

$= \lambda^4 - 4\lambda^3 - 73\lambda^2 + 260\lambda + 568$ in the neighborhood of the stated roots and showed that the error in the eigenvalues was less than 1×10^{-7} .

The last component of $x^{(4)}$ in [1] should read $-0.38333123+$ in place of $-0.3833123+$.

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¹E. BODEWIG, "A practical refutation of the iteration method for the algebraic eigenproblem," *MTAC*, v. 8, 1954, p. 237-239.

²R. T. GREGORY, "Computing eigenvalues and eigenvectors of a symmetric matrix on the ILLIAC," *MTAC*, v. 7, 1953, p. 215-220.

³C. G. J. JACOBI, "Ein leichtes verfahren . . .", *Jn. reine angew. Math.*, v. 30, 1846, p. 51-95.

A Note on the Summation of Chebyshev Series

In a recent note [1] I gave tables of the numerical coefficients in the Chebyshev expansions of some common functions, and suggested that these should be used to provide polynomial approximations by truncation and rearrangement in powers of the independent variable.

The purpose of the present note is to show that the truncated Chebyshev series may be evaluated directly without rearrangement, and without using tables of Chebyshev polynomials. The process used is one of recurrence, and is well suited for use with automatic computing machines.

Let the truncated series be denoted by

$$(1) \quad f(x) = A_0 + A_1T_1^*(x) + A_2T_2^*(x) + \cdots + A_NT_N^*(x),$$

where, following the notation of LANCZOS [2] which is used throughout,

$$T_n^*(x) = \cos n\theta, \quad \cos \theta = 2x - 1, \quad 0 \leq x \leq 1.$$

In order to evaluate $f(x)$ for a given value of x we construct the sequence B_N, B_{N-1}, \dots, B_0 , given by $B_{N+2} = B_{N+1} = 0$ and

$$(2) \quad B_n - (4x - 2)B_{n+1} + B_{n+2} = A_n, \quad n = N, N - 1, \dots, 1, 0.$$

Then we have

$$(3) \quad f(x) = B_0 - (2x - 1)B_1.$$

This result may be verified by using the recurrence relation

$$T_n^*(x) - (4x - 2)T_{n+1}^*(x) + T_{n+2}^*(x) = 0.$$

It may be thought that the coefficient $(4x - 2)$ in (2) can give rise to considerable building-up error in $f(x)$. To investigate this point, we observe that the general solution of the equation

$$u_n - (4x - 2)u_{n+1} + u_{n+2} = 0$$

is given by

$$u_n = \alpha T_n^*(x) + \beta U_n^*(x),$$

where $U_n^*(x) = \sin(n + 1)\theta \operatorname{cosec} \theta$, $\cos \theta = 2x - 1$, and α, β are arbitrary constants.

Hence a rounding error $\epsilon(n)$ in A_n or B_n introduces an error $\epsilon_s(n)$ in $B_s (s \leq n)$, given by

$$(4) \quad \epsilon_s(n) = \alpha T_s^*(x) + \beta U_s^*(x),$$

where

$$\left. \begin{aligned} \epsilon(n) &= \alpha T_n^*(x) + \beta U_n^*(x) \\ 0 &= \alpha T_{n+1}^*(x) + \beta U_{n+1}^*(x) \end{aligned} \right\}.$$

Solving the last pair of equations, we obtain

$$\alpha = \epsilon(n) \cdot U_{n+1}^*(x) / T_1^*(x), \quad \beta = -\epsilon(n) \cdot T_{n+1}^*(x) / T_1^*(x)$$

and substituting in (4) we find that

$$(5) \quad \begin{aligned} \epsilon_s(n) &= \{T_s^*(x) \cdot U_{n+1}^*(x) - U_s^*(x) \cdot T_{n+1}^*(x)\} \epsilon(n) / T_1^*(x) \\ &= U_{n-s}^*(x) \cdot \epsilon(n). \end{aligned}$$

