The Determination of the Chebyshev Approximating Polynomial for a Differentiable Function

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If $f(x)$ is continuous over any interval, which we may take, without loss of generality, to be the interval $-1 \leqq x \leqq 1$, there exists a unique polynomial $P_n{}^*(x)$, of given maximum degree n, which is such that the maximum of $|f(x) - P_n{}^*(x)|$ over $-1 \leqq x \leqq 1$ is less than the maximum of $|f(x) - P_n(x)|$ over $-1 \leqq x \leqq 1$, where $P_n(x)$ is any other polynomial of degree not exceeding n. The polynomial $P_n{}^*(x)$ is known as the Chebyshev approximation, of maximum degree n, to $f(x)$ over $-1 \leqq x \leqq 1$. It is characterized by the fact that $f(x) - P_n{}^*(x)$ assumes extreme values at $n + 2$ points, at least, of the interval $-1 \leqq x \leqq 1$, these extreme values being equal in magnitude and alternating in sign [1]. We refer to the points of any such set of $n + 2$ points as critical points, and we denote them by $(x_1{}^*, \cdots, x_{n+2}^*)$, where $-1 \leqq x_1{}^* < \cdots < x_{n+2}^* \leqq 1$. Thus, the end points, ± 1, of the interval $-1 \leqq x \leqq 1$ may be critical points, but at least n of the $n + 2$ critical points, namely, $x_2{}^*, \cdots, x_{n+1}^*$, are interior points of this interval.

We assume that $f(x)$ is not only continuous, but also differentiable, over $-1 \leqq x \leqq 1$, and so the derivative of $f(x) - P_n{}^*(x)$ is zero at each of the n points $x_2{}^*, \cdots, x_{n+1}^*$. If the derivative of $f(x) - P_n{}^*(x)$ cannot be zero more than n times, it follows that $x_1{}^* = -1$, $x_{n+2}^* = 1$ and that the derivative of $f(x) - P_n{}^*(x)$ has precisely n zeros that are interior points of the interval $-1 \leqq x \leqq 1$.

The polynomial $P_n{}^*(x)$ is an odd function of x when $f(x)$ is odd, and is an even function of x when $f(x)$ is even. Thus, when $f(x)$ is odd we may take n to be even, the maximum degree of $P_n{}^*(x)$ being $n - 1$, and when $f(x)$ is even we may take n to be odd, the maximum degree of $P_n{}^*(x)$ being again $n - 1$. In these cases the critical points are distributed symmetrically about the mid-point $x = 0$ of the interval $-1 \leqq x \leqq 1$, and we may confine our attention to the part $0 \leqq x \leqq 1$ of this interval. When $f(x)$ is odd, so that n is even, the number of critical points is even and $x = 0$ is not a critical point; on the other hand, when $f(x)$ is even, the number of critical points is odd and $x = 0$ is a critical point. When $f(x)$ is odd, or even, and $x = 1$ is a critical point, we change our notation and denote the positive interior critical points by $x_1{}^* < x_2{}^* < \cdots < x_k{}^*$, where $n = 2k$ in the first case, and $n = 2k + 1$ in the second. For example, when $f(x) = \arctan x$, $P_{2k}^*(x)$ is an odd polynomial of degree $\leqq 2k - 1$, and so the derivative of $\arctan x - P_{2k}^*(x)$ cannot vanish more than $2k$ times; this implies that the points ± 1 are critical points, and, in addition, since this derivative must vanish $2k$ times, that $P_{2k}^*(x)$ is of degree $2k - 1$. Similarly, when $f(x) = \cos mx$, $m > 0$, P_{2k+1}^* is an even polynomial of degree $2k$, the points 0 and 1 being critical points.

It is clear that $P_n{}^*(x)$ is easily determined if any set $x_1{}^* < x_2{}^* < \cdots < x_{n+2}^*$

