On the Time-Step to be Used for the Computation of Orbits by Numerical Integration

W. J. Eckert [1] has adapted Cowell's method of numerical integration to the determination of orbits on punched card machines. Eckert, Brouwer, and Clemence [2] have used Cowell's method on a large-scale computer. We shall assume that Cowell's method as modified by Eckert is the method of numerical integration best suited for the determination of orbits on large-scale computers. What we are concerned with in this paper is how to make use of this method in such a way that we do the least work (have the fewest arithmetic operations) per unit of advance in the time. Our results apply also to the computation of orbits on ordinary desk computers, but additional factors, such as the work involved in transcribing, must be considered.

In Cowell's method as modified by Eckert the attractions and the central differences of the attractions which are required at the "new time-step" are obtained from the values at previous time-steps by an extrapolation process. This extrapolation process may be shown to be equivalent to extrapolating the attractions ahead in the time using Newton's backward difference interpolation formula, so that Cowell's integration formula may be written in terms of backward differences instead of central differences. Let x_n^i, $i = 1, 2, 3$, be rectangular coordinates and let X_n^i be the attractions at time $t = n\Delta t$. Then we obtain for x^i_{n+1} in terms of backward differences,

$$(1)\qquad x^i_{n+1} = {}''X^i_{n+1} + \Delta t^2\left(\frac{1}{12}X_n^i + \frac{1}{12}\nabla X_n^i + \frac{19}{240}\nabla^2 X_n^i + \frac{3}{40}\nabla^3 X_n^i + \frac{863}{12{,}096}\nabla^4 X_n^i + \frac{275}{4{,}032}\nabla^5 X_n^i + \frac{33{,}953}{518{,}400}\nabla^6 X_n^i + \frac{8{,}183}{129{,}600}\nabla^7 X_n^i + \frac{3{,}250{,}433}{53{,}222{,}400}\nabla^8 X_n^i + \frac{4{,}671}{78{,}848}\nabla^9 X_n^i + \frac{301{,}307{,}139{,}941}{5{,}230{,}697{,}472{,}000}\nabla^{10} X_n^i + \cdots\right),$$

$$(2)\qquad {}''X^i_{n+1} = 2{}''X_n^i - {}''X^i_{n-1} + \Delta t^2 X_n^i.$$

Here ∇ is a backward difference operator, i.e.,

$$\nabla X_n^i = X_n^i - X^i_{n-1}, \quad \nabla^2 X_n^i = \nabla X_n^i - \nabla X^i_{n-1}, \quad \text{etc.}$$

The $''X_n^i$ defined by equation (1) are called "second-summations" [2]. In order to start the integration, $''X_0^i$ and $''X^i_{-1}$ and all the backward differences for $n = 0$ must be obtained in such a way that the initial conditions are satisfied. One way of doing this is to start the integration using a less accurate but "easy-starting" method with time-step a product of an appropriate negative power of

two times Δt. Then from this solution backward differences can be evaluated and then the second summations for starting Cowell's method can be obtained from (1).

Let m be the order of the last backward difference retained in (1). We shall call $m - 2$ the order of the integration formula (1) and we label the coefficient of the last backward difference retained, c_m. For a computer with large memory we may construct a sequence of instructions for integrating with (1) and (2) in which m and Δt are parameters, i.e., we may compute using any value of m between, say, one and one hundred and any value of Δt. We would like to choose the combination of m and Δt which minimizes the work performed by the computer in obtaining the orbit to a prescribed degree of accuracy. Assuming that most of the work is involved in computing the attractions and not in evaluating the backward difference series we would at first expect to choose Δt close to the limiting value for which the backward difference series in (1) converges. For a circular orbit this limiting value of Δt may be shown to be the time required to traverse an arc of 30° so that we would choose approximately 12 steps per period. However, as will be shown, with 12 steps per period, the integration would become unstable for m greater than 6, so that the radius of convergence of the backward difference series in (1) is not what really determines how large a Δt we should use. It turns out that if m is large enough or if Δt is large enough, spurious solutions of (1) and (2) initiated by rounding errors grow exponentially. In this case we say that the integration procedure is unstable. The spurious solutions are solutions of (1) and (2) which have no analogue in the differential equations of motion, i.e., which vanish as $\Delta t \to 0$. Brouwer [3] has discussed the accumulation of rounding error when we integrate using (1) and (2) neglecting the backward difference series, i.e., when the spurious solutions damp out. Here we wish to estimate the value of Δt at which the integration first becomes unstable as a function of m.

