Synthetic division was not used this time because it was felt that the round-off error would be carried into the reduced polynomial. Observing the erratic property of the last digits it was concluded that the limit of accuracy in the worst case is about 12 decimal places. There seems to be no question for values of m from 1 to 5 where $n = 0, 1$, and 2.

A check of the b_j was made through the fact that

$$\sum_{j=1}^{m} b_j = \int_0^1 x^n dx = \frac{1}{n+1}$$

for each polynomial of degree m associated with x^n. Here again the limit of accuracy is put at about the twelfth decimal place for the worst possible cases (*i.e.*, with m and n both large). If one wishes to round off values to ten or eleven decimal places there should be no doubt of the accuracy.

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1. R. COURANT & D. HILBERT, *Methods of Mathematical Physics*, First English edition, Translated and Revised from the German Original, Interscience Publishers, Inc., New York, v. 1, 1953, p. 90–91.
2. P. C. HAMMER, O. J. MARLOWE, & A. H. STROUD, "Numerical integration over simplexes and cones," *MTAC*, v. 10, 1956, p. 130–137.
3. A. ERDÉLYI (editor), *Higher Transcendental Functions*, v. 2, McGraw-Hill Book Co., Inc., New York, 1953, p. 168–170.
4. WILLIAM VERNON LOVITT, *Linear Integral Equations*, McGraw-Hill Book Co., Inc., New York, 1924.

Tables of the Exponential Integral Ei(x)

In some molecular structure calculations it is desirable to have values of the integral Ei(x) to higher accuracy than is provided by the standard tables [1]. Direct computation of the values needed is extremely tedious over a wide range of the argument intermediate between the regions where expansions about zero and infinity are useful. However, if a net of sufficiently accurate values of Ei(x) has already been computed, then the generation of additional values intermediate between these becomes a problem of a considerably lesser magnitude. A few such values have been published [2, 3]. Kotani [4] has given a considerable number of values in his most valuable compilation, but unfortunately they are of somewhat fewer significant figures than under discussion here. A prohibitive amount of labor would be required to make such an extensive accurate table that only simple interpolation formulas need be used. However, a most useful intermediate objective could be achieved if a table were prepared with entries sufficiently closely spaced that rapidly convergent series expansions could be used to obtain further values. Accordingly, an 18-significant digit table was prepared by the means described below.

The two usual ways of obtaining values of $\mathrm{Ei}(x)$ *ab initio* are from the Taylor series

$$\mathrm{Ei}(x) = \gamma + \ln|x| + x + \frac{x^2}{2\cdot 2!} + \frac{x^3}{3\cdot 3!} + \cdots \tag{1}$$

and from the asymptotic formula

$$\mathrm{Ei}(x) \sim e^x\left[\frac{1}{x} + \frac{1}{x^2} + \frac{2!}{x^3} + \frac{3!}{x^4} + \cdots\right]. \tag{2}$$

The Taylor series becomes inconvenient beyond about $|x| = 4$, and the asymptotic formula is incapable of giving sufficiently accurate values for $|x| < 50$. It is necessary to generate values in the range $4 < |x| < 50$ with the aid of a suitable recursion formula.

Two recursion formulas have been employed recently for this purpose. The first, used by Ruedenberg, Roothaan, and Jaunzemis [5], may be written in the form

$$\mathrm{Ei}(x) = \mathrm{Ei}(x-h) + \sum_{k=1}^{\infty} \frac{(-h)^k}{k!} A_{k-1}(-x) \tag{3}$$

where $A_n(x)$ is defined as

$$A_n(x) = x^{-n-1}e^{-x}n! \sum_{k=0}^{n} \frac{x^k}{k!}. \tag{4}$$

The integrals $A_n(x)$ have been tabulated by Kotani [4] for positive x, though not all to sufficient accuracy for use in (3). $A_n(x)$ for negative x can be obtained, also not with sufficient accuracy for use in (3) from Kotani's tables of $B_n(x)$ and the relation $B_n(x) = (-1)^{n+1}A_n(-x) - A_n(x)$.

