TECHNICAL NOTES AND SHORT PAPERS

Remark on Determination of Characteristic Roots by Iteration

In this journal [1] one example was given of a 4th order symmetric matrix for which a certain algorithm did not quickly or conveniently furnish the values of the characteristic roots; indeed one of the trial values converged to a negative limit by "passing through $+\infty$."

The same matrix was subjected to an established machine routine in this laboratory for the determination of characteristic roots and corresponding characteristic vectors by iterated orthogonal transformation. The computing procedure is due essentially to Jacobi [3] and is very similar to that reported in this journal [2] by ROBERT T. GREGORY of the University of Illinois; it was called to our attention by Dr. H. H. GOLDSTINE of the Institute for Advanced Study.

The computational procedure employed is that of applying sets of successive plane rotations, each of which reduces to zero one of the non-diagonal elements of the matrix, until the sum of squares of non-diagonal elements fails to decrease. For a matrix of order n, the number of plane rotations in each set is $\frac{1}{2}n(n-1)$, one for each of the non-diagonal element positions. The computation for each of the rotations requires multiplication by two matrices which differ in only four places from the identity matrix, and, because of the symmetric nature of the original matrix, the number of multiplications needed for each rotation is only $4n$.

The matrix in question was

$$A = \begin{pmatrix} 2 & 1 & 3 & 4 \\ 1 & -3 & 1 & 5 \\ 3 & 1 & 6 & -2 \\ 4 & 5 & -2 & -1 \end{pmatrix}$$

and five sets of six rotations each, i.e., not more than 30 plane rotations, produced the eigenvalues

$$\lambda_1 = +5.6688643, \quad \lambda_2 = -1.5731907, \quad \lambda_3 = +7.9329047, \quad \lambda_4 = -8.0285783;$$

an independent computation showed that the error in each value is less than 1×10^{-7}. A simultaneously performed auxiliary computation involving $4n$ additional multiplications gave the eigenvectors as columns of the matrix

$$U = \begin{pmatrix} +.3787\ 0268 & -.6880\ 4793 & +.5601\ 4450 & -.2634\ 6239 \\ +.3624\ 1904 & +.6241\ 2285 & +.2116\ 3276 & -.6590\ 4071 \\ -.5379\ 3516 & +.2598\ 0086 & +.7767\ 0826 & +.1996\ 3352 \\ +.6601\ 9881 & +.2637\ 5026 & +.1953\ 8161 & +.6755\ 7335 \end{pmatrix}.$$

An independent (hand) computation verified that the matrix product

$$U\,(\text{Diag. }(+.5.6688643,\ -1.5731907,\ +7.9329047,\ -8.0285783))\,U^T = A_1$$

differed from A in all sixteen elements by less than $4 \cdot 10^{-7}$.

Similar (hand) computation evaluated the characteristic polynomial $f(\lambda)$

$= \lambda^4 - 4\lambda^3 - 73\lambda^2 + 260\lambda + 568$ in the neighborhood of the stated roots and showed that the error in the eigenvalues was less than 1×10^{-7}.

The last component of $x^{(4)}$ in [1] *should read* $-0.38333123+$ *in place of* $-0.3833123+$.

J. L. Brenner
G. W. Reitwiesner

Computing Laboratory
Ballistic Research Laboratories
Ordnance Corps, U. S. Army
Aberdeen Proving Ground, Maryland

Sponsored by the Office of Ordnance Research, U. S. Army.

[1] E. Bodewig, "A practical refutation of the iteration method for the algebraic eigenproblem," *MTAC*, v. 8, 1954, p. 237–239.
[2] R. T. Gregory, "Computing eigenvalues and eigenvectors of a symmetric matrix on the ILLIAC," *MTAC*, v. 7, 1953, p. 215–220.
[3] C. G. J. Jacobi, "Ein leichtes verfahren . . .", *Jn. reine angew. Math.*, v. 30, 1846, p. 51–95.

A Note on the Summation of Chebyshev Series

In a recent note [1] I gave tables of the numerical coefficients in the Chebyshev expansions of some common functions, and suggested that these should be used to provide polynomial approximations by truncation and rearrangement in powers of the independent variable.

The purpose of the present note is to show that the truncated Chebyshev series may be evaluated directly without rearrangement, and without using tables of Chebyshev polynomials. The process used is one of recurrence, and is well suited for use with automatic computing machines.

Let the truncated series be denoted by

$$f(x) = A_0 + A_1T_1^*(x) + A_2T_2^*(x) + \cdots + A_NT_N^*(x), \tag{1}$$

where, following the notation of Lanczos [2] which is used throughout,

$$T_n^*(x) = \cos n\theta, \quad \cos\theta = 2x - 1, \quad 0 \leqslant x \leqslant 1.$$

In order to evaluate $f(x)$ for a given value of x we construct the sequence B_N, $B_{N-1}, \cdots, B_0$, given by $B_{N+2} = B_{N+1} = 0$ and

$$B_n - (4x - 2)B_{n+1} + B_{n+2} = A_n, \quad n = N, N-1, \cdots, 1, 0. \tag{2}$$

Then we have

$$f(x) = B_0 - (2x - 1)B_1. \tag{3}$$

This result may be verified by using the recurrence relation

$$T_n^*(x) - (4x - 2)T^*_{n+1}(x) + T^*_{n+2}(x) = 0.$$

It may be thought that the coefficient $(4x - 2)$ in (2) can give rise to considerable building-up error in $f(x)$. To investigate this point, we observe that the general solution of the equation

$$u_n - (4x - 2)u_{n+1} + u_{n+2} = 0$$