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TECHNICAL NOTES AND SHORT PAPERS

On The Numerical Evaluation of Cauchy Principal Values of Integrals

By I. M. Longman

Introduction. This note demonstrates a simple method for the numerical evaluation of the Cauchy principal value of an integral

$$P \int_a^b f(x)dx \quad (x \text{ real}),$$

when the integrand $f(x)$ has singularities at one or more points in the finite interval $[a, b]$ of integration. The necessity for such numerical evaluation arises sometimes in the solution of problems in applied mathematics. Such integrals can occur, for example, in the solution of wave propagation problems. For instances of this see Lamb [1], and Pekeris [2]. Sometimes a singularity can be avoided by a suitable transformation in the complex plane. The present note presents a simple direct method of dealing with such a singularity.

Formulation of the method. The method is quite general and can be applied separately to each singularity of $f(x)$, and so, for simplicity, we will suppose that $f(x)$ has only one singularity which, without loss of generality, we may suppose to be at $x = 0$. Suppose then we require to evaluate by some numerical quadrature method the integral

$$I = P \int_{-a}^{a} f(x)dx \tag{1}$$

where the range of integration is made symmetrical about the origin by splitting up the integral into two ranges if necessary. We split up the function $f(x)$ into its odd and even components

$$f(x) = \tfrac{1}{2}[f(x) - f(-x)] + \tfrac{1}{2}[f(x) + f(-x)] = g(x) + h(x), \text{ say.} \tag{2}$$

Then

$$I = P \int_{-a}^{a} g(x)dx + P \int_{-a}^{a} h(x)dx. \tag{3}$$

Now since $g(x)$ is an odd function the first integral vanishes, and so

$$I = 2 \int_0^a h(x)dx, \tag{4}$$

where either there is now no singularity at $x = 0$, or, if there is, it is not such as to make the integral (4) diverge, since we suppose the Cauchy principal value

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(1) to exist. If (4) still contains a singularity (e.g. of the form $\log x$) the problem is no longer that of finding a Cauchy principal value, but the entirely different one of the numerical evaluation of an infinite integral. In such a case the singularity in (4) can usually be removed by a substitution of the form $x = y^n$, for a sufficiently large value of n. But, in general, such a singularity arises from the presence of a pole on the path of integration and this must be of odd order if the Cauchy principal value is to exist. In this case the numerical evaluation has been reduced to the determination of (4) which no longer contains a singularity. However, in (4), at least in the first instance, the integrand may reduce to the indeterminate form $\infty - \infty$ or equivalently $0/0$, at the lower limit of integration, but this is not a serious difficulty. In the first place it may be that $\lim_{x\to 0} h(x)$ is known (as in examples 1 and 2), or it may be possible to eliminate the indeterminate form by algebraic manipulation as in example 3. In any case if $h(x)$ or at least the part of it which gives the indeterminateness is an analytic function, values at and near $x = 0$ can be obtained by expansion. Another way of overcoming the difficulty is the use an integration formula, e.g. Gaussian [3], [4], which does not involve the end points of the range of integration.

Examples. The method is illustrated by three examples.

(1) To evaluate

$$I = P\int_{-1}^{1} (e^x/x)dx.$$

We have

$$I = P\int_{-1}^{1} (1/x)\cosh x\,dx + P\int_{-1}^{1} (1/x)\sinh x\,dx = 2\int_{0}^{1} (1/x)\sinh x\,dx$$

which can now be evaluated numerically with no disturbing singularity.

(2) To evaluate

$$I = P\int_{0}^{\pi/2} \frac{e^x}{\sin x - \cos x}\,dx.$$

Here the singularity is at $x = \pi/4$, and making the substitution $x = \pi/4 + y$ a little manipulation shows that the integral is reduced to

$$I = 2^{\frac{1}{2}}e^{\pi/4}\int_{0}^{\pi/4} (\sinh y)/(\sin y)dy.$$

(3) To evaluate

$$I = P\int_{-\pi/2}^{\pi/2} \frac{dy}{\sqrt{1 + \sin y} - 1}.$$

The singularity is at $y = 0$, and proceeding on the same lines as above we obtain the form

$$I = 2\int_{0}^{\pi/2} \frac{dy}{\sqrt{1 - \sin y} + \sqrt{1 + \sin y}}$$

which does not contain any singularities.

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Mersenne Numbers

By Hans Riesel

During 1957 the author of this note had the opportunity of running the Swedish electronic digital computer BESK in order to examine Mersenne numbers. The intention of the author's investigation on the BESK was to check some known results, and to examine some Mersenne numbers not previously examined.

Mersenne numbers are numbers $M_p = 2^p - 1$, where p is a prime. See [1], which contains a more complete list of references. The Mersenne numbers have attained interest in connection with digital computers because there is a simple test to decide whether they are prime or composite. This is Lucas's test [1]. Furthermore the number $2^{p-1}M_p$ is a perfect number, if M_p is a prime, and all known perfect numbers are of this form.

In the beginning of 1957 a program for testing the primeness of the Mersenne numbers on the BESK was worked out by the author. This program, using Lucas's test, works for all $p < 10000$. As a test, this program was run for the following values of p: 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, and 2281. The result was that the numbers M_p are prime for these values of p, thus confirming known results.

After these tests, values of $p > 2300$ were to be tested. These values had not been tested before. But since the testing of M_p for one value of p of this order takes several hours on the BESK, a special program for calculating the smallest factor of M_p, if this factor is $< 10 \cdot 2^{20} = 104\ 85760$, was worked out. This special program is based on the following well-known theorem:

All prime factors q of M_p $(p > 2)$ are of the form $q = 2kp + 1$, and of one of the two forms $q = 8l \pm 1$.

The proof of the theorem is quite simple. If q is a factor of M_p, $2^p \equiv 1 \pmod{q}$. Since p is a prime, since all numbers n for which $2^n = 1 \pmod{q}$ form a module, and since $2^p \equiv 1 \pmod{q}$, this module consists of all integral multiples of the prime p. Now $2^{q-1} \equiv 1 \pmod{q}$ if q is a prime, and hence $q - 1$ is a multiple of p, and in fact an even multiple (since q must be an odd number). This is the first part of the theorem: $q - 1 = 2kp$ $(k = 1, 2, 3, \cdots)$. The second part follows immediately from the theory of quadratic residues. Since $x^2 \equiv 2 \pmod{q}$ has the solution $x \equiv 2^{(p+1)/2} \pmod{q}$, we see that 2 is a quadratic residue mod q, hence $q \equiv \pm 1 \pmod{8}$.

By the above mentioned special program for small factors of M_p the values of $M_p \pmod{q}$ for all primes q of the theorem were now calculated. When this residue was $\equiv 0$, the factor q was printed out. When no factor $q < 10 \cdot 2^{20}$ was found, the BESK turned to the next value of p. This program was run for all

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