# TECHNICAL NOTES AND SHORT PAPERS

## Wolfram, Vega, and Thiele

In my article "New information concerning ISAAC WOLFRAM's life and calculations," *MTAC*, v. 4, 1950, p. 185–200, special consideration is given to his extraordinary table of *ln x* to 48 D, as published in J. C. SCHULZE, *Recueil de Table Logarithmiques*, v. 1, 1778, p. 190-258. In this table are 3457 arguments, not 3462 as stated on p. 193, line −7, and p. 197, line 8. This erroneous statement was caused by overlooking the fact that in the 69 pages of the table there were ten groups of figures on every page *except* 256, where there were only nine groups. Thus this page reduced the estimated number of arguments by 5. Hence certain changes must be made in the text.

Following the change indicated above on p. 193, *for* 2230, *read* 2225; *for* 928, *read* 904; *for* 533, *read* 552. In addition to the necessary change indicated on p. 197 are others. First of all 9579 was a *misprint* for 9599. The reduction of the number of arguments in WOLFRAM by 5 means that there should be 79 arguments in THIELE, not 74, which are not in WOLFRAM; the additional arguments to the 74 listed are: 6049, 7453, 9707, 9821, 9877. In the last seven lines of p. 196, *for* 3456, *read* 3457; *for* 2280, *read* 2225; *for* 74, *read* 79.

In referring to VEGA's 1794 reprint of WOLFRAM's table, p. 194–195, I failed to note that VEGA gave only 3451 arguments, that is, 6 less than WOLFRAM. This fact was brought to my attention in June 1954, by Dr. ALAN FLETCHER, of the University of Liverpool. I now find that the 6 WOLFRAM arguments omitted in VEGA are the composite numbers 2215, 2225, 2233, 2299, 2387, 2401. Since no one of these omissions is a prime, PETERS' and STEIN's Table 13, based on VEGA, is unchanged.

Next, I refer to two matters in a letter of April 6, 1953, from my friend Mr. C. R. COSENS of the University of Cambridge. On p. 197, I had written concerning THIELE's 1908 table, "Curiously enough WOLFRAM's error in no. 28 (7853) is corrected. This is indeed a major mystery; the only explanation which I can offer is that the typesetter substituted an 8 for a 7, by mistake which THIELE did not observe!" The correction of this error in WOLFRAM was published by BURCKHARDT in 1817. I agree with COSENS that a better explanation of THIELE's achievement in this regard may have been that BURCKHARDT's correction had been brought to his attention.

The second matter which Mr. COSENS discusses at some length in his letter, has reference to the *Caliberstabe*, p. 189, lines 2–5, and footnote 34. Mr. COSENS shows that such a *Caliper Rule*, with scales, was used in connection with artillery; that the weight of a round shot equals the cube of its diameter. The diameter (bore of the gun) would be the cube root of the weight.

R. C. ARCHIBALD

Brown University
Providence, Rhode Island

## A Note on Approximating Polynomials for Trigonometric Functions

High speed automatic digital calculators have two means available for the evaluation of $\sin x$ and $\cos x$ when $x$ is given. Either a table of values of the required function may be held in the store[1] of the machine, and the given value obtained by interpolation; or the machine can calculate the required value from a number of terms of an infinite series. The former procedure is likely to be unsatisfactory if a high degree of accuracy is required since it is rarely possible to store enough function values to make linear, or even quadratic, interpolation feasible.

If function values are to be calculated directly from a series it is well known that the ordinary Taylor series is not the best possible. Chebyshev polynomials have the property that $n$ terms of their series expansion of any function define a polynomial of degree $n$ which minimizes the absolute value of the difference between any function value and the corresponding polynomial value in the interval $(-1, +1)$. Likewise, the Legendre polynomial expansion minimizes the integral square difference between function and approximating polynomial in $(-1, +1)$.

The two types of expansion find distinct applications in programming for a high speed computer.

(1) If the accuracy of the work is the maximum of which the machine is normally capable (often 9 or 10 decimal places) the Chebyshev series is appropriate, since it can give function values to the required precision with the minimum complexity.

(2) If the required accuracy is less than the full capacity of the machine (say 3–6 decimal places) and if the results of a number of calculations are to be combined additively as in summing a Fourier series, then the Legendre polynomial series will be the best basis of approximation.

In the work of this laboratory both of the above applications are of frequent occurrence, and since the numerical coefficients of the series required were apparently not to be found in the literature it is thought that they may be of interest to other workers in the field.

