certain to yield further roots. If $\lambda_1 = \bar{\lambda}_1$ are a pair of conjugate complex roots, taking $k = -$ real part of λ_1 will minimize $|\lambda_1 + k|$ and will possibly yield further roots. By forming the polynomial $g(x) = x^r f\left(\frac{1}{x}\right)$ one obtains the roots of minimum modulus of $f(x)$. These devices have the advantage that the precision obtained in any new root is independent of the errors in previously obtained roots. It is only after these devices have become exhausted, that it may become necessary to remove from $f(x)$ factors corresponding to computed roots.

Concluding Remarks. The method described here fails only in the case where $f(x)$ has more than one pair of roots of the same maximum modulus (e.g., the equation $x^r - 1 = 0$). It could be refined without difficulty to take in these more general cases, as indeed was done by Rutishauser [3] and Aitken [1]. The results are considerably more complicated in form and it is doubtful if they have any practical value.

N. S. MENDELSOHN

Canadian Armament Res. and Dev. Est.
and
The University of Manitoba
Winnipeg, Canada

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A Method of Inverting Large Matrices of Special Form

1. Introduction. A method is suggested to invert large matrices, M_n, of the form

$$M_n = \begin{pmatrix} \alpha_0 & -P_1 & 0 & 0 & \cdots & 0 \\ -P_1 & \alpha_1 & -P_2 & 0 & \cdots & 0 \\ 0 & -P_2 & \alpha_2 & -P_3 & & \\ 0 & 0 & -P_3 & \alpha_3 & & \\ & & & & & -P_n \\ 0 & 0 & & & -P_n & \alpha_n \end{pmatrix} \tag{1}$$

in which the elements are themselves special square matrices. The diagonal elements, α_i, are symmetric matrices. The minus signs are not required; they were a convenience in the particular problems we studied. This type of matrix arises, e.g., in the least-square fitting of survey data or in the difference equation approximation, to some common partial differential equations.

The method suggested takes advantage of the special properties of the large matrix and involves only algebraic operations and inversions of matrices the size of the elements. It is thus particularly applicable for use by "medium" sized

computers. The routine will be derived using the simpler matrix,

$$M_n = \begin{pmatrix} \alpha_0 & -I & 0 & & 0 \\ -I & \alpha_1 & -I & & \\ 0 & -I & \alpha_2 & -I & \\ & & & & 0 \\ & & & & -I \\ 0 & & 0 & -I & \alpha_n \end{pmatrix} \tag{2}$$

where I is the identity matrix.

This arises in the difference equation solution to Poisson's equation over a rectangle as shown by Stein and Peck [3]. The algorithm is then generalized to include the case where P_i are diagonal matrices. It could be further generalized to include P_i with no special properties if desired.

A method of partitioning is used. The basic algebra of the method is presented in Frazer, Duncan, and Collar [2]. The method of partitioning by one row and column at a time was suggested by Wagner [4]. A comparison of partitioning with other methods of matrix inversion is given by Fox [1], who also gives a comprehensive bibliography.

2. Method of Inversion. Let

$$M = \left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right) \tag{3}$$

where a and d are square matrices. Let the inverse of M be given by the equation

$$M^{-1} = \left(\begin{array}{c|c} A & C \\ \hline B & D \end{array}\right). \tag{4}$$

Then according to H. M. Wagner [4],

$$\begin{aligned} D &= (d - ca^{-1}b)^{-1} \\ C &= -a^{-1}bD \\ B &= -Dca^{-1} \\ A &= a^{-1} - a^{-1}bB. \end{aligned}$$

Now define the set of sub-matrices

$$M_1 = \begin{pmatrix} \alpha_0 & -I \\ -I & \alpha_1 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \tag{5}$$

$$M_2 = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{M_1} & 0 \\ & & -I \\ \hline 0 & -I & \alpha_2 \end{array}\right) = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix},$$

and, in general,

$$M_i = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{M_{i-1}} & 0 \\ & & -I \\ \hline 0 & -I & \alpha_i \end{array}\right) = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}. \tag{6}$$

Application of equations (5) obtains

$$M_1^{-1} = \begin{pmatrix} \alpha_0^{-1} + \alpha_0^{-1}(\alpha_1 - \alpha_0^{-1})^{-1}\alpha_0^{-1} & \alpha_0^{-1}(\alpha_1 - \alpha_0^{-1})^{-1} \\ (\alpha_1 - \alpha_0^{-1})^{-1}\alpha_0^{-1} & (\alpha_1 - \alpha_0^{-1})^{-1} \end{pmatrix}.$$