Applying ∇^2 to equation (2) and substituting $\Delta t^2 X_n{}^i$ for $\nabla^{2\prime\prime} X^i{}_{n+1}$ we obtain

$$x^i{}_{n+1} = 2x_n{}^i - x^i{}_{n-1} + \Delta t^2(X_n{}^i + \tfrac{1}{12}\nabla^2 X_n{}^i + \cdots + c_m \nabla^m X_n{}^i). \tag{3}$$

The integration formula (3) is derived by Collatz [4] with coefficients through the fifth order and is attributed by Collatz to Störmer [5]. The solution of (1) and (2) depends on the starting values of $''X_n{}^i$ and $'X_n{}^i$ as well as $m - 1$ values of the coordinates. The solution of (3) depends on $m + 1$ values of the coordinates. Hence the solution of equations (1) and (2) depends on the same number of arbitrary constants as the solution of (3). By choosing the starting conditions properly any solution of (1) and (2) is a solution of (3) and vice versa.

Let us assume that we have a solution to (3) and that then we make a small perturbation in this solution. Let $\epsilon_n{}^i$ be the difference between the coordinates for the new perturbed orbit and the old orbit. Let us suppose that the perturbation is so small that terms in $(\epsilon_n{}^i)^2$ are negligible. Let us assume further that we have only one body in the field of a fixed mass M. Then $X_n{}^i = -\gamma M x_n{}^i / r_n{}^3$. Let us use the notation $X_n{}^{i,j} = \partial X_n{}^i / \partial x_n{}^j$. We obtain equations for $\epsilon_n{}^i$ by subtracting the equations for the perturbed orbit from the equations for the unperturbed orbit and then expanding the attractions for the perturbed orbit about the attrac-

tions for the unperturbed orbit keeping first order terms in ϵ_n^i only. We obtain

$$(4)\quad \epsilon^i_{n+1} - 2\epsilon_n^i + \epsilon^i_{n-1} = \Delta t^2 \sum_{j=1}^{3} [X_n^{i,j}\epsilon_n^j + \tfrac{1}{12}\nabla^2(X_n^{i,j}\epsilon_n^j) + \cdots + c_m\nabla^m(X_n^{i,j}\epsilon_n^j)].$$

The $X_n^{i,j}$ are understood to be given functions of the coordinates x_n^i of the unperturbed orbit. Let the components of the orthonormal characteristic vectors of the 3×3 matrix $X_n^{i,j}$ belonging to characteristic values U_n, V_n, W_n be u_n^i, v_n^i, w_n^i so that $\sum_j X_n^{i,j}u_n^j = U_n u_n^i$, $\sum_j X_n^{i,j}v_n^j = V_n v_n^i$, and $\sum_j X_n^{i,j}w_n^j = W_n w_n^i$. Written out explicitly, the matrix $X_n^{i,j}$ appears as follows:

