is indebted to Dr. J. W. CARR III and the staff of the Willow Run Research Center for kindly making the MIDAC available for this study (which is part of Army Ordnance Project DA-20-018-ORD-12332).

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[1] Compare D. H. LEHMER, *Guide to tables in the theory of numbers*. National Research Council, 1941, p. 75–77, O. TAUSSKY, *Some computational problems in algebraic number theory*. National Bureau of Standards *Report* (to appear).

[2] For theoretical background consult H. HASSE, *Arithmetische Bestimmung von Grundeinheit*. Berlin, 1950, p. 70.

[3] These were taken from a table of minimum positive g for $p < 3000$ in I. M. VINOGRADOV, *Osnovy Teorii Chisel* [*Fundamentals of the Theory of Numbers*]. Moscow, 1940, p. 110.

[4] The case $p = 163$ was discovered through another procedure by E. ARTIN, according to a private communication.

167.—CULLEN NUMBERS. These are numbers of the form $n2^n + 1$ and are remarkable in that they seem to be composite for $n > 1$, although there is no *a priori* reason for this. CUNNINGHAM & WOODALL[1] made a study of these numbers and found them all composite with a small factor for $1 < n < 141$. No factor of $141 \cdot 2^{141} + 1$ is known. I have completely factored the following cases left incomplete by Cunningham. The case $n = 46$ is due to R. A. LIÉNARD of Lyons.

n	$n2^n + 1$	n	$n2^n + 1$
33	$47 \cdot 6031230671$	42	$23 \cdot 43 \cdot 83 \cdot 2250270487$
35	$37 \cdot 32502455213$	43	$3 \cdot 5 \cdot 163 \cdot 2633 \cdot 58752797$
37	$3 \cdot 5 \cdot 339016085231$	45	$11 \cdot 47 \cdot 2437 \cdot 1256655529$
38	$3^2 \cdot 20879 \cdot 55586743$	46	$5 \cdot 31 \cdot 47 \cdot 139297 \cdot 3189821$
39	$41 \cdot 3433 \cdot 152326961$	47	$7 \cdot 11 \cdot 43 \cdot 3593 \cdot 556021079$
40	$41 \cdot 131611 \cdot 8150491$	48	$7 \cdot 379 \cdot 997 \cdot 5107973329$
41	$13 \cdot 43 \cdot 1291 \cdot 124932557$	66	$5^3 \cdot 13 \cdot 67 \cdot 107 \cdot 131 \cdot 8353 \cdot 382030403$

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[1] A. J. C. CUNNINGHAM & H. J. WOODALL, "Factorisation of $Q = (2^q \mp q)$ and $(q \cdot 2^q \mp 1)$," *Messenger Math.*, v. 47, 1917, p. 1–38.

CORRIGENDA

V. 6, p. 225, l. 11, *for* monomial *read* elementary.
V. 7, p. 34, l. 6, *for* 6 *read* 1.
V. 7, p. 175, l. 17, *for* 9 *read* 8.