A second recursion formula for $\mathrm{Ei}(x)$, used by Kotani [4], may be cast in the form

$$e^{-x}\,\mathrm{Ei}(x) = e^{-h}[e^{-(x-h)}\,\mathrm{Ei}(x-h)] + x^{-1}\sum_{k=0}^{\infty} x^{-k}R_k(h) \tag{5}$$

where the coefficients $R_n(h)$ are related to the A_n by

$$R_n(h) = n! - h^{n+1}A_n(h). \tag{6}$$

This second expansion is far more convenient than the first, as very little computation is required to apply it for given h to different values of x. Moreover, the coefficients $R_n(h)$ may be obtained without great labor by consideration of the recursion relation between $R_n(h)$ and $R_{n-1}(h)$,

$$R_n(h) = nR_{n-1}(h) - h^n e^{-h}, \quad n \geq 1, \tag{7}$$

together with the value $R_0(h) = 1 - e^{-h}$. Equation (7) may be verified by induction. A third recursion formula, intermediate in character between (3) and (5), was given by Gram [3].

Equations (5) and (7) were employed to compute the values of $e^{-x}\,\mathrm{Ei}(x)$ presented here for $|x| \geq 4$. By using the formula to proceed from smaller to larger x, no loss in significant figures accompanies repeated use of the formula, and the final accuracy is determined by the number of figures carried in the R_n. Starting values of Ei(-50) and Ei(4) were computed by (2) and (1), respectively, and the values of Ei(-4) and Ei(50) obtained by repeated use of the recursion formulas were checked against values computed, respectively, from (1) and (2). Twenty-one significant figures were carried throughout the calculations, and the final results were rounded off to 18. Some intermediate values were also obtained in more than one way as a check. The same work sheets were used to compute for both positive and negative values of x, thus further reducing the possibility of error in formulation of the $x^{-k}R_k$. All check conditions were satisfied to 20 or more significant digits.

The 21-figure values of $e^{-x}\,\mathrm{Ei}(x)$ were multiplied by 19-figure values of e^x [6] to obtain the values of Ei(x) presented to 18 figures. Table 1 gives our final values of Ei$(-x)$ and $-e^x\,\mathrm{Ei}(-x)$, together with additional values obtained by Bretschneider [2]. In table 2 are presented the corresponding quantities for positive x. Table 3 gives values of the $R_n(h)$ to sufficient precision to interpolate from the nearest table value anywhere for $|x| > 4$. Miscellaneous constants used are listed in table 4.

TABLE 1. *Exponential Integrals, Negative Arguments*

The numbers in parentheses are the powers of 10 by which the entries so marked must be multiplied.

x	$-\mathrm{Ei}(-x)$		$-e^x\,\mathrm{Ei}(-x)$
1.0	0.21938 39343 95520 274	(0)	0.59634 73623 23194 0743
2.0	0.48900 51070 80611 196	(− 1)	0.36132 86168 88222 5846
3.0	0.13048 38109 41970 374	(− 1)	0.26208 37402 55318 4961
4.0	0.37793 52409 84890 648	(− 2)	0.20634 56499 01055 8331
4.4	0.23360 10040 65582 899	(− 2)	0.19027 00470 21504 7599
4.8	0.14529 93936 87832 422	(− 2)	0.17655 38999 22275 4908
5.2	0.90862 16124 48659 585	(− 3)	0.16470 78767 04783 2961
5.6	0.57084 01696 48211 674	(− 3)	0.15437 02562 92501 3756
6.0	0.36008 24521 62658 659	(− 3)	0.14526 76292 33886 8938
6.4	0.22794 79559 29372 847	(− 3)	0.13718 93461 69177 0926
6.8	0.14475 77922 44899 578	(− 3)	0.12997 03917 12376 9249
7.2	0.92188 11688 71620 423	(− 4)	0.12347 95998 70253 3931
7.6	0.58858 77207 53838 951	(− 4)	0.11761 13567 51900 9850
8.0	0.37665 62284 39249 018	(− 4)	0.11227 96392 53499 3118
9.0	0.12447 35417 80062 721	(− 4)	0.10086 19555 80640 9291
10.0	0.41569 68929 68532 428	(− 5)	0.09156 33339 39788 0819

