TECHNICAL NOTES AND SHORT PAPERS

Note upon the Numerical Evaluation of Limits of Sequences

1. Introduction. It is well known that, if a sequence s_n tends to a limit s such that

$$s = s_n + r_n$$

where

$$r_n = \frac{A}{n^2} + \frac{B}{n^3} + \cdots,$$

A, B, etc., independent of n, it is possible, using Richardson's deferred approach to the limit [1], to find a better approximation to s by calculating s_n for two different values of n and solving for s by eliminating A and neglecting terms of order less than $O(n^{-2})$; e.g.,

$$s_n' = \frac{4s_{2n} - s_n}{3} + O(n^{-3})$$

is a better approximation to s than was s_n. This formula suffers from the disadvantage that it is necessary to calculate the value of the sequence at n and also at $2n$. Furthermore this formula applies only when the remainder term is of the form indicated. It is the purpose of this note to show that, under certain conditions, an improved approximation may be derived from the values of the sequence at n and at $n + 1$.

2. Analysis. Let $r_n = Kp_n(1 + q_n)$ where $q_n = o(1)$ and p_n is a simple function of n which satisfies this relation. K is a possibly unknown constant. The meaning of the term simple in this connection may be understood from the following examples.

(1) $$r_n = \int_n^\infty \frac{dy}{y^4 + ay^3 + b}, \qquad p_n = \frac{1}{n^3},$$

(2) $$r_n = \int_{x_1}^{x_2} \frac{dx}{x^2 + n^2}, \qquad p_n = \frac{1}{n},$$

(3) $$r_n = \frac{1}{2^n}\frac{1}{n}\int_0^n \tanh(x + e^{-x})dx, \quad p_n = \frac{1}{2^n}.$$

The exact form of p_n is unimportant. If r_n were known exactly, there would of course be no need to carry out this process. If the order of r_n can be found, a suitable p_n can also be found. We have

$$s = s_n + Kp_n(1 + q_n)$$
$$s = s_{n+1} + Kp_{n+1}(1 + q_{n+1}).$$

Multiplying the first equation by p_{n+1} and the second by p_n, it follows that

$$s = s_n' + r_n'$$

where

$$s_n' = \frac{p_n s_{n+1} - p_{n+1} s_n}{p_n - p_{n+1}}$$

$$r_n' = \frac{K p_n p_{n+1}(q_{n+1} - q_n)}{p_n - p_{n+1}}.$$

Thus a better approximation to s will have been obtained if

$$\left|\frac{r_n'}{r_n}\right| = \left|\frac{q_{n+1} - q_n}{p_{n+1} - p_n} \cdot \frac{p_{n+1}}{1 + q_n}\right| < 1.$$

This inequality must be verified before we are able to use the method. In section 3, a full discussion will be given of the calculation of $\pi/2$.

3. Calculation of $\pi/2$. A formula for $\pi/2$ is $\pi/2 = \lim s_n$, where

$$s_n = \int_0^n \frac{dx}{1 + x^2}.$$

It follows that

$$r_n = \int_n^\infty \frac{dx}{1 + x^2} = \int_0^{1/n} \frac{dy}{1 + y^2}.$$

Clearly a suitable p_n is $1/n$ and

$$r_n = \int_0^{1/n} \frac{dy}{1 + y^2} = \frac{1}{n}(1 + q_n) < \frac{1}{n}$$

as

$$q_n = -n \int_0^{1/n} \frac{y^2 dy}{1 + y^2} > -\frac{1}{3n^2}$$

and

$$1 + q_n > 1 - \frac{1}{3n^2}.$$

Also

$$s_n' = (n + 1)s_{n+1} - n s_n = s_{n+1} + n(s_{n+1} - s_n).$$

(It will be observed that $p_n = 1/n$ gives rise to a particularly convenient form for s_n'.) Furthermore

$$\frac{p_n - p_{n+1}}{p_{n+1}} = \frac{1}{n}.$$

It may be shown that

$$q_{n+1} - q_n = \int_0^1 \frac{2n + 1}{n^2(n + 1)^2} \frac{z^2 dz}{\{1 + (z/n)^2\}\{1 + (z/n + 1)^2\}} < \frac{2n + 1}{3n^2(n + 1)^2}.$$

It follows that

$$\left|\frac{r_n'}{r_n}\right| < \frac{2n^2 + n}{(3n^2 - 1)(n + 1)^2}$$

$$r_n' < \frac{2n + 1}{(3n^2 - 1)(n + 1)^2}$$

(r_n' and r_n are both positive). It follows therefore that the convergence of the method is fairly rapid. If $n = 1$ the remainder ratio is 3/8, whereas if $n = 10$ it is .006.

$$\begin{aligned} s_{10} &= 1.47113 \\ s_{11} &= 1.48014 \\ s_{10}' &= s_{11} + 10(s_{11} - s_{10}) \\ &= 1.5702. \end{aligned}$$

It will be observed that, because $s_{11} - s_{10}$ is multiplied by 10, a decimal place of accuracy is lost. Also the figures given for s_{10} and s_{11} may be in error by .5 units of the last place, and so $s_{10}' = 1.5702 \pm .0001$, it being possible for $s_{11} - s_{10}$ to have an error 1 in the last place. Also $r_{10}' < 21/299 \times 121 = .0006$ and is positive. Thus $1.5702 - .0001 < \pi/2 < 1.5702 + .0001 + .0006$ or $1.5701 < \pi/2 < 1.5709$, comparing well with the true value 1.5708. The agreement is remarkably good for the comparative roughness of the approximation.

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1. Sir Harold Jeffreys & Bertha Swirles Jeffreys, *Methods of Mathematical Physics*, Cambridge Univ. Press, New York, 1956, p. 265.

Factors of Fermat Numbers

The writer has prepared a multiple precision routine for the SWAC which tests numbers of the form $N = k \cdot 2^n + 1$ for primeness. If N is prime, it is then tested to find whether it divides any Fermat number $F_m = 2^{2^m} + 1$. The running time for either test is about a minute and a half for n near 500, and about seven minutes for n near 1000. There is also a preliminary sieve routine which examines N for small factors. If there is no small factor, the smallest positive number a for which $(a/N) = -1$ is found. The congruence $a^{(N-1)/2} \equiv -1 \pmod N$ is then a necessary and sufficient condition for primeness, at least if $k < 2^n$ [1].

During the period September–November 1956, the cases $k = 3, 5, 7$ were run for $n < 1024$, and the odd values of k from 9 to 57 were run for $n < 512$. Some isolated larger values of k have also been used. The cases for k up to 17 and the other results listed below have been checked by a second run. The work is con-