The interest in pseudo-squares is heightened by the fact that they may be used in tests for primality, as shown by MARSHALL HALL.[4] The operation of the SWAC and the reduction and checking of the output data was done by JOHN SELFRIDGE.

D. H. L.

[1] M. KRAITCHIK, *Recherches sur la Theorie de Nombres*, v. 1, Paris, 1924, p. 41–46.
[2] D. H. LEHMER, "The mechanical combination of linear forms," *Amer. Math. Monthly*, v. 35, 1928, p. 114–121.
[3] D. H. LEHMER, "The sieve problem for all-purpose computers," *MTAC*, v. 7, 1953, p. 6–14.
[4] M. HALL, "Quadratic residues in factorization," Amer. Math. Soc., *Bull.*, v. 39, 1933, p. 578–763.

171.—L. F. RICHARDSON (1881–1953). This English mathematician made several notable contributions to numerical analysis. A brief account of his life by P. A. SHEPPARD appears in *Nature* (v. 172, 1953, p. 1127–8).

His work on numerical analysis (apart from that appearing incidentally in his book[1]) was mainly contained in three long papers:

A: "The approximate arithmetical solution by finite differences of physical problems, involving differential equations with an application to the stresses in a masonry dam." Royal. Soc. *Phil. Trans.*, v. 210 A, 1910, p. 307–357.
B: "The deferred approach to the limit"—Part I, L. F. RICHARDSON, Part II, J. A. GAUNT, Royal Soc. *Phil. Trans.*, v. 226 A, 1926, p. 299–361.
C: "A purification method for computing the latent columns of numerical matrices and some integrals of differential equations," Royal Soc. *Phil. Trans.*, v. 242 A, 1950, p. 439–491.

There were minor contributions in

D: "Theory of the measurement of wind by shooting spheres upward," Royal Soc. *Phil. Trans.*, v. 223 A, 1923, p. 345–361.

An introduction to some of the material of A, B appeared in

E: "How to solve differential equations approximately by arithmetic," *Math. Gazette* v. 12, 1925, p. 415–421.

and one to some of the material of C in

F: "A method for computing principal axes," *British Jn. of Psychology*, v. 3, 1950, p. 16–20.

His work is highly individualistic and his language and symbolism picturesque. For instance he introduced the terms "marching" problem, for initial value problems of the form

$$y'' = ky, \quad y(0), \quad y'(0) \text{ given,}$$

and "jury" problem for a problem of the form

$$y^{vi} - 3y^{iv} + 3y'' - y = \lambda y, \; y = y'' = y''' = 0 \text{ for } x = \pm 1.$$

We shall discuss briefly two of Richardson's contributions to numerical analysis.

Richardson early in his life recognized the importance of the use of central difference operators in numerical applications, a fact which had been pointed out somewhat earlier by W. F. SHEPPARD. The use of central-difference approximation to derivatives suggested that the (local) difference between the solution to a discrete problem and that of the continuous problem would be a power series in h^2, h being the mesh-length. Taking account of the first term only, it is possible by solving the discrete problem for two values of h, and then eliminating the h^2-contribution to obtain a better approximate solution. This he called the "deferred approach to the limit." Among his examples were the extrapolation from the perimeter of a square and hexagon to that of a circle (see *MTAC* v. 2 1946, p. 114 and p. 223–4) and the evaluation of e. He also indicated the passage to the fundamental frequency of a continuous string from that of strings of beads. Richardson is mainly concerned with applications of his method to the numerical solution of differential equations; there are, however, discussions in D of a quadrature and the solution of a Volterra type integral equation.

In B he examines in some detail the justification of this process, considering two main questions: (1) Are the odd-powers of h always absent? (2) How small must h be in order that h^2-extrapolation may make an improvement? We shall not discuss this paper in detail: Undoubtedly the process is a valuable practical tool, but there are certainly cases where it is unreliable.[2] Richardson was fully aware of the possible failures and difficulties which might be encountered in its application and discussed various bad examples.

The latter part of A is concerned with a detailed study of the stresses in a dam, with particular reference to a model of the Assuan dam.

An interesting remark in A concerns the cost of computation about 1910. The unit operation was the evaluation of

$$v_N + v_W + v_S + v_E - 4v_0$$

and for this the rate was $n/18$ pence where n was the number of digits carried. An average output in the case $n = 3$ was of the order of 2,000 *correct* units per week, paying about 28 shillings or about 5 dollars at the then current rate.

The second contribution of Richardson which we shall discuss is an algorithm for the solution of a system of n linear equations, $A\mathbf{x} = \mathbf{b}$. This is to choose an arbitrary $\mathbf{x}^{(0)} = \{x_i^{(0)}\}$ and then put

$$\mathbf{x}^{(r+1)} = \mathbf{x}^{(r)} + \beta_r\{A\mathbf{x}^{(r)} - \mathbf{b}\},$$

where the factors β_r are to be chosen. Some suggestions for their choice was given in A; an up-to-date study has been given by D. M. YOUNG.[3]

The success of this algorithm can be established by expressing the errors in terms of the characteristic values of A. Following Young we assume that A has linearly independent characteristic vectors $\mathbf{v}_1, \cdots, \mathbf{v}_n$ with characteristic values $\lambda_1, \cdots, \lambda_n$. Then the error vector $\boldsymbol{\varepsilon}^{(r)} = \mathbf{x}^{(r)} - \mathbf{x}$ satisfies

$$\boldsymbol{\varepsilon}^{(r+1)} = \boldsymbol{\varepsilon}^{(r)} + \beta_r A \boldsymbol{\varepsilon}^{(r)}.$$