$$(5)\quad X_n^{i,j} = \frac{\gamma M}{r_n^3}\begin{bmatrix} -1+3\left(\frac{x_n^1}{r_n}\right)^2 & 3\left(\frac{x_n^1}{r_n}\right)\left(\frac{x_n^2}{r_n}\right) & 3\left(\frac{x_n^1}{r_n}\right)\left(\frac{x_n^3}{r_n}\right) \\ 3\left(\frac{x_n^2}{r_n}\right)\left(\frac{x_n^1}{r_n}\right) & -1+3\left(\frac{x_n^2}{r_n}\right)^2 & 3\left(\frac{x_n^2}{r_n}\right)\left(\frac{x_n^3}{r_n}\right) \\ 3\left(\frac{x_n^3}{r_n}\right)\left(\frac{x_n^1}{r_n}\right) & 3\left(\frac{x_n^3}{r_n}\right)\left(\frac{x_n^2}{r_n}\right) & -1+3\left(\frac{x_n^3}{r_n}\right)^2 \end{bmatrix}.$$

By solving the determinantal equation belonging to this matrix we find that $X_n^{i,j}$ has a single root $2\gamma M/r_n^3$ and a double root $-\gamma M/r_n^3$. Label the single root U_n and its corresponding characteristic vector u_n^i. We obtain for u_n^i,

$$(6)\qquad u_n^i = x_n^i/r_n,$$

i.e., u_n^i is a unit vector in the direction of the radius vector. The remaining characteristic vectors of $X_n^{i,j}$, v_n^i and w_n^i, may be chosen arbitrarily in the plane perpendicular to the radius vector. We expand ϵ_n^i in u_n^i, v_n^i, w_n^i. Let the expansion coefficients be α_n, β_n, γ_n so that

$$(7)\qquad \epsilon_n^i = \alpha_n u_n^i + \beta_n v_n^i + \gamma_n w_n^i.$$

In order to carry our analysis further we have found it necessary to make the assumption that the orbit is circular. Our results will hold approximately for non-circular orbits in terms of the local radius of curvature provided that the radius of curvature of the orbit does not change too rapidly. Let r, θ be polar coordinates in the plane of motion. Let θ_n be the value of θ corresponding to x_n^i. It is easily shown that $\Delta\theta^2 = \dfrac{\gamma M}{r^3}\Delta t^2$, where $\Delta\theta$ is the angle between two successive points on the orbit separated in time by Δt. We assume that the unperturbed orbit lies in the x, y plane and that $y = 0$ and $x = r$ when $t = 0$. Then

$$(8)\qquad x_n^1 = x_n = r\cos n\Delta\theta, \quad x_n^2 = y_n = r\sin n\Delta\theta,$$

$$(9)\qquad u_n = \begin{pmatrix} \cos n\Delta\theta \\ \sin n\Delta\theta \\ 0 \end{pmatrix}.$$

We choose for v_n^i, w_n^i,

$$v_n = \begin{pmatrix} -\sin n\Delta\theta \\ \cos n\Delta\theta \\ 0 \end{pmatrix}, \quad w_n = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{10}$$

Let S be the rotation matrix for rotating a vector through angle $\Delta\theta$ in the (r, θ) plane. Then

$$S = \begin{pmatrix} \cos\Delta\theta & -\sin\Delta\theta & 0 \\ \sin\Delta\theta & \cos\Delta\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{11}$$

We have the following relations involving u_n, v_n, w_n, and S:

$$\begin{aligned} u_n &= S^n u_0, \quad v_n = S^n v_0, \quad w_n = S^n w_0, \\ \bar{u}_0 S^n u_0 &= \bar{v}_0 S^n v_0 = \cos n\Delta\theta, \\ \bar{v}_0 S^n u_0 &= -\bar{u}_0 S^n v_0 = \sin n\Delta\theta, \\ \bar{w}_0 S^n u_0 &= \bar{w}_0 S^n v_0 = \bar{u}_0 S^n w_0 = \bar{v}_0 S^n v_0 = 0, \\ \bar{w}_0 S^n w_0 &= 1. \end{aligned} \tag{12}$$

