Equally-Weighted Quadrature Formulas for Inversion Integrals

In a previous article [1] the author considered Gaussian-type quadrature formulas for the numerical evaluation of inversion integrals $\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^p}{p} F(p)dp$, where an n-point formula was exact whenever $F(p)$ was a polynomial of degree $(2n - 1)$ in $1/p$. In this present note we consider equally weighted (i.e., Chebyshev type) quadrature formulas of the form

$$(1) \qquad \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^p}{p} F(p)dp = \frac{1}{n}\sum_{j=1}^{n} F(p_j),$$

where (1) is exact whenever $F(p)$ is any polynomial of degree n in $1/p$. The analogous set of equally weighted quadrature formulas for evaluating infinite integrals that are direct Laplace transforms has already been considered in one of the author's earlier papers [2]. Similar to the derivation given there, it is easily seen here that the well-known relation

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^p}{p}\left(\frac{1}{p}\right)^r dp = \frac{1}{r!},$$

by choosing $r = 0$, establishes the factor $1/n$ outside the summation in (1), and the choice of $r = 1, 2, \cdots, n$ establishes the following necessary and sufficient conditions on p_j, in order that (1) should hold whenever $F(p)$ is an arbitrary nth degree polynomial in $1/p$

$$(2) \qquad \sum_{j=1}^{n}\left(\frac{1}{p_j}\right)^r = \frac{n}{r!}, \quad r = 1, 2, \cdots, n.$$

Following the usual methods [2], one determines for the n-point case of (1) the coefficients of the polynomials $\phi_n(z)$ whose zeros z_j are the reciprocals of the required points p_j. Thus from $\phi_n(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1}z + a_n$, the coefficients a_k are found successively from

$$(3) \quad ka_k + a_{k-1}\sum_{j=1}^{n} z_j + a_{k-2}\sum_{j=1}^{n} z_j^2 + \cdots + a_1\sum_{j=1}^{n} z_j^{k-1} + \sum_{j=1}^{n} z_j^k = 0,$$

$$k = 1, 2, \cdots, n.$$

Below are the general formulas for the first ten coefficients $a_1, a_2, \cdots, a_{10}$ for any n, having meaning, of course, only for an a_k where $k \leq n$

$$a_1 = -n,$$

$$a_2 = \frac{n^2}{2} - \frac{n}{4},$$

$$a_3 = -\frac{n^3}{6} + \frac{n^2}{4} - \frac{n}{18},$$

$$a_4 = \frac{n^4}{24} - \frac{n^3}{8} + \frac{25}{288}n^2 - \frac{n}{96},$$

$$a_5 = -\frac{n^5}{120} + \frac{n^4}{24} - \frac{17}{288}n^3 + \frac{7}{288}n^2 - \frac{n}{600},$$

$$a_6 = \frac{n^6}{720} - \frac{n^5}{96} + \frac{43}{1728}n^4 - \frac{25}{1152}n^3 + \frac{1507}{2\ 59200}n^2 - \frac{n}{4320},$$

$$a_7 = -\frac{n^7}{5040} + \frac{n^6}{480} - \frac{13}{1728}n^5 + \frac{13}{1152}n^4 - \frac{1741}{2\ 59200}n^3 + \frac{53}{43200}n^2 - \frac{n}{35280},$$

$$a_8 = \frac{n^8}{40320} - \frac{n^7}{2880} + \frac{61}{34560}n^6 - \frac{7}{1728}n^5 + \frac{17627}{41\ 47200}n^4 - \frac{3779}{20\ 73600}n^3 + \frac{15787}{677\ 37600}n^2 - \frac{n}{3\ 22560},$$

$$a_9 = -\frac{n^9}{3\ 62880} + \frac{n^8}{20160} - \frac{7}{20736}n^7 + \frac{19}{17280}n^6 - \frac{22289}{124\ 41600}n^5 + \frac{2887}{20\ 73600}n^4 - \frac{4\ 87471}{10973\ 49120}n^3 + \frac{12317}{3048\ 19200}n^2 - \frac{n}{32\ 65920},$$

$$a_{10} = \frac{n^{10}}{36\ 28800} - \frac{n^9}{1\ 61280} + \frac{79}{14\ 51520}n^8 - \frac{11}{46080}n^7 + \frac{69353}{1244\ 16000}n^6 - \frac{11347}{165\ 88800}n^5 + \frac{22\ 37339}{54867\ 45600}n^4 - \frac{26791}{2709\ 50400}n^3 + \frac{5\ 90383}{9\ 14457\ 60000}n^2 - \frac{n}{362\ 88000}.$$

The first ten polynomials $\phi_n(z)$ are

$$\phi_1(z) = z - 1,$$

$$\phi_2(z) = z^2 - 2z + \frac{3}{2},$$

$$\phi_3(z) = z^3 - 3z^2 + \frac{15}{4}z - \frac{29}{12},$$

$$\phi_4(z) = z^4 - 4z^3 + 7z^2 - \frac{62}{9}z + \frac{289}{72},$$

$$\phi_5(z) = z^5 - 5z^4 + \frac{45}{4}z^3 - \frac{535}{36}z^2 + \frac{1805}{144}z - \frac{1627}{240},$$

$$\phi_6(z) = z^6 - 6z^5 + \frac{33}{2}z^4 - \frac{82}{3}z^3 + \frac{481}{16}z^2 - \frac{4537}{200}z + \frac{27769}{2400},$$

$$\phi_7(z) = z^7 - 7z^6 + \frac{91}{4}z^5 - \frac{1631}{36}z^4 + \frac{4417}{72}z^3 - \frac{1\ 06351}{1800}z^2 + \frac{53\ 02619}{1\ 29600}z - \frac{180\ 44381}{9\ 07200},$$

