

TECHNICAL NOTES AND SHORT PAPERS

Table Pertaining to Solutions of a Hill Equation

1. **Introduction.** The electronic digital computer of the Graduate College of the University of Illinois has been employed to determine the characteristic exponent and a quantity related to the phase and amplitude functions for solutions of a Hill equation of the form

$$d^2Y/dt^2 + (A + B \cos 2t + C \cos 4t + D \cos 6t)Y = 0.$$

Results were obtained for the following values of the coefficients:

$$D = 0$$

$$C: -0.5(0.1)0.5$$

$$B: 0(0.2)0.8(0.1)1.6(0.2)2.0$$

$$A: -0.5(0.1) - 0.2(0.05) - 0.15(0.03)0.15(0.05)0.2(0.1)0.5;$$

and

$$D: -0.05(0.05)0.10$$

$$C = 0$$

$$B: \text{as before}$$

$$A: \text{as before.}$$

2. **The Quantities Determined.** The quantities which were determined can be defined by reference to the matrix $M(\tau)$ which relates the value and slope of a general solution at $t = \tau + \pi$ to the value and slope at $t = \tau$:

$$\begin{pmatrix} Y(\tau + \pi) \\ Y'(\tau + \pi) \end{pmatrix} = M(\tau) \begin{pmatrix} Y(\tau) \\ Y'(\tau) \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} Y(\tau) \\ Y'(\tau) \end{pmatrix}.$$

(i) The first quantity of interest is the invariant

$$\frac{1}{2} \text{Tr } M = \cosh \pi\mu,$$

where μ is the characteristic exponent of the Floquet solutions [1]. In cases such that $-1 < \frac{1}{2} \text{Tr } M < 1$, corresponding to stable solutions of the differential equation, it is convenient to write

$$\frac{1}{2} \text{Tr } M = \cos \sigma.$$

In terms of the phase-amplitude representation of such solutions, $Y(t) = w(t)e^{\pm i\Phi(t)} = w(t)e^{\pm i[(\sigma/\pi)t + \Psi(t)]}$ (with $w(t)$ and $\Psi(t)$ periodic in t , mod π , and with $w^2\Phi'$ a constant), σ is interpretable as the advance of the phase Φ when t increases by π .

(ii) The second quantity of interest is the matrix element $M_{12}(\tau)$, which may

be identified with $\frac{\sin \sigma}{\Phi'(\tau)}$. In terms of $M_{12}(\tau)$ the complete matrix may be written

$$M(\tau) = \begin{pmatrix} \cos \sigma - \frac{1}{2}M'_{12} & M_{12} \\ -\frac{\frac{1}{4}(M'_{12})^2 + \sin^2 \sigma}{M_{12}} & \cos \sigma + \frac{1}{2}M'_{12} \end{pmatrix},$$

while the amplitude function $w(\tau)$ may be taken as proportional to $\sqrt{M_{12}}$. Because of the properties mentioned it was felt that, in the interests of brevity, it would suffice to characterize the solutions by tabulating $M_{12}(\tau)$. Since this quantity is an even periodic function, it is only necessary to tabulate values in the range $0 \leq \tau \leq \pi/2$.

3. Method of Computation. For each set of coefficients A, B, C, D two solutions of the differential equation were obtained in the course of the computation. The solutions were calculated by direct integration of the equation, using the Runge-Kutta procedure with a step size $h = \pi/64$. The initial conditions imposed on these solutions were

$$Y_1(0) = 1, \quad Y_1'(0) = 0,$$

and

$$Y_2(0) = 0, \quad Y_2'(0) = 1.$$

Because of the symmetry of the problem, the integration was restricted to the range $0 \leq t \leq \pi/2$.

The quantities which were tabulated were then determined by the equations

$$(1) \quad \frac{1}{2} \text{Tr } M = Y_1(\pi/2)Y_2'(\pi/2) + Y_1'(\pi/2)Y_2(\pi/2),$$

$$(2) \quad M_{12}(\tau) = 2\{Y_2(\pi/2)Y_2'(\pi/2)[Y_1(\tau)]^2 - Y_1(\pi/2)Y_1'(\pi/2)[Y_2(\tau)]^2\}.$$

(As is shown in the introduction to the completed table, these relations may be readily established by an appeal to symmetry which permits matrices which carry solutions from $\pi/2$ to π and from t to $\pi/2$ to be written in terms of those which apply to the range 0 to $\pi/2$ and 0 to t .) The matrix element M_{12} was thus computed for the nine values of τ : $0(\pi/16)\pi/2$. In those cases for which $-1 < \frac{1}{2} \text{Tr } M < 1$ the quantity $\frac{\sigma}{\pi} = \frac{1}{\pi} \cos^{-1}(\frac{1}{2} \text{Tr } M)$ was also computed, the inverse cosine being taken to lie in either the first or second quadrant.

4. Estimated Accuracy. The chief error in the computation was regarded as arising from truncation error in the integration. As is outlined in the introduction of the table, this error was estimated (i) by considering the absolute error in a single integration step to be $h^5 Y^v/120$, and (ii) by repeating representative calculations with a step size $h' = \frac{1}{2}h$. These two methods of estimating the error led to the conclusion that the tabulated values of $\cos \sigma$ and of M_{12} are in error by no more than 4×10^{-6} , or by several units in the last decimal place retained. In the case that $|\cos \sigma|$ is very close to unity, a small error in $\cos \sigma$ will of course be magnified in the evaluation of σ/π .

5. Availability of Tables. The table described in the present article has been deposited in the file of Unpublished Mathematical Tables which *MTAC* maintains. This material may be made available on loan to interested people [2].

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1. E. T. WHITTAKER & G. N. WATSON, *A Course of Modern Analysis*, Cambridge Univ. Press, 1946, p. 412 ff.

2. G. BELFORD, L. JACKSON LASLETT, & J. N. SNYDER, *Table Pertaining to Solutions of a Hill Equation* [*MTAC*, this issue, Review 59, p. 110].

Rational Formulae for the Production of a Spherically Symmetric Probability Distribution

Von Neumann [1] has described the following method for producing a uniform distribution on the circumference of the unit circle: Select two random numbers x_1, x_2 from a uniform distribution on $(-1, +1)$. Reject the pair if $x_1^2 + x_2^2 \geq 1$, and re-select until a pair is obtained satisfying $x_1^2 + x_2^2 < 1$. Then, by the trigonometric double-angle identities,

$$x = \frac{x_1^2 - x_2^2}{x_1^2 + x_2^2}, \quad y = \frac{2x_1x_2}{x_1^2 + x_2^2}$$

are the co-ordinates of a point having the desired distribution.

The following is an extension of this method for points distributed on a sphere uniformly with respect to area. Select four random numbers x_0, x_1, x_2, x_3 from a uniform distribution on $(-1, +1)$. Reject the quadruplet if $x_0^2 + x_1^2 + x_2^2 + x_3^2 \geq 1$, and re-select until a quadruplet is obtained satisfying $x_0^2 + x_1^2 + x_2^2 + x_3^2 < 1$. Then

$$x = \frac{2(x_1x_3 + x_0x_2)}{x_0^2 + x_1^2 + x_2^2 + x_3^2}, \quad y = \frac{2(x_2x_3 - x_0x_1)}{x_0^2 + x_1^2 + x_2^2 + x_3^2}, \quad z = \frac{x_0^2 + x_3^2 - x_1^2 - x_2^2}{x_0^2 + x_1^2 + x_2^2 + x_3^2}$$

are the co-ordinates of a point uniformly distributed over the surface of the unit sphere.