We obtain equations for the expansion coefficients α_n, β_n, γ_n by first substituting (7) into (4), then substituting for u_n, $S^n u_0$; for v_n, $S^n v_0$; and for w_n, $S^n w_0$, and finally multiplying the equations obtained from the preceding steps successively by $\bar{u}_0 S^{-n}$, $\bar{v}_0 S^{-n}$, $\bar{w}_0 S^{-n}$. We then obtain

$$\begin{aligned} \alpha_{n+1}\cos\Delta\theta - 2\alpha_n + \alpha_{n-1}\cos\Delta\theta - \beta_{n+1}\sin\Delta\theta + \beta_{n-1}\sin\Delta\theta \\ = 2\Delta\theta^2 A(\alpha_n) + \Delta\theta^2 B(\beta_n), \end{aligned} \tag{13}$$

$$\begin{aligned} \alpha_{n+1}\sin\Delta\theta - \alpha_{n-1}\sin\Delta\theta + \beta_{n+1}\cos\Delta\theta - 2\beta_n + \beta_{n-1}\cos\Delta\theta \\ = 2\Delta\theta^2 B(\alpha_n) - \Delta\theta^2 A(\beta_n), \end{aligned} \tag{14}$$

$$\gamma_{n+1} - 2\gamma_n + \gamma_{n-1} = -\Delta\theta^2 c(\gamma_n), \tag{15}$$

where A and B are the real and imaginary parts of I and

$$I(\omega_n) = e^{-in\Delta\theta}[\omega_n e^{in\Delta\theta} + \tfrac{1}{12}\nabla^2(\omega_n e^{in\Delta\theta}) + \cdots + c_m\nabla^m(\omega_n e^{in\Delta\theta})], \tag{16}$$

$$c(\gamma_n) = \gamma_n + \tfrac{1}{12}\nabla^2\gamma_n + \cdots + c_m\nabla^m\gamma_n. \tag{17}$$

In (13) and (14) the argument of A and B indicates that α_n or β_n is to be substituted for ω_n in $I(\omega_n)$. We consider first equation (15). Equation (15) has m solutions of the form $\gamma_n = \lambda^n$, λ being obtained by substituting $\gamma_n = \lambda^n$ in (15), multiplying by λ^{m-n-1}, and solving the resulting polynomial of degree m in λ. It may be shown that the two roots λ which approximate the solution of the variational differential equation obtained from (15) when $\Delta\theta \to 0$ lie almost on the unit circle close to $+1$, whereas if $\Delta\theta$ is large enough some roots have negative real parts. As $\Delta\theta$ is increased, one of these roots with negative real part emerges from the unit circle at $\lambda = -1$ on the real axis. The value of $\Delta\theta$ for which $\lambda = -1$

is the largest value of $\Delta\theta$ for which the modulus of all roots is less than or equal to one, and is the largest value of $\Delta\theta$ for which the integration is stable. Let $\Delta\theta_\gamma{}^m$ be this largest value of $\Delta\theta$ for which the integration formula of order m is stable. Substituting $\lambda = -1$ in (15) we obtain for $\Delta\theta_\gamma{}^m$,

$$\Delta\theta_\gamma{}^m = 2\left[1 + \frac{2^2}{12} + \frac{2^3}{12} + \cdots + c_m 2^m\right]^{-\frac{1}{2}}. \tag{18}$$

Values of $\Delta\theta_\gamma{}^m$ are tabulated as a function of m in Table 1 at the end of this paper.

