The integrals

$$F(y) = \int_0^T f(t)K(y,t)dt \quad \text{or} \quad \int_0^y f(t)K(y,t)dt$$

are evaluated by means of high speed analog computer elements for various values of y. The variable y is given a sequence of values and by a suitable switching procedure the values of $F(y)$ are evaluated at a rate of 60 per second and plotted with a reference value on a cathode ray tube. The functions $f(t)$ and $K(y,t)$ for y fixed are obtained by function generators or by differential analyzer techniques. During the generation of $F(y)$, y which should remain fixed does actually vary. The error due to this cause and the errors due to the multiplier and function generator are discussed.

F. J. M.

1164. J. R. Ragazzini & G. Reynolds, "The electronic complex plane scanner," *Rev. Sci. Instruments*, v. 24, 1953, p. 523–527.

A rational function $F(z)$ is supposed given in the form of a quotient of products of linear factors $z - \lambda_i$. It is required to obtain $\log |F(jw)|$ and $\arg F(jw)$ where $j = \sqrt{-1}$ and w is real. $\operatorname{Log} |F(jw)|$ is the sum of $\pm \log |(jw - \lambda_i)|$. The quantity $jw - \lambda_i$ is represented as a complex voltage and $\log |(jw - \lambda_i)|$ is obtained by a suitable circuit. The argument of $F(jw)$ is obtained by differentiation using the Cauchy-Riemann equations.

F. J. M.

1165. J. Tadayon, "Measurement of the angle between two curves," *Rev. Sci. Instruments*, v. 24, 1953, p. 871–872.

An optical instrument for measuring the angle between two curves on a graph is based on the mirror principle for finding the normal to a curve.

F. J. M.

NOTES

168.—A Practical Refutation of the Iteration Method for the Algebraic Eigenproblem. In the second part of my paper on algebraic eigenproblems[2] I have proved that the computation by means of the formation of the characteristic equation requires less computational work than the iteration method, and that this holds even when nothing but the first eigenvalue has to be calculated. This advantage grows with every accessory eigenvalue or vector. Further one has no trouble with deflation which requires a lot of multiplications. Also one can compute every eigenvalue and vector apart from the others, and do this to any desired accuracy, by the more powerful methods for algebraic equations. At last, there arise no difficulties from an unfavorable quotient of the two dominant eigenvalues.

The reason these facts are not yet universally acknowledged is that the iteration method seems to be simpler and more mechanical in its application. But one has to consider that iteration does not converge quickly enough in practice, unless the quotient of the two dominant eigenvalues is $\frac{1}{4}$ or less. This last will be rarely the case. For the eigenvalues must lie somewhere between two circles around the origin in the complex plane. The radius of the inner circle has a value quite different from zero (if the determinant of the matrix is $\neq 0$). In this ring can lie only some of the n eigenvalues having a quotient less than $\frac{1}{4}$. The other eigenvalues must cluster around some circles. In proportion as the situation is more favourable for e.g. the two dominant eigenvalues, the smaller the room will be for the other ones. There is no escape from this situation, and sooner or later the then dominant eigenvalues will lie close together.

That iteration may give trouble in the most simple cases is a well known fact from numerous calculations. Yet the following example of a symmetric 4×4 matrix, the eigenproblem of which is therefore real, is amazing.

$$A = \begin{vmatrix} 2 & 1 & 3 & 4 \\ 1 & -3 & 1 & 5 \\ 3 & 1 & 6 & -2 \\ 4 & 5 & -2 & -1 \end{vmatrix}$$

Let μ be the dominant eigenvalue. Following AITKEN,[1] we begin the iteration with the usual starting vector $v = (1,1,1,1)'$ and so compute $v_m = Av_{m-1}$. The quotient μ_m of the first components of v_m and v_{m-1} yields for μ the "approximations":

$$S: \quad \mu_2 = 7.2,\ \mu_3 = 7.5,\ \mu_4 = 7.85,\ \mu_5 = 7.56,\ \mu_6 = 8.0,\ \mu_7 = 7.6,$$
$$\mu_8 = 8.08,\ \mu_9 = 7.667,\ \mu_{10} = 8.122,\ \mu_{11} = 7.683.$$

The convergence is not impressive, especially if one considers that the sequence S converges to about -8 (minus eight)! On the contrary, S seems to diverge or to "converge" to two limits.