$$\phi_8(z) = z^8 - 8z^7 + 30z^6 - \frac{628}{9}z^5 + \frac{4037}{36}z^4 - \frac{3277}{25}z^3 + \frac{9\ 22919}{8100}z^2 - \frac{29\ 22187}{39690}z + \frac{1455\ 11171}{42\ 33600},$$

$$\phi_9(z) = z^9 - 9z^8 + \frac{153}{4}z^7 - \frac{407}{4}z^6 + \frac{3027}{16}z^5 - \frac{1\ 03911}{400}z^4 + \frac{6\ 50239}{2400}z^3 - \frac{84\ 99571}{39200}z^2 + \frac{414\ 78457}{3\ 13600}z - \frac{15146\ 11753}{254\ 01600},$$

$$\phi_{10}(z) = z^{10} - 10z^9 + \frac{95}{2}z^8 - \frac{1280}{9}z^7 + \frac{43235}{144}z^6 - \frac{1\ 70381}{360}z^5 + \frac{14\ 90251}{2592}z^4 - \frac{173\ 63761}{31752}z^3 + \frac{4151\ 58089}{10\ 16064}z^2 - \frac{32552\ 25203}{137\ 16864}z + \frac{14\ 23249\ 22009}{13716\ 86400}.$$

The values of p_j and the zeros $z_j \equiv 1/p_j$ of the above polynomials $\phi_n(z)$, for $n = 1(1)10$, are given in Table 1.

The zeros $z_j \equiv 1/p_j$ of $\phi_n(z)$ were calculated for $n = 3, 4, \cdots, 10$, by first obtaining an initial approximation, using a procedure that had been employed upon the Univac Scientific Computer (ERA 1103) at the Convair Digital Computing Laboratory. The initial approximations to the complex zeros were then used to construct approximate real quadratic factors, which were refined by Bairstow's method (Milne [3], Olver [4]), using only a desk calculator. The initial approximations to the real zeros were refined by Newton's method. All factors of $\phi_n(z)$ were checked by four different formulas (see [4], p. 414). Also a final functional check was performed upon the values of $1/p_j$ by substituting into equation (2) for $n = 1(1)10$, and $r = 1(1)n$.

The 8-decimal values of p_j and $1/p_j$ in Table 1 are guaranteed as far as the seventh decimal place. But they are believed to be correct to within around two units in the eighth decimal place for $n = 1(1)7$ and have a high probability of being correct to within several units in the eighth decimal place for $n = 8(1)10$.

TABLE 1. p_j and $1/p_j$

n	j	p_j	$1/p_j$
1	1	1.00000 000 + .00000 000i	1.00000 000 + .00000 000i
2	1, 2	.66666 667 ± .47140 452i	1.00000 000 ∓ .70710 678i
3	1, 2	.46343 318 ± .66891 655i	.69981 792 ∓ 1.01011 279i
	3	.62485 778 + .00000 000i	1.60036 417 + .00000 000i
4	1, 2	.31209 699 ± .78442 870i	.43788 772 ∓ 1.10059 277i
	3, 4	.54603 449 ± .22670 497i	1.56211 228 ∓ .64856 456i
5	1, 2	.19029 304 ± .86260 499i	.24387 201 ∓ 1.10548 034i
	3, 4	.46724 697 ± .36843 448i	1.31966 923 ∓ 1.04058 814i
	5	.53392 634 + .00000 000i	1.87291 752 + .00000 000i
6	1, 2	.08786 626 ± .92009 404i	.10285 254 ∓ 1.07702 331i
	3, 4	.39416 727 ± .46819 799i	1.05229 916 ∓ 1.24993 725i
	5, 6	.49826 825 ± .14769 920i	1.84484 830 ∓ .54685 928i
7	1, 2	−.00076 496 ± .96470 825i	−.00082 196 ∓ 1.03658 217i
	3, 4	.32727 973 ± .54346 944i	.81317 581 ∓ 1.35033 175i
	5, 6	.45588 935 ± .25464 118i	1.67190 107 ∓ .93385 569i
	7	.49224 949 + .00000 000i	2.03149 016 + .00000 000i
8	1, 2	−.07902 919 ± 1.00066 480i	−.07843 500 ∓ .99314 110i
	3, 4	.26601 917 ± .60293 762i	.61252 402 ∓ 1.38829 762i
	5, 6	.41223 251 ± .33698 985i	1.45409 421 ∓ 1.18868 595i
	7, 8	.47182 912 ± .10911 533i	2.01181 677 ∓ .46525 329i
9	1, 2	−.14919 526 ± 1.03046 752i	−.13761 844 ∓ .95050 835i
	3, 4	.20966 304 ± .65149 353i	.44761 307 ∓ 1.39088 424i
	5, 6	.36931 455 ± .40305 392i	1.23580 348 ∓ 1.34870 244i
	7, 8	.44525 659 ± .19444 915i	1.88616 975 ∓ .82371 403i
	9	.46815 071 + .00000 000i	2.13606 428 + .00000 000i
10	1, 2	−.21284 773 ± 1.05570 953i	−.18351 682 ∓ .91023 036i
	3, 4	.15754 418 ± .69213 469i	.31266 794 ∓ 1.37363 580i
	5, 6	.32790 360 ± .45764 025i	1.03454 187 ∓ 1.44386 337i
	7, 8	.41610 417 ± .26374 950i	1.71443 371 ∓ 1.08670 151i
	9, 10	.45488 509 ± .08636 297i	2.12187 330 ∓ .40285 183i

The refinement of the factors and zeros of $\phi_n(z)$ by Bairstow's method was done by Mrs. Genevieve Mullin Kimbro.

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