TABLE 1—*Continued*

x	$-\mathrm{Ei}(-x)$	$-e^x\,\mathrm{Ei}(-x)$
11.0	0.14003 00304 24744 178 (− 5)	0.08384 17788 60345 9505
12.0	0.47510 81824 67249 393 (− 6)	0.07732 61331 38919 2306
13.0	0.16218 66218 80143 287 (− 6)	0.07175 35335 24462 3954
14.0	0.55656 31111 14518 212 (− 7)	0.06693 25181 83439 6292
15.0	0.19186 27892 14786 698 (− 7)	0.06272 02791 07409 2418
16.0	0.66404 87249 44104 278 (− 8)	0.05900 81036 08556 4362
17.0	0.23064 31989 82165 449 (− 8)	0.05571 17557 43476 7274
18.0	0.80360 90344 82867 766 (− 9)	0.05276 49444 02625 0680
19.0	0.28078 29097 06079 527 (− 9)	0.05011 47797 95479 9196
20.0	0.98355 25290 64988 169 (−10)	0.04771 85454 95960 8417
21.0	0.34532 01267 14675 627 (−10)	0.04554 13616 54505 1706
22.0	0.12149 37895 62043 727 (−10)	0.04355 44646 92617 8827
23.0	0.42826 84795 66567 262 (−11)	0.04173 39215 56000 6238
24.0	0.15123 05893 99970 577 (−11)	0.04005 96555 23840 3225
25.0	0.53488 99755 34021 664 (−12)	0.03851 46988 44904 0221
26.0	0.18946 85885 67497 824 (−12)	0.03708 46128 39349 6946
27.0	0.67206 37435 26204 038 (−13)	0.03575 70332 31537 5283
28.0	0.23869 41511 93373 316 (−13)	0.03452 13102 36847 3608
29.0	0.84877 59778 35356 284 (−14)	0.03336 82211 22561 0743
30.0	0.30215 52010 68881 254 (−14)	0.03228 97387 58980 1252
31.0	0.10767 67038 61623 826 (−14)	0.03127 88438 29256 7742
32.0	0.38409 61801 22506 683 (−15)	0.03032 93713 77333 2475
33.0	0.13713 84348 44874 657 (−15)	0.02943 58845 81341 5053
34.0	0.49006 76118 39278 771 (−16)	0.02859 35702 76001 3970
35.0	0.17527 05938 99473 720 (−16)	0.02779 81519 71952 9646
36.0	0.62733 39009 76224 159 (−17)	0.02704 58170 44635 3756
37.0	0.22470 20697 58857 122 (−17)	0.02633 31554 69674 7075
38.0	0.80541 06914 29074 987 (−18)	0.02565 71080 22614 4181
39.0	0.28887 79301 52270 100 (−18)	0.02501 49222 79330 0217
40.0	0.10367 73261 45165 697 (−18)	0.02440 41150 79628 5763
41.0	0.37231 66776 45997 772 (−19)	0.02382 24403 72451 8529
42.0	0.13377 90881 00117 751 (−19)	0.02326 78615 63199 8260
43.0	0.48094 96556 95001 785 (−20)	0.02273 85276 44261 4456
44.0	0.17299 59874 28164 776 (−20)	0.02223 27525 18155 0389
45.0	0.62256 90809 46238 364 (−21)	0.02174 89970 25785 2704
46.0	0.22415 31759 74429 975 (−21)	0.02128 58532 75605 0273
47.0	0.80741 97842 72581 395 (−22)	0.02084 20309 37076 1867
48.0	0.29096 64190 40584 234 (−22)	0.02041 63452 16965 1554
49.0	0.10489 81164 23680 237 (−22)	0.02000 77062 82189 8425
50.0	0.37832 64029 55045 902 (−23)	0.01961 51099 30114 8704