To analyze the sequence S more deeply, all eigenvalues and vectors were computed from the characteristic equation, the vectors by the method described in BODEWIG,[2] part II, section 3, p. 1–3. These are to 8D

$$\mu^{(1)} = -8.02857835,\ \mu^{(2)} = 7.93290472,\ \mu^{(3)} = 5.66886436,\ \mu^{(4)} = -1.57319074.$$
$$x^{(1)} = (1,\ 2.50146030,\ -0.75773064,\ -2.56421169)'$$
$$x^{(2)} = (1,\ 0.37781815,\ 1.38662122,\ 0.34880573)'$$
$$x^{(3)} = (1,\ 0.95700152,\ -1.42046826,\ 1.74331693)'$$
$$x^{(4)} = (1,\ -0.90709211,\ -0.37759122,\ -0.38331238)'.$$

Since S does converge to $-8.0\ldots$ the situation seems to be that the μ_i diverge in the beginning. The μ_{2i} decrease and the μ_{2i+1} increase, and in order to converge to -8, the μ_{2i} go through zero and the μ_{2i+1} through ∞. By a special method the index has been determined where the signs change. This is the case at the 363rd or 364th iteration.

To demonstrate this conclusion we have computed the values of μ_i which would have resulted if the iterations had been really effected. The 100th to 102nd iteration would yield the "approximations":

$$\mu_{100} = 7.28514, \quad \mu_{101} = 8.64677.$$

The 200th to 202nd iteration would yield:

$$\mu_{200} = 5.96936, \quad \mu_{201} = 10.57380.$$

The 300th to 302nd would be:

$$\mu_{300} = 2.86532, \quad \mu_{301} = 22.13220.$$

Our above conjecture concerning the μ_{2i} and μ_{2i+1} is therefore confirmed, as is also apparent from theoretical reasons. Near the critical point 363 we should get

$$\mu_{360} = 0.13587, \mu_{361} = 468.66407, \mu_{362} = 0.04022, \mu_{363} = 1583.336,$$
$$\mu_{364} = - 0.055448, \mu_{365} = - 1148.735.$$

The 400th to 402nd iteration would give

$$\mu_{400} = - 1.75101, \quad \mu_{401} = - 36.46902,$$

the 800th to 802nd:

$$\mu_{800} = - 7.94348. \quad \mu_{801} - 8.11356,$$

and at last the 1200th to 1202nd:

$$\mu_{1200} = - 8.02787279, \quad \mu_{1201} = - 8.02927758.$$

Thus 1200 iterations will scarcely yield 4 figures of the dominant eigenvalue! And this for a simple matrix of order 4.

This amazing conduct can afterwards be explained from the knowledge of the eigenvalues and vectors. The slow convergence has two sources. Not only have the dominant eigenvalues nearly the same absolute value, but also the starting vector v is nearly orthogonal to $x^{(1)}$. In fact writing $x^{(1)}$ in the approximate form $(4,10,-3,-10)'$, the cosine between v and $x^{(1)}$ is only 1/30 which is about cos 88°.

Yet the situation would not considerably improve if another starting vector, e.g. $(1,0,0,0)'$ would be taken. Nor should we waste time by computing the eigenproblem of matrices of the form $A + c$ with varying c's. The time is better spent by computing the characteristic equation as is pointed out in BODEWIG[2] (Part II in "Zusammenfassung"). Even the use of quadratic equations would not be very efficient in our case as $\mu^{(3)}$ lies close to $\mu^{(2)}$. Nor would this be the case for other matrices if one wants to have more than the two dominant eigenvalues, and even then the determination of the vectors is far from agreeable.

We owe to VON MISES[3] the discovery of the iteration method for finding the dominant value and vector and to AITKEN its application to all complicated cases more or less, and to HOTELLING the discovery of deflation. But times have changed. In our days also the higher eigenvalues and vectors must be computed and with extra accuracy. A deeper analysis shows that the iteration method in its present form is not appropriate and that even in cases when, by exception, the first eigenvalue is furnished quickly the later eigenvalues present the greater troubles.

So the whole eigenproblem must be considered anew.