Next consider equations (13) and (14). These equations have solutions of the form $\alpha_n = a\lambda^n$, $\beta_n = b\lambda^n$, λ being obtained by substituting $a\lambda^n$, $b\lambda^n$ for α_n, β_n in (13) and (14); multiplying the resulting equations by λ^{m-n-1}, setting the determinant of coefficients of a, b to zero; and solving the resulting determinantal equation of order $2m$. We assume as before that $\lambda = -1$ gives the largest value of $\Delta\theta$ for which all roots have modulus less than one. A proof of this statement for the polynomial equation obtained from (13) and (14) is not easy, and, indeed, the statement may not even be true. However, it is certainly true that if $\lambda < -1$, the integration is unstable, so that the assumption $\lambda = -1$ gives us a bound on $\Delta\theta$ which may not be exceeded. We have had an opportunity to test the stability of various order time-step combinations on a computer and have found that the equations of motion do become unstable at values of $\Delta\theta$ close to those predicted here, for the order time-step combinations tested. This gives us confidence that as $\Delta\theta$ is increased, the first "spurious" root to leave the unit circle must emerge at a point close to minus one. Substituting $\omega_n = \omega(-1)^n$ and making use of the relation

$$\nabla^p(-1)^n e^{in\Delta\theta} = (-1)^n e^{i(n-p/2)\Delta\theta}\left(2\cos\frac{\Delta\theta}{2}\right)^p,$$

we obtain

$$I(\omega) = (-1)^n\left[1 + \tfrac{1}{12}e^{-i\Delta\theta}\left(2\cos\frac{\Delta\theta}{2}\right)^2 + \tfrac{1}{12}e^{-i\frac{3}{2}\Delta\theta}\left(2\cos\frac{\Delta\theta}{2}\right)^3 + \cdots + c_m e^{-i\frac{m}{2}\Delta\theta}\left(2\cos\frac{\Delta\theta}{2}\right)^m\right]\omega. \tag{19}$$

Then substituting $\alpha_n = a(-1)^n$, $\beta_n = b(-1)^n$ in (13), (14), and multiplying through by $(-1)^n$ we obtain

$$\begin{aligned} -2(1+\cos\Delta\theta)a = {} & 2\Delta\theta^2\left[1 + \tfrac{1}{12}\cos\Delta\theta\left(2\cos\frac{\Delta\theta}{2}\right)^2 + \tfrac{1}{12}\cos\tfrac{3}{2}\Delta\theta\left(2\cos\frac{\Delta\theta}{2}\right)^3 + \cdots + c_m\cos\frac{m}{2}\Delta\theta\left(2\cos\frac{\Delta\theta}{2}\right)^m\right]a \\ & - \Delta\theta^2\left[\tfrac{1}{12}\sin\Delta\theta\left(2\cos\frac{\Delta\theta}{2}\right)^2 + \tfrac{1}{12}\sin\tfrac{3}{2}\Delta\theta\left(2\cos\frac{\Delta\theta}{2}\right)^3 + \cdots + c_m\sin\frac{m}{2}\Delta\theta\left(2\cos\frac{\Delta\theta}{2}\right)^m\right]b, \end{aligned} \tag{20}$$

$$(21)\quad -2(1+\cos\Delta\theta)b = -\Delta\theta^2\left[1+\tfrac{1}{12}\cos\Delta\theta\left(2\cos\frac{\Delta\theta}{2}\right)^2\right.$$
$$\left.+\tfrac{1}{12}\cos\tfrac{3}{2}\Delta\theta\left(2\cos\frac{\Delta\theta}{2}\right)^3+\cdots+c_m\cos\frac{m}{2}\Delta\theta\left(2\cos\frac{\Delta\theta}{2}\right)^m\right]b$$
$$-2\Delta\theta^2\left[\tfrac{1}{12}\sin\Delta\theta\left(2\cos\frac{\Delta\theta}{2}\right)^2+\tfrac{1}{12}\sin\tfrac{3}{2}\Delta\theta\left(2\cos\frac{\Delta\theta}{2}\right)^3+\cdots\right.$$
$$\left.+c_m\sin\frac{m}{2}\Delta\theta\left(2\cos\frac{\Delta\theta}{2}\right)^m\right]a.$$

We solved (20) and (21) for $\Delta\theta$ for several values of m. The results obtained are tabulated under the heading $\Delta\theta^m{}_{\alpha,\beta}$ in Table 1. These solutions were obtained by successive approximation starting with $\Delta\theta_\gamma{}^m$ and then substituting for the trigonometric functions in (20), (21); then solving the determinantal equation for $\Delta\theta^2$, then resubstituting for the trigonometric functions, etc. Two or three iterations were sufficient to obtain $\Delta\theta^m{}_{\alpha,\beta}$ to the number of places given in Table 1.