TABLE 2. *Exponential Integrals, Positive Arguments*

The numbers in parentheses are the powers of 10 by which the entries so marked must be multiplied.

x	Ei(x)		e^{-x} Ei(x)
1.0	0.18951 17816 35593 676	(1)	0.69717 48832 35066 0688
2.0	0.49542 34356 00189 016	(1)	0.67048 27097 90073 2810
3.0	0.99338 32570 62541 656	(1)	0.49457 64013 48641 2350
4.0	0.19630 87447 00562 200	(2)	0.35955 20078 63620 6962
4.4	0.26008 97327 16051 460	(2)	0.31932 10053 85318 4003
4.8	0.34697 88987 37753 276	(2)	0.28555 48567 95924 4435
5.2	0.46624 85050 57967 478	(2)	0.25720 89914 23568 2427
5.6	0.63101 78597 42992 615	(2)	0.23334 18047 99632 7448
6.0	0.85989 76214 24392 048	(2)	0.21314 73100 81593 6031
6.4	0.11793 48657 00181 887	(3)	0.19595 55338 64928 3941
6.8	0.16270 70875 71431 415	(3)	0.18121 91105 15268 3455
7.2	0.22568 78077 01063 810	(3)	0.16849 53143 53259 2881
7.6	0.31457 18784 98083 578	(3)	0.15742 79475 14471 5487
8.0	0.44037 98995 34838 269	(3)	0.14773 09983 73400 9966
9.0	0.10378 78290 71708 959	(4)	0.12808 43565 23213 8681
10.0	0.24922 28976 24187 776	(4)	0.11314 70204 73410 7780
11.0	0.60714 06374 09861 151	(4)	0.10140 28126 36185 3129
12.0	0.14959 53266 63975 289	(5)	0.09191 45454 08896 5894
13.0	0.37197 68849 06890 356	(5)	0.08407 90291 67225 1376
14.0	0.93192 51363 39653 713	(5)	0.07749 22514 92093 0115
15.0	0.23495 58524 90768 304	(6)	0.07187 35404 92410 7085
16.0	0.59556 09986 70837 002	(6)	0.06702 15610 41399 0873
17.0	0.15166 37894 04251 688	(7)	0.06278 78642 32855 1665
18.0	0.38779 04330 59744 350	(7)	0.05906 04044 06932 4489
19.0	0.99509 07251 04684 476	(7)	0.05575 29076 96429 0258
20.0	0.25615 65266 40565 888	(8)	0.05279 77952 79648 1322
21.0	0.66127 18635 54849 213	(8)	0.05014 13386 46825 6145
22.0	0.17114 46713 00363 668	(9)	0.04774 02599 85690 4452
23.0	0.44396 63698 30271 221	(9)	0.04555 92944 77218 8886
24.0	0.11541 15391 84918 295	(10)	0.04356 94088 38540 5760
25.0	0.30059 50906 52554 869	(10)	0.04174 64774 50664 5301
26.0	0.78429 40991 89818 637	(10)	0.04007 02837 69455 1129
27.0	0.20496 49711 98808 124	(11)	0.03852 37569 74926 3125
28.0	0.53645 11859 23146 942	(11)	0.03709 23813 73152 0594
29.0	0.14059 91957 58406 905	(12)	0.03576 37344 29948 2813
30.0	0.36897 32094 07274 197	(12)	0.03452 71217 92361 8461
31.0	0.96945 55759 68393 966	(12)	0.03337 32862 79497 7246
32.0	0.25500 43566 35778 693	(13)	0.03229 41738 81757 5328
33.0	0.67146 40184 07649 756	(13)	0.03128 27441 22948 5202
34.0	0.17698 03724 41162 685	(14)	0.03033 28152 54435 4435
35.0	0.46690 55014 46615 954	(14)	0.02943 89370 25758 9265