E. BODEWIG

Borweg 7
The Hague
Holland

[1] A. C. AITKEN, "The evaluation of the latent roots and latent vectors of a matrix," Roy. Soc. Edinburgh, *Proc.*, v. 57, 1936–1937, p. 269–304.

[2] E. BODEWIG, "Bericht über die Methoden zur numerischen Lösung algebraischer Eigenwertprobleme," Seminario matematico e fisico dell' Universita di Modena, *Atti*, v. 4, 1949–1950, v. 5, 1950–1951.

[3] R. VON MISES, "Praktische Methoden zur Gleichungsauflösung," *Zeit. angw. Math. Mech.*, v. 9, 1929, p. 62–77.

169.—ANALYTICAL APPROXIMATIONS. [See also NOTE **157.**] The following are approximations for the exponential integral and certain Bessel functions.

(56) $0 \le x \le 1 \quad |\epsilon|_{\max} = 2 \times 10^{-7}$

$$-\mathrm{Ei}(-x) + \log_e x \doteq -.57721566 + .99999193\,x - .24991055\,x^2 + .05519968\,x^3 - .00976004\,x^4 + .00107857\,x^5$$

(57) $-3 \le x \le 3 \quad |\epsilon|_{\max} = 10^{-7}$

$$J_0(x) \doteq 1 - 2.2499997\,(x/3)^2 + 1.2656208\,(x/3)^4 - .3163866\,(x/3)^6 + .0444479\,(x/3)^8 - .0039444\,(x/3)^{10} + .0002100\,(x/3)^{12}$$

(58) $0 \le x \le 3 \quad |\epsilon|_{\max} = 2 \times 10^{-8}$

$$Y_0(x) - \frac{2}{\pi}\log_e\frac{x}{2}\,J_0(x) \doteq .36746691 + .60559366\,(x/3)^2 - .74350384\,(x/3)^4 + .25300117\,(x/3)^6 - .04261214\,(x/3)^8 + .00427916\,(x/3)^{10} - .00024846\,(x/3)^{12}$$

(59, 60) $3 \le x < \infty$

$$J_0(x) = x^{-\frac{1}{2}} f_0(3/x) \cos\{x - \varphi_0(3/x)\}$$
$$Y_0(x) = x^{-\frac{1}{2}} f_0(3/x) \sin\{x - \varphi_0(3/x)\}$$

$|\epsilon|_{\max} = 10^{-8}$

$$f_0(3/x) \doteq .79788456 - .00000077\,(3/x) - .00552740\,(3/x)^2 - .00009512\,(3/x)^3 + .00137237\,(3/x)^4 - .00072805\,(3/x)^5 + .00014476\,(3/x)^6$$

$|\epsilon|_{\max} = 5 \times 10^{-8}$

$$\varphi_0(3/x) \doteq .78539816 + .04166397\,(3/x) + .00003954\,(3/x)^2 - .00262573\,(3/x)^3 + .00054125\,(3/x)^4 + .00029333\,(3/x)^5 - .00013558\,(3/x)^6$$

(61) $0 \le x \le 3 \quad |\epsilon|_{\max} = 5 \times 10^{-9}$

$$J_1(x)/x \doteq .5 - .56249985\,(x/3)^2 + .21093573\,(x/3)^4 - .03954289\,(x/3)^6 + .00443319\,(x/3)^8 - .00031761\,(x/3)^{10} + .00001109\,(x/3)^{12}$$

(62) $0 \le x \le 3 \quad |\epsilon|_{\max} = 5 \times 10^{-8}$

$$\left\{Y_1(x) - \frac{2}{\pi}\log_e\frac{x}{2}\,J_1(x)\right\}x \doteq -.6366198 + .2212091\,(x/3)^2 + 2.1682709\,(x/3)^4 - 1.3164827\,(x/3)^6 + .3123951\,(x/3)^8 - .0400976\,(x/3)^{10} + .0027873\,(x/3)^{12}$$

(63, 64) $3 \le x < \infty$

$$J_1(x) = x^{-\frac{1}{2}} f_1(3/x) \sin\{x - \varphi_1(3/x)\}$$
$$Y_1(x) = -x^{-\frac{1}{2}} f_1(3/x) \cos\{x - \varphi_1(3/x)\}$$

$|\epsilon|_{\max} = 3 \times 10^{-8}$