We note that $\Delta\theta^m{}_{\alpha,\beta}$ is close to $\Delta\theta_\gamma{}^m$. We use $\Delta\theta_\gamma{}^m$ as the limiting value of $\Delta\theta$ for stable integration in subsequent discussion. This approximation is justified since in practice we would not want to choose $\Delta\theta$ close to its limiting value because this would make the rounding error large, and it also is justified because orbits obtained by numerical integration are rarely really circular so that we can only know the limiting value of $\Delta\theta$ approximately. In Table 1 we tabulate $N_m = \dfrac{2\pi}{\Delta\theta_\gamma{}^m}$. N_m is the smallest number of steps per period giving stable integration of order m.

In order to make use of our results we require an estimate of the truncation and rounding error. We assume that Δt has been chosen small enough so that the backward difference series in (3) converges. Let $y_n{}^i$, $Y_n{}^i$ be coordinates and attractions corresponding to the exact solution of the differential system. Then $y_n{}^i$, $Y_n{}^i$ satisfy

$$(22)\qquad \frac{y^i{}_{n+1}-2y_n{}^i+y^i{}_{n+1}}{\Delta t^2}-\sum_{k=0}^{m}c_k\nabla^k Y_n{}^i=\sum_{k=m+1}^{\infty}c_k\nabla^k Y_n{}^i.$$

Our numerical solution $x_n{}^i$ satisfies

$$(23)\qquad \frac{x^i{}_{n+1}-2x_n{}^i+x^i{}_{n+1}}{\Delta t^2}-\sum_{k=0}^{m}c_k\nabla^k X_n{}^i=\rho_n.$$

Here ρ_n is the rounding error incurred at step n. In order to keep as much accuracy as possible without performing extended accuracy multiplication, the term in parenthesis in (3) should be evaluated with such scaling as to give all precision possible using "single-precision" arithmetic, then extra digits should be carried in the summation of the coordinates in (3). Now let $\epsilon_n{}^i = y_n{}^i - x_n{}^i$. Assuming that $\epsilon_n{}^i$ is small, we obtain from (22) and (23) (in manner analogous to the derivation of (4))

$$(24)\qquad \frac{\epsilon^i{}_{n+1}-2\epsilon_n{}^i+\epsilon^i{}_{n+1}}{\Delta t^2}-\sum_{k=0}^{m}\sum_{j=1}^{3}c_k\nabla^k(Y_n{}^{i,j}\epsilon_n{}^j)=-\rho_n+c_{m+1}\nabla^{m+1}Y_n{}^i.$$

In (24) we retain only the first term on the right-hand side of (22). $c_{m+1}\nabla^{m+1}Y_n^i$ is called the local truncation error. For the circular orbit defined by (8)

$$
\begin{aligned}
\nabla^{m+1}Y_n^1 &= -\frac{\gamma M}{r^2}\left(2\sin\frac{\Delta\theta}{2}\right)^{m+1}\cos(\omega t_n - \varphi_m),\\
\nabla^{m+1}Y_n^2 &= -\frac{\gamma M}{r^2}\left(2\sin\frac{\Delta\theta}{2}\right)^{m+1}\sin(\omega t_n - \varphi_m),\\
\nabla^{m+1}Y_n^3 &= 0,
\end{aligned}
\tag{25}
$$

where $\omega = \sqrt{\dfrac{\gamma M}{r^3}}$ and φ_m equals $\dfrac{m+1}{2}(\Delta\theta - \pi)$. We let $\Delta_{m+1} = c_{m+1}\left(2\sin\dfrac{\Delta\theta}{2}\right)^{m+1}$.