TABLE 2—*Continued*

x	Ei(x)	e^{-x} Ei(x)
36.0	0.12328 52079 91209 769 (15)	0.02859 62854 56693 8000
37.0	0.32579 88998 67226 400 (15)	0.02780 05752 89820 6486
38.0	0.86163 88199 96578 654 (15)	0.02704 79867 47528 2957
39.0	0.22804 46200 30190 260 (16)	0.02633 51039 35588 4305
40.0	0.60397 18263 61124 158 (16)	0.02565 88627 85975 1452
41.0	0.16006 64914 32450 411 (17)	0.02501 65068 56911 8321
42.0	0.42447 96092 13685 076 (17)	0.02440 55496 39127 1203
43.0	0.11263 48290 16696 676 (18)	0.02382 37422 76697 9580
44.0	0.29904 44718 63233 668 (18)	0.02326 90458 15478 3594
45.0	0.79439 16035 70445 377 (18)	0.02273 96072 54528 2793
46.0	0.21113 42388 64782 419 (19)	0.02223 37388 05633 9048
47.0	0.56143 29680 81034 311 (19)	0.02174 98998 70127 3613
48.0	0.14936 30213 11299 314 (20)	0.02128 66813 26253 2629
49.0	0.39754 42747 90374 484 (20)	0.02084 27917 88506 7972
50.0	0.10585 63689 71316 910 (21)	0.02041 70455 55943 9873

TABLE 3. *Interpolation Coefficients*

n	$R_n(1)$	$R_n(0.5)$	$R_n(0.4)$
0	0.63212 05588 28557 67840	0.39346 93402 87366 57640	0.32967 99539 64360 69926
1	0.26424 11176 57115 35681	0.09020 40104 31049 86459	0.06155 19355 50104 97896
2	0.16060 27941 42788 3920	0.02877 53559 33941 3733	0.01585 26637 34507 6698
3	0.11392 89412 56922 854	0.01050 97353 37744 942	0.00465 75082 57242 094
4	0.08783 63238 56249 10	0.00413 07751 18940 18	0.00146 98398 50456 01
5	0.07130 21781 09803 2	0.00169 97924 78681 1	0.00048 51219 80875 1
6	0.05993 36274 87377	0.00072 17133 14077	0.00016 51009 76689
7	0.05165 59512 4019	0.00031 34724 1953	0.00005 74544 7340
8	0.04536 81687 501	0.00013 85189 668	0.00002 03348 418
9	0.04043 40775 80	0.00006 20405 06	0.00000 72931 98
10	0.03646 13346 2	0.00002 80899 6	0.00000 26438 3
11	0.03319 52397	0.00001 28320 4	0.00000 09668 6
12	0.03046 3435	0.00000 59057	0.00000 03562
13	0.02814 522	0.00000 2735	0.00000 0132
14	0.02615 36	0.00000 127	0.00000 005
15	0.02442 4	0.00000 06	
16	0.02291		
17	0.0216		
18	0.020		
19	0.02		

n	$R_n(0.3)$	$R_n(0.2)$	$R_n(0.1)$
0	0.25918 17793 18282 13393	0.18126 92469 22018 14133	0.09516 25819 64040 42684
1	0.03693 63131 13766 77411	0.01752 30963 06421 76960	0.00467 88401 60444 46952
2	0.00719 89863 66178 9403	0.00229 69624 89724 2648	0.00030 93061 40529 34331
3	0.00159 48671 40130 438	0.00034 10414 44548 9397	0.00002 30810 03552 07035
4	0.00037 88409 72999 84	0.00005 41965 73270 988	0.00000 18402 72404 6854
5	0.00009 40165 88742 6	0.00000 89890 25369 98	0.00000 01529 87843 068
6	0.00002 40430 49579	0.00000 15353 84022 91	0.00000 00130 89640 37
7	0.00000 62844 0219	0.00000 02679 34521 0	0.00000 00011 43740 8
8	0.00000 16701 340	0.00000 00475 25440 2	0.00000 00001 01552 1
9	0.00000 04496 81	0.00000 00085 38816	0.00000 00000 09131
10	0.00000 01223 6	0.00000 00015 5013	0.00000 00000 0083
11	0.00000 00335 8	0.00000 00002 8382	0.00000 00000 0008
12	0.00000 00093	0.00000 00000 523	
13	0.00000 0003	0.00000 00000 097	
14		0.00000 00000 02	