The Chebyshev polynomials are defined, following LANCZOS,[2] by:

$$T_0(x) = 1, \quad T_n(x) = \cos(n \cos^{-1} x),$$

and it may be shown that[3] the required expansions are:

$$\sin(\pi x/2) = 2 \sum_{m=0}^{\infty} (-1)^m J_{2m+1}(\pi/2) T_{2m+1}(x),$$

$$\cos(\pi x/2) = J_0(\pi/2) + 2 \sum_{m=1}^{\infty} (-1)^m J_{2m}(\pi/2) T_{2m}(x).$$

The Legendre polynomials are conveniently defined by RODRIGUES' formula:

$$P_0(x) = 1, \quad P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n,$$

and it can be shown that:[4]

$$\sin(\pi x/2) = 3 \cdot J_{3/2}(\pi/2) \cdot P_1(x) - 7 \cdot J_{7/2}(\pi/2) \cdot P_3(x) + 11 \cdot J_{11/2}(\pi/2) \cdot P_5(x) - \cdots,$$

$$\cos(\pi x/2) = 1 \cdot J_{1/2}(\pi/2) \cdot P_0(x) - 5 \cdot J_{5/2}(\pi/2) \cdot P_2(x) + 9 \cdot J_{9/2}(\pi/2) \cdot P_4(x) - \cdots.$$

To make use of these expansions, for finding optimum polynomials for use with an automatic digital calculator, it is necessary to have numerical values for the functions $J_n(\pi/2)J_{n+1/2}(\pi/2)$. Since these do not appear to have been tabulated it was thought worth constructing the table given below. Values were obtained by means of the well-known series:

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)}\left\{1 - \frac{(z/2)^2}{1!(\nu+1)} + \frac{(z/2)^4}{2!(\nu+1)(\nu+2)} - \cdots\right\}$$

using the value ($z = \pi/2$) taken to 20 decimal places. The resulting values were then rounded off to 11 decimal places and the resulting table checked by an application of the recursion formula:

$$J_\nu(z) = z/2\nu\{J_{\nu+1}(z) + J_{\nu-1}(z)\}.$$

In addition, the value of $J_{11.5}(\pi/2)$ was calculated directly from the recursion formula using the explicit value derived from the initial values:

$$J_{1/2}(\pi/2) = 2/\pi,$$

$$J_{3/2}(\pi/2) = 4/\pi^2,$$

and the table of values of $\pi^{-n}$ to 25 decimal places computed by GLAISHER.[5]

| $n$ | $J_n(\pi/2)$ | $n$ | $J_n(\pi/2)$ |
|---|---|---|---|
| 0 | 0.47200 12157 7 | .5 | .63661 97723 7 |
| 1 | 0.56682 40889 1 | 1.5 | .40528 47345 7 |
| 2 | 0.24970 16291 4 | 2.5 | .13741 70540 3 |
| 3 | 0.06903 58882 9 | 3.5 | .03212 73337 1 |
| 4 | 0.01399 60398 1 | 4.5 | .00575 32170 8 |
| 5 | 0.00224 53571 2 | 5.5 | .00083 61720 0 |
| 6 | 0.00029 83476 0 | 6.5 | .00010 23428 0 |
| 7 | 0.00003 38506 4 | 7.5 | .00001 08228 5 |
| 8 | 0.00000 33522 0 | 8.5 | .00000 10077 8 |
| 9 | 0.00000 02945 7 | 9.5 | .00000 00838 4 |
| 10 | 0.00000 00232 7 | 10.5 | .00000 00063 0 |
| 11 | 0.00000 00016 7 | 11.5 | .00000 00004 3 |
| 12 | 0.00000 00001 1 | 12.5 | .00000 00000 3 |

A. D. BOOTH

University of London
England

[1] A. D. BOOTH and K. H. V. BOOTH, *Automatic Digital Calculators.* Butterworths (London), 1953, p. 180.
[2] C. LANCZOS, *Tables of Chebyshev Polynomials.* NBS Applied Math. Ser. 9, Washington, D. C., 1952, p. V.
[3] G. N. WATSON, *A Treatise on the Theory of Bessel Functions.* 2nd ed., Cambridge, 1944, p. 21.
[4] J. BAUER, *Crelle's Journal*, v. LVI, 1859, p. 113.
[5] J. W. L. GLAISHER, London Math. Soc., *Proc.*, v. 8, 1877, p. 140.

## Continued Fraction Expansion of $2^{\frac{1}{3}}$

The Institute for Advanced Study computer is being used to compute extensive continued fraction expansions of certain real algebraic numbers. The