As can be seen by examining equations (22) and (23), if the problem is scaled so that $(Y_n^i)_{\max} = \gamma M/r^2$ occupies the full number size of the computer, then the local truncation error is the same order as the rounding error when Δ_{m+1} is the same order of magnitude as ρ_n. This assumes that extra accuracy is carried in the coordinates as mentioned previously. Δ_{m+1} is tabulated in Table 1.

We may now apply our results and choose an appropriate time-step size and order, m, for various particular cases. For example, suppose we wish to integrate for orbits of the planets on a computer with a 14 decimal-digit number size and we wish to obtain the orbits with all the precision of which the machine is capable without performing extended accuracy operations other than extended accuracy summations. We see from Table 1 that we are going to need between 60 and 70 steps with $m = 12$ to keep the truncation error small enough and that then we are too close to the unstable value of $\Delta\theta$, so that we must use more steps, say 100 steps per period. But then the truncation error is sufficiently small with $m = 10$, 11, or 12. $m = 11$ and 100 steps per period would appear to be a good compromise. In the integrations of the orbits of the five outer planets, Eckert, Brouwer, and Clemence [2] used $m = 11$ and they had approximately 113 steps per period in Jupiter, the planet with the smallest period. Hence this theoretical analysis gives results in good agreement with what has been found appropriate in practice.

Many computers have a 10 decimal-digit word size. On these $m = 10$ with 60 steps per revolution would appear to be a good choice.

We see that we would not expect to use very high orders ($m > 14$) under any practical circumstances. An extreme case would be one in which we wished to follow a very large number of revolutions, say 10^{10} revolutions. We would then wish to perform extended accuracy computations on, say, a 14 digit machine thus obtaining effectively 28 decimal-digit number size. In this case, to keep the local truncation error below 10^{-28} and to be assured of stability we would want to choose about 180 time-steps per period and $m = 13$. Assuming we could perform the computations for one time-step in a millisecond (this is probably somewhat faster than we could expect from any computer today) it would require 56 years to complete the computation. Hence even in an extreme case m is less than 14.

Of course we do not always wish to integrate obtaining the maximum precision of which a particular computer with its given number size is capable. Instead, we often wish the error in the orbit to be less than a given bound over the total

interval of the integration. In order to be assured of this it is necessary to estimate the accumulated truncation error and the accumulated rounding error. From our prescribed bound in the maximum allowable accumulated truncation error we obtain the maximum allowable Δt for each order m. Then for least work by the computer we choose the $(m, \Delta t)$ combination which gives the largest value of Δt for which the integration is stable according to Table 1. We obtain the minimum allowable number size from our estimate of the accumulated rounding error. Then we have obtained the maximum Δt and minimum number size which may be used to solve our problem using Cowell's method.

Let us estimate the accumulated truncation error. It is easily seen that equation (24) approximates the differential system

$$d^2\epsilon^i/dt^2 - \sum_{j=1}^{3} Y^{i,j}\epsilon^j = -\rho + c_{m+1}\nabla^{m+1}Y^i \tag{26}$$

where ϵ^i, $Y^{i,j}$, ρ, and $c_{m+1}\nabla^m Y^i$ now stand for continuous quantities. For the circular orbit we can obtain exactly a particular solution of equation (26) corresponding to the local truncation error. It is

$$\epsilon^i \text{ (truncation)} = r\Delta_{m+1}[(\tfrac{1}{3}\cos\varphi_m + 2\theta\sin\varphi_m)u^i - \tfrac{3}{2}\theta^2\sin\varphi_m v^i]. \tag{27}$$

Here φ_m is the phase factor defined following equation (25) and u^i and v^i are defined in equations (9) and (10) except that here $n\Delta\theta$ is replaced by a continuous variable $\theta = \omega t$. We see that after a large number, P, of periods, the last term in (27) dominates and the amplitude of the truncation error approaches $|6\pi^2 \sin\varphi_m P^2\Delta_{m+1}r|$. The way the rounding error accumulates can also be investigated starting with (26). One is led to the same result obtained by Brouwer [3], namely that after n steps the root-mean-square rounding error increases by a factor of $n^{\frac{1}{2}}$. The method used here to estimate the accumulated truncation error is similar to that discussed by Rademacher [6].