TABLE 3—*Continued*

n	$R_n(0.05)$	$R_n(0.04)$	$R_n(0.03)$
0	0.04877 05754 99285 99091	0.03921 05608 47676 79056	0.02955 44664 51491 82307
1	0.00120 91042 74250 29045	0.00077 89832 81583 86218	0.00044 11004 45036 57776
2	0.00004 01349 87248 79589	0.00002 07034 60524 00723	0.00000 87999 09879 49816
3	0.00000 15012 83683 7984	0.00000 06198 57466 2730	0.00000 01977 00232 6848
4	0.00000 00599 50832 064	0.00000 00198 08900 862	0.00000 00047 40048 565
5	0.00000 00024 94965 16	0.00000 00006 59665 74	0.00000 00001 18416 36
6	0.00000 00001 06831 23	0.00000 00000 22600 90	0.00000 00000 03043 36
7	0.00000 00000 04670 6	0.00000 00000 00790 6	0.00000 00000 00079 9
8	0.00000 00000 00207 5	0.00000 00000 00028 1	0.00000 00000 00002 1
9	0.00000 00000 00009	0.00000 00000 00001	

n	$R_n(0.02)$	$R_n(0.01)$
0	0.01980 13266 93244 69778	0.00995 01662 50831 94643
1	0.00019 73532 27109 59173	0.00004 96679 13340 26589
2	0.00000 26269 84896 4813	0.00000 03308 43305 6150
3	0.00000 00393 65302 9900	0.00000 00024 80083 0958
4	0.00000 00006 29424 231	0.00000 00000 19834 046
5	0.00000 00000 10485 40	0.00000 00000 00165 24
6	0.00000 00000 00179 69	0.00000 00000 00001 42
7	0.00000 00000 00003 1	
8	0.00000 00000 00000 1	

TABLE 4. *Constants*

γ = 0.57721 56649 01532 86060 6512 (Euler's constant)
$\log_{10} e$ = 0.43429 44819 03251 82765 1129

h	e^h	e^{-h}
0.01	1.01005 01670 84168 05754 21655	0.99004 98337 49168 05357 39060
0.02	1.02020 13400 26755 81016 01439	0.98019 86733 06755 30222 08141
0.03	1.03045 45339 53516 85561 24400	0.97044 55335 48508 17693 25284
0.04	1.04081 07741 92388 22675 70448	0.96078 94391 52323 20943 92107
0.05	1.05127 10963 76024 03969 75176	0.95122 94245 00714 00909 14253
0.10	1.10517 09180 75647 62481 17078	0.90483 74180 35959 57316 42491
0.20	1.22140 27581 60169 83392 10720	0.81873 07530 77981 85866 99355
0.30	1.34985 88075 76003 10398 37443	0.74081 82206 81717 86606 68738
0.40	1.49182 46976 41270 31782 48530	0.67032 00460 35639 30074 44329
0.50	1.64872 12707 00128 14684 86508	0.60653 06597 12633 42360 37995
1.00	2.71828 18284 59045 23536 02875	0.36787 94411 71442 32159 55238

Our tables agree entirely with the values Bretschneider [2] and Gram [3] give for integers from −10 to 20. They differ from Kotani's table [4] by 1 in his last place for a number of values of the argument, and by larger amounts for the following values of x: 6.0, 8.0, 9.0, 10.0, 12.0, −18.0, −24.0, −32.0, −33.0, −43.0, −48.0.