If we expand $\boldsymbol{\varepsilon}^{(0)}$ in terms of the $\mathbf{v}_i$ as: $\boldsymbol{\varepsilon}^{(0)} = \sum_{i=1}^{n} c_i \mathbf{v}_i$ we find

$$\boldsymbol{\varepsilon}^{(r)} = \sum_{i=1}^{n} c_i \mathbf{v}_i \prod_{k=1}^{r} (1 + \beta_k \lambda_i).$$

This gives

$$\|\epsilon^{(r)}\|^2 = \sum_{i=1}^{n} c_i^2 \|\mathbf{v}_i\|^2 \left\{ \prod_{k=1}^{r} (1 + \beta_k \lambda_i)^2 \right\} \leqslant \|\epsilon^{(0)}\|^2 M^{(r)}$$

where

$$M^{(r)} = \max_{1 \leq i \leq n} \left\{ \prod_{k=1}^{r} (1 + \beta_k \lambda_i)^2 \right\}.$$

For convergence we have to show that $M^{(r)} \to 0$. In practice the sequence of β_r will often be taken as periodic and then it will be sufficient if the product taken over a period, is less than unity. In this case, if we know, for instance, that

$$0 < \lambda_r \leqslant b, \quad r = 1,2,\cdots,n$$

and choose β_r so that $0 > \beta_r > - 2b^{-1}$, then each factor will be less than unity and convergence is assured.

It is clear that these ideas can also be used in the practical determination of the characteristic vectors of a matrix. Suppose we have an approximation $\mathbf{v}$ to a characteristic vector $\mathbf{v}_1$ of a matrix A and, for simplicity, assume that the only contamination is a component of the characteristic vector $\mathbf{v}_2$. Suppose we have $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$. Let λ_1,λ_2 be the characteristic values corresponding to $\mathbf{v}_1,\mathbf{v}_2$. Then, for any $\alpha(\neq\lambda_1)$

$$\begin{aligned} \mathbf{v}^{(1)} &= (A - \alpha I)\mathbf{v} = c_1(\lambda_1 - \alpha)\mathbf{v}_1 + c_2(\lambda_2 - \alpha)\mathbf{v}_2 \\ &= (\lambda_1 - \alpha)[c_1\mathbf{v}_1 + c_2(\lambda_2 - \alpha)(\lambda_1 - \alpha)^{-1}\mathbf{v}_2]. \end{aligned}$$

The strength of the component of $\mathbf{v}_2$ will therefore be reduced if $|(\lambda_2 - \alpha| < |\lambda_1 - \alpha|$, i.e. if α is nearer λ_2 than λ_1. If this is so successive repetition of the multiplication $A - \alpha I$ will purify $\mathbf{v}$. This method is a generalization of the familiar "power" method for the determination of the characteristic value of largest absolute value.

In C, Richardson exploits this idea, with a wealth of numerical examples, including cases when the matrix is unsymmetric or has non-linear elementary divisors. He discusses the use of purifiers of the form

$$(A - \alpha_1 I)(A - \alpha_2 I)\cdots(A - \alpha_k I)$$

and the optimal choice of the α_i. It is clear that information about the location of the characteristic roots is essential for satisfactory choice of the α_i. Richardson makes use of the bounds given by Hirsch, Rayleigh's quotient, and the comparison of the ratios of corresponding components of $A\mathbf{v}$ and $\mathbf{v}$. The latter is used in an intuitive way, no mention being made of the result of Collatz, that there is always at least one characteristic root between the greatest and least of the ratios of the components.

J. T.

[1] L. F. RICHARDSON, *Weather Prediction by Numerical Processes.* Cambridge, England, 1923. For the following comment on this, I am indebted to G. E. FORSYTHE, "It is a monumental attempt to forecast for six hours, from almost no initial condition, and (I understand) a poor balance of Δt and Δx, Δy, Δz. It is superbly written and the author has (in my opinion) the most elegant English style of any mathematical writer of the century. [See p. 219 of this book, or the first page or two of C.] The Preface speaks for itself of the troubles encountered by the author."

[2] Cf. W. WASOW, "Discrete approximations to elliptic differential equations." *Zeit. f. ang. Math. u. Phys.*, v. 5, 1954.

[3] D. M. YOUNG, "On Richardson's method for solving linear equations with positive definite matrices." *Jn. Math. and Physics*, v. 32, 1953, p. 243–255. Experiments on the solution of the Laplace equation by this method, on ORDVAC, have been carried out by D. M. Young, and C. H. WARLICK.

CORRIGENDA

V. 8, p. 93, l.-3, *for* $12\mu = \mu^3$ *read* $12\mu + \mu^3$.
V. 8, p. 106, l. 8, *for* PEARCY *read* PEARCEY.
V. 8, p. 121, l. 20 *for* $+3(2 + i)$ *read* $-3(2 + i)$.

EDITORIAL NOTE. With this issue of *MTAC* the present Editorial Committee rounds out its fifth year and resigns. It is a pleasure to thank our many contributors, reviewers and referees for their cooperative assistance to the Committee and to *MTAC*. Future editorial correspondence should be addressed to

Dr. C. B. TOMPKINS
Numerical Analysis Research
University of California
Los Angeles, 24
California