The corresponding error in $f(x)$ may be found from (3), and is given by

$$\begin{aligned} \epsilon_0(n) - (2x - 1)\epsilon_1(n) &= \{U_n^*(x) - T_1^*(x) \cdot U_{n-1}^*(x)\} \epsilon(n) \\ &= T_n^*(x) \cdot \epsilon(n), \end{aligned}$$

which is the same as that produced by an error of $\epsilon(n)$ in A_n when the series is summed in the usual way. Hence the rounding-off of B_n to the same number of decimals as A_n can only double the maximum rounding error in $f(x)$. If one or two guarding figures are retained in A_n and B_n , this error may be made negligible compared with the truncation error which has been introduced in replacing the infinite Chebyshev series by $f(x)$.

Thus, although the errors in B_n may become quite large if N is large (see (5)), the error in $f(x)$ is not serious.

We may note that a similar device may be used to evaluate other series in terms of functions which satisfy a linear recurrence relation; for example, Neumann series of Bessel functions.

Let

$$f(x) = a_0 p_0(x) + a_1 p_1(x) + \dots + a_N p_N(x),$$

where

$$(6) \quad p_{n+1}(x) + \alpha_n p_n(x) + \beta_n p_{n-1}(x) = 0,$$

and α_n, β_n may be functions of x as well as of n .

We construct the sequence b_N, b_{N-1}, \dots, b_0 , where $b_{N+1} = b_{N+2} = 0$ and

$$b_n + \alpha_n b_{n+1} + \beta_{n+1} b_{n+2} = a_n, \quad n = N, N - 1, \dots, 0.$$

Then application of the recurrence relation (6) shows that

$$f(x) = b_0 p_0(x) + b_1 \{p_1(x) + \alpha_0 p_0(x)\}.$$

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¹ C. W. CLENSHAW, "Polynomial approximations to elementary functions," *MTAC*, v. 8, 1954, p. 143-147.

² NBS Applied Mathematics Series 9, *Tables of Chebyshev Polynomials $S_n(x)$ and $C_n(x)$* . U. S. Govt. Printing Office, Washington, 1952.

Conjectures Concerning the Mersenne Numbers

Conjectures concerning the Mersenne numbers are appropriate since they were inaugurated with one. A conjecture [1] that seems likely to be false, but unlikely to be proved false, is that all numbers p_n are prime ($n = 1, 2, 3, \dots$), where, for example, p_4 is

$$\begin{array}{r} 2^2 - 1 \\ 2^2 - 1 \\ 2^2 - 1 \\ 2^2 - 1 \end{array}$$

Recursively, $p_1 = 3$, $p_{n+1} = 2^{p_n} - 1$. The first four are 3, 7, 127 and $2^{127} - 1$, all known to be prime. Any factor of p_5 is congruent to 1 modulo p_4 , so p_5 certainly has no factor less than 2^{127} . Similarly

$$2^{2^{281}} - 1$$

is not divisible by any known prime, if $2^{2281} - 1$ is still the largest known prime [2]. One can try to argue about the probability that a number of the form $2^p - 1$ is prime, when p is known to be prime. The probability that a whole number x is prime is about $1/\log x$, and is close to

$$(1) \quad \frac{1}{2} e^{\gamma} \left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) \left(1 - \frac{1}{5}\right) \dots \left(1 - \frac{1}{q}\right)$$

where $q \doteq \sqrt{x}$, so the factors $(1 - \frac{1}{2})$, $(1 - \frac{1}{3})$, etc., can be regarded as probabilities that are not far from independent. But if $x = 2^p - 1$, only every p th factor of (1) should be taken, and the probability apparently ought to be about the p th root of $1/p \cdot \log 2$, which is approximately 1 when p is large. But this argument is also invalid, as we may see from the statistics of Mersenne primes [2]. We may see from these statistics (assuming them to contain no gaps), that, if m_n denotes the n th Mersenne prime ($m_1 = 3$), then

$$2.18 \log \log m_n < n < 2.72 \log \log m_n \quad (3 \leq n \leq 17)$$

while

$$2.31 \log \log m_{17} = 17.$$