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On the Computation of log Z and arc tan Z

In a previous note, Clenshaw [1] has given numerical values of coefficients for the expansion of some transcendental functions in Chebyshev polynomials. In particular, he tabulates the coefficients for log $(1 + x)$ and arc tan x to nine decimal places. Here, treating these functions in a more general form, we determine precise theoretical coefficients and show that the development leads to formulas for computation in the complex domain.

The Chebyshev polynomials of the first kind which we use in the range $-1 \leq x \leq 1$ are defined as

$$T_n(x) = \cos n\theta, \quad x = \cos \theta. \tag{1}$$

For a discussion of the properties of these functions, see the work of Lanczos [2]. If $f(x)$ is bounded and continuous in a given range, then the expansion

$$f(x) = \tfrac{1}{2}C_0 + \sum_{k=1}^{\infty} C_k T_k(x) \tag{2}$$

is convergent, and

$$C_k = \frac{2}{\pi}\int_{-1}^{1} \frac{f(x)T_k(x)dx}{\sqrt{1 - x^2}}. \tag{3}$$

The transformation $x = 2y - 1$ shifts the range to $0 \leq y \leq 1$, and we denote the "shifted" polynomial as $B_n(y)$.

Consider $f(y) = 1/(y + a)$. Utilizing the above together with a known formula [3], we find

$$1/(y + a) = [1 + 2\sum_{k=1}^{\infty} (-)^k q^k B_k(y)]/(a^2 + a)^{\frac{1}{2}}; \; q = 2a + 1 - 2(a^2 + a)^{\frac{1}{2}} \tag{4}$$

$$0 \leq y \leq 1; \quad |\arg a| \leq \pi/2, \quad a \neq 0.$$

From a numerical point of view, (4) is pathologic. If $a = 1$, the Taylor series development of $1/(y + 1)$ is slowly convergent near $y = 1$; at $y = 1$ it is divergent. However, (4) is precise and rapidly convergent in this region. The above is a striking example to show the strength of a Chebyshev expansion and the comparative weakness of a Taylor series expansion. The integral of (4) is not pathologic, and we find

$$(5)\qquad \log(y+a) - \log a = 2\sum_{k=1}^{\infty} \frac{q^k}{k}\{1 - (-)^k B_k(y)\}$$

$$= -2\log(1-q) - 2\sum_{k=1}^{\infty}\frac{(-)^k q^k}{k} B_k(y).$$

In particular, if $y = 1$,

$$(6)\qquad \log(a+1) - \log a = 4\sum_{k=0}^{\infty}\frac{q^{2k+1}}{2k+1},$$

and if further $a = 1$, we have the interesting series

$$(7)\qquad \log 2 = 4\sum_{k=0}^{\infty}\frac{(3-2\sqrt{2})^{2k+1}}{2k+1}.$$

If in the first expansion of (5) we retain N terms, then a bound for the error committed is

$$(8)\qquad |E_N| \leq \frac{4|q^{N+1}|}{(N+1)|(1-q)|}.$$

It is of interest to compare (6) with the well known expression

$$(9)\qquad \log(a+1) - \log a = 2\sum_{k=0}^{\infty}\frac{(2a+1)^{-2k-1}}{(2k+1)}.$$

Examination of the factor $r = q(2a+1)$ shows that the convergence of (6) is much more rapid than (9). Indeed if $a = 1$, r is quite close to its asymptotic value for large a which is $\frac{1}{2}$. Consider the error bounds for (6) and (9) which are easily derived after the manner of (8). If $a = 1$, the ratio of these bounds is about 2^{-2N-2}, and the economy of (6) is evident.

We now turn to the expansion for arc tan z. In (4), replace a by a^2, and y by y^2. Noting that $B_k(y^2) = T_{2k}(y)$, and integrating, we find that

$$(10)\qquad \arctan y/a = 2\sum_{k=0}^{\infty}\frac{(-)^k R^{2k+1}}{2k+1} T_{2k+1}(y);\quad R = (a^2+1)^{\frac{1}{2}} - a,$$

$$-1 \leq y \leq 1.$$

This series converges for all values of a in the right half plane except on the line joining the points $a = \pm i$. A convenient bound for the error committed after using N terms is

$$(11)\qquad |F_N| \leq \frac{2|R^{2N+3}|}{(2N+3)|1-R^2|}.$$

We also have

$$(12)\quad \arctan y = \pi/4 + 2\sum_{k=0}^{\infty}\frac{(-)^k(\sqrt{2}-1)^{2k+1}}{(2k+1)} T_{2k+1}\left(\frac{y-1}{y+1}\right);\quad 0 \leq y \leq \infty,$$

and the interesting series

$$(13)\qquad \pi/8 = \sum_{k=0}^{\infty} \frac{(-)^k(\sqrt{2}-1)^{2k+1}}{(2k+1)}; \quad \pi/12 = \sum_{k=0}^{\infty} \frac{(-)^k(2-\sqrt{3})^{2k+1}}{(2k+1)}.$$

Since

$$(14)\qquad \text{arc tan } z = \pi/2 - \text{arc tan } 1/z,$$

we see that (10) can be used for computation everywhere in the right half plane except when a is near the imaginary axis and $|a| \leq 1$. In this region we use the fact that

$$(15)\qquad \text{arc tan } z = \text{arc tan } (x + iy) = \alpha + i\beta$$

$$\alpha = \tfrac{1}{2} \text{arc tan} \frac{2x}{1 - x^2 - y^2}; \quad \beta = \tfrac{1}{4} \log \frac{(1+y)^2 + x^2}{(1-y)^2 + x^2},$$

and the appropriate series given above. Of course, arc tan z can always be computed in this way, but (10) seems advantageous since only a single formula is needed. Our expansions are also useful in connection with the formula

$$(16)\qquad \log z = \tfrac{1}{2} \log |z^2| + i \text{ arc tan } y/x; \quad z = x + iy$$

For numerical purposes it may be desirable to truncate the infinite expansion and rearrange the terms so that a polynomial approximation is obtained. Details are left to the reader. However, this procedure is not necessary since the Chebyshev polynomials satisfy simple recurrence relations. Also, a previous suggestion by Clenshaw [4] is valuable. With $a = 1$, we have truncated the expansions arising from (5) and (6) for several values of N, and expressed the finite series as a polynomial in y. We find that the corresponding coefficients given in a recent volume by Hastings [5] are quite close to ours as are also the error bounds. In the latter reference, the treatment is semi-theoretical, as the development is based in part on known tabular data for the functions. The present treatment is more general and seems more pleasing as the need for tabular data is obviated.

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