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1. NYMTP (A. N. LOWAN, technical director), *Tables of Sine, Cosine and Exponential Integrals*, v. 1, NBS, 1940, p. 31; *Circular and Hyperbolic Functions, Exponential and Sine and Cosine Integrals, Factorial Function and Allied Functions, Hermitian Probability Functions*, BAAS Mathematical Tables, v. 1, second edition, 1946, p. viii and 31–33.
2. C. A. BRETSCHNEIDER, "Über die abgeleiteken Vierecke usw.," *Archiv der Math. u. Phys.*, first series, v. 3, 1843, p. 27–34.
3. J. P. GRAM, "Undersøgelser angaaende Maengden af Primtal under en given Graense," *Danske Vidensk. Selsk. Skr.*, 6 Raekke, naturvid. og math. Afd., v. 2, 1884, p. 184–288.

4. M. Kotani, A. Amemiya, E. Ishiguro, & T. Kimura, *Table of Molecular Integrals*, Maruzen Co., Ltd., Tokyo, 1955.
5. K. Ruedenberg, C. C. J. Roothaan, & W. Jaunzemis, "Study of two-center integrals useful in calculations of molecular structure. III. A unified treatment of the hybrid, Coulomb, and one-electron integrals," *J. Chem. Phys.*, v. 24, 1956, p. 201–20.
6. NBS, Applied Mathematics Series, No. 14, *Tables of the Exponential Function e^x*, U. S. Gov. Printing Office, Washington, D. C., 1951.
7. G. Placzek, "The Functions $E_n(x) = \int_1^\infty e^{-xu}u^{-n}du$," with Appendices by G. Blanch and the MTP, NBS Applied Mathematics Series, No. 37, p. 57–111.

On the Computation of log Z and arc tan Z

In a previous note, Clenshaw [1] has given numerical values of coefficients for the expansion of some transcendental functions in Chebyshev polynomials. In particular, he tabulates the coefficients for log $(1 + x)$ and arc tan x to nine decimal places. Here, treating these functions in a more general form, we determine precise theoretical coefficients and show that the development leads to formulas for computation in the complex domain.

The Chebyshev polynomials of the first kind which we use in the range $-1 \leq x \leq 1$ are defined as

$$(1) \qquad T_n(x) = \cos n\theta, \quad x = \cos\theta.$$

For a discussion of the properties of these functions, see the work of Lanczos [2]. If $f(x)$ is bounded and continuous in a given range, then the expansion

$$(2) \qquad f(x) = \tfrac{1}{2}C_0 + \sum_{k=1}^{\infty} C_k T_k(x)$$

is convergent, and

$$(3) \qquad C_k = \frac{2}{\pi}\int_{-1}^{1}\frac{f(x)T_k(x)dx}{\sqrt{1 - x^2}}.$$

The transformation $x = 2y - 1$ shifts the range to $0 \leq y \leq 1$, and we denote the "shifted" polynomial as $B_n(y)$.

Consider $f(y) = 1/(y + a)$. Utilizing the above together with a known formula [3], we find

$$(4) \quad 1/(y + a) = [1 + 2\sum_{k=1}^{\infty}(-)^k q^k B_k(y)]/(a^2 + a)^{\frac{1}{2}}; \; q = 2a + 1 - 2(a^2 + a)^{\frac{1}{2}}$$
$$0 \leq y \leq 1; \quad |\arg a| \leq \pi/2, \quad a \neq 0.$$

From a numerical point of view, (4) is pathologic. If $a = 1$, the Taylor series development of $1/(y + 1)$ is slowly convergent near $y = 1$; at $y = 1$ it is divergent. However, (4) is precise and rapidly convergent in this region. The above is a striking example to show the strength of a Chebyshev expansion and the comparative weakness of a Taylor series expansion. The integral of (4) is not patho-