This suggests a convenient notation

$$\begin{aligned} [0] &= \alpha_0^{-1} \\ [1] &= (\alpha_1 - [0])^{-1} \\ &\cdot \\ &\cdot \\ &\cdot \\ [n] &= (\alpha_n - [n-1])^{-1}. \end{aligned} \tag{7}$$

In this notation,

$$M_1^{-1} = \begin{pmatrix} [0] + [0][1][0] & [0][1] \\ [1][0] & [1] \end{pmatrix}.$$

Using this expression and applying equations (5) to M_2 gives

$$M_2^{-1} = \begin{pmatrix} [0] + [0][1][0] + [0][1][2][1][0], & [0][1] + [0][1][2][1], & [0][1][2] \\ [1][0] + [1][2][1][0], & [1] + [1][2][1], & [1][2] \\ [2][1][0], & [2][1], & [2] \end{pmatrix}. \tag{8}$$

Using mathematical induction and equation (8) as a guide, it is easy to write down M_n^{-1} in terms of the matrices $[i]$ for any value of n. An explicit expression for a general term in the inverse is not apparent. The following recurrence relations proved convenient for machine programming. Let the elements of M_n be $m_{ij}(i, j = 0, 1, 2 \cdots n)$.

Then

$$\begin{aligned} m_{nn} &= [n] \\ m_{ij} &= [i]m_{i+1,j}, \quad i > j \\ m_{ii} &= [i] + m_{i,i+1}[i] \\ m_{ij} &= \text{transpose } m_{ij}, \quad i < j. \end{aligned} \tag{9}$$

3. Generalization of Result. By similar analysis, an expression for the inverse of the matrix of equation (1) may be derived. The derivation itself is not of interest. For the general case in which the matrices P_i are general, equations (7) are replaced by

$$\begin{aligned} [0] &= \alpha_0^{-1} \\ [1] &= (\alpha_1 - P_1[0]P_1)^{-1} \\ &\cdot \\ &\cdot \\ &\cdot \\ [n] &= (\alpha_n - P_n[n-1]P_n)^{-1}. \end{aligned} \tag{7'}$$

Equation (8) becomes (omitting brackets)

$$(8') \quad M_2^{-1} = \begin{pmatrix} 0+0P_11P_10+0P_11P_22P_21P_10, & 0P_11+0P_11P_22P_21, & 0P_11P_22 \\ 1P_10+ \quad 1P_22P_21P_10, & 1+ \quad 1P_22P_21, & 1P_22 \\ 2P_21P_10, & 2P_21, & 2 \end{pmatrix}.$$

Recurrence formulas similar to (9) may be induced from this array.

4. Computation Procedure. We have inverted matrices of both described types using a Datatron computer without magnetic tape storage. It was found convenient to use interpretive matrix algebra commands of a three-address form. Storage locations were reserved both for square and for diagonal matrices. These locations were given pseudo-addresses. Typical single word commands would perform the following operations:

(i) Multiply the matrix in pseudo-address A by that in B and store the product in C
(ii) Invert the matrix in A and store the inverse in B
(iii) Punch onto cards the matrix (or its transpose) stored in A.

A standard punch card format is used.

The sub-matrices [0] to [n] are first computed and punched. These results are used by a second program to compute the inverse. Using the interpretive three-address commands, a large inversion may be programmed in a few minutes. Our programs limit us to 20 × 20 matrices (α_i and P_i) and to a 200 × 200 matrix M. It required about $2\frac{1}{2}$ hours to invert an 80 × 80 matrix by this method on our computer.

H. I. Meyer
B. J. Hollingsworth

United Gas Corporation
Shreveport, Louisiana

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Note on the General Solution of the Confluent Hypergeometric Equation

The note [1] on a Webb & Airey-Adams-Bateman-Olsson error by Murlan S. Corrington, which is concerned with the confluent hypergeometric differential equation

$$(1) \qquad xy'' + (\gamma - x)y' - \alpha y = 0,$$

does not quite cover all the possibilities for its general solution. A statement of the complete situation does not seem to have appeared in print before now, though in the Fletcher, Miller, and Rosenhead *Index of Mathematical Tables* [2], the statement is deliberately cautious and correct. It seems worth while to give a full statement here.