TABLE 1. *Values of* $\Delta\theta_\gamma{}^m$, $\Delta\theta^m{}_{\alpha,\beta}$, N_m, Δ_{m+1} *for* $m = 6(1)14$

m	$\Delta\theta_\gamma{}^m$	$\Delta\theta^m{}_{\alpha,\beta}$	N_m	Δ_{m+1}	m	$\Delta\theta_\gamma{}^m$	$\Delta\theta^m{}_{\alpha,\beta}$	N_m	Δ_{m+1}
6	.6252	.629	10.05	10^{-3}	11	.1260		49.86	10^{-12}
7	.4593		13.68	10^{-4}	12	.0905	.0893	69.39	10^{-15}
8	.3346	.3211	18.78	10^{-6}	13	.0650		96.70	10^{-18}
9	.2424		25.92	10^{-8}	14	.0466		134.88	10^{-21}
10	.1750	.1701	35.90	10^{-10}					

Note: Values listed for $m = 13$, 14, and Δ_{13} are based on estimated values of C_{13}, C_{14}, and C_{15}.

J. W. SHELDON

Computer Usage Company
New York, New York

B. ZONDEK

Naval Proving Ground
Dahlgren, Virginia

M. FRIEDMAN

Institute of Flight Structures
Columbia University
New York, New York

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Optimum Quadrature Formulas In s Dimensions

Although numerical procedures for functions of more than one variable are of considerable practical importance, they have received relatively little attention. So far as numerical integration is concerned, aside from successive application of appropriate one-dimensional results, as discussed in standard texts, and a few isolated special results for two dimensions, there has been little systematic study of the problem. Tyler [1] has collected a set of formulas of up to seventh degree for integration over rectangular regions in the plane, and up to fifth degree in three dimensions, and also gives a $(2s + 1)$-point third-degree formula for s dimensions. Hammer, Marlowe, Stroud, and Wymore [2, 3, 4] have pointed out useful methods of extending available results for spaces of a given dimension to higher dimensions, and to regions which are related to the given region by affine transformations. They have also developed a number of formulas for simplexes, cones, and spheres in s dimensions.

It is the purpose of the present paper to describe a general approach to the problem which may often yield useful results and to use it to derive formulas in s dimensions using $(s + 1)$ points for second-degree accuracy, and $2s$ points for third-degree accuracy. This method is applicable to any region, although its advantages are most pronounced for regions with a center of symmetry, and in our detailed calculations we will limit ourselves to a hypercube with center at the origin. We will be concerned with formulas of the form

$$(1) \qquad \int_S \cdots \int \varphi(x^{(1)}, \cdots, x^{(s)}) dx^{(1)} \cdots dx^{(s)} = \sum_{i=1}^{m} c_i \varphi(x_i^{(1)}, \cdots, x_i^{(s)}) - R\varphi$$

where $R\varphi$ denotes the error of the formula for a particular function φ. A formula will be said to be of degree n if $R\varphi$ is zero whenever φ is a polynomial of degree n or less in the s variables $x^{(j)}$.

If a formula is of degree n, the c_i and $x_i^{(j)}$ must satisfy the set of equations

$$(2) \qquad \sum_{i=1}^{m} c_i \prod_{j=1}^{s} (x_i^{(j)})^{n_j} = \int_S \cdots \int \prod_{j=1}^{s} (x^{(j)})^{n_j} dx^{(j)} \equiv I_{n_1, \cdots, n_s}$$