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of critical points is known; indeed, the $n + 2$ equations $f(x_k^*) - P_n^*(x_k^*) = (-1)^{k-1}\omega$, $k = 1, \cdots, n + 2$, constitute a set of $n + 2$ linear equations for ω and the $n + 1$ coefficients of $P_n^*(x)$. If $x_1 < x_2 < \cdots < x_{n+2}$ is any set of $n + 2$ points of the interval $-1 \leqq x \leqq 1$, and we write $f(x_k) - P_n(x_k) = (-1)^{k-1}E$, $k = 1, \cdots, n + 2$, these equations determine E and the $n + 1$ coefficients of $P_n(x)$, and the function E of the $n + 2$ variables $(x_1, \cdots, x_{n+2})$ has an absolute maximum at $(x_1^*, \cdots, x_{n+2}^*)$. Thus, the derivative of E with respect to each of the n variables $x_2, \cdots, x_{n+1}$ is zero at $(x_1^*, \cdots, x_{n+2}^*)$, and this implies that the derivative of each of the $n + 1$ coefficients of $P_n(x)$ with respect to each of the n variables $x_2, \cdots, x_{n+1}$ is zero at $(x_1^*, \cdots, x_{n+2}^*)$. Hence, these coefficients are insensitive to small changes of the variables $(x_2, \cdots, x_{n+1})$ when these variables have the values $x_2^*, \cdots, x_{n+1}^*$ and $x_1 = x_1^*$, $x_{n+2} = x_{n+2}^*$.

The method by which we determine $P_n^*(x)$ is an iterative one. Let us suppose that the points ± 1 are critical points, so that there are n interior critical points, which we denote, changing slightly our previous notation, by $x_1^* < x_2^* < \cdots < x_n^*$. Let us suppose further, that we are in possession of a polynomial, $P_n^{(0)}(x)$, of degree $\leqq n$, which we term our entering polynomial and which possesses the following property: The difference $f(x) - P_n^{(0)}(x)$ assumes extreme values of alternating sign at $n + 2$ points of the interval $-1 \leqq x \leqq 1$. We denote by $x_1^{(1)}, \cdots, x_n^{(1)}$ approximations to the second, third, $\cdots$, $(n + 1)$st of these points and we regard $x_1^{(1)}, \cdots, x_n^{(1)}$ as approximations, in the first cycle of an iterative procedure, to $x_1^*, \cdots, x_n^*$. We determine the approximating polynomial of degree $\leqq n$, $P_n^{(1)}(x)$, with which we end the first, and begin the second, cycle of this procedure by means of the $n + 1$ linear equations obtained by eliminating $E^{(1)}$ from the $n + 2$ linear equations

$$f(-1) - P_n^{(1)}(-1) = E^{(1)}; \qquad f(x_k^{(1)}) - P_n^{(1)}(x_k^{(1)}) = (-1)^k E^{(1)},$$
$$k = 1, \cdots, n; \qquad f(1) - P_n^{(1)}(1) = (-1)^{n+1}E^{(1)}.$$

Denoting $f(-1) - P_n^{(0)}(-1)$ by $\delta_0^{(1)}$, $f(x_k^{(1)}) - P_n^{(0)}(x_k^{(1)})$ by $\delta_k^{(1)}$, $k = 1, \cdots, n$, and $f(1) - P_n^{(0)}(1)$ by $\delta_{n+1}^{(1)}$, we can write these $n + 2$ equations as

$$\delta P_n^{(0)}(-1) = \delta_0^{(1)} - E^{(1)}; \qquad \delta P_n(x_k^{(1)}) = \delta_k^{(1)} - (-1)^k E^{(1)}, \ k = 1, \cdots, n;$$
$$\delta P_n^{(0)}(1) = \delta_{n+1}^{(1)} - (-1)^{n+1}E^{(1)},$$

where $\delta P_n^{(0)}(x)$ denotes the polynomial, of degree $\leqq n$, $P_n^{(1)}(x) - P_n^{(0)}(x)$. $E^{(1)}$ is conveniently eliminated by combining the last $n + 1$ of these $n + 2$ equations alternately by addition and subtraction with the first, and the $n + 1$ coefficients of $\delta P_n^{(0)}(x)$ are obtained by solving the resulting $n + 1$ linear equations. Then the coefficients of $P_n^{(1)}(x)$ are obtained by adding each of the coefficients of $\delta P_n^{(0)}(x)$ to the corresponding coefficients of $P_n^{(0)}(x)$.

The first step in the second cycle of the iterative procedure is the determination of new approximations $x_1^{(2)}, \cdots, x_n^{(2)}$ to $x_1^*, \cdots, x_n^*$. Just as $x_1^{(1)}, \cdots, x_n^{(1)}$ were approximations to the zeros of $D[f(x) - P_n^0(x)]$, where D denotes differentiation with respect to x, so $x_1^{(2)}, \cdots, x_n^{(2)}$ are approximations to the zeros of $D[f(x) - P_n^{(1)}(x)]$. Writing $x_k^{(2)} = x_k^{(1)} + \delta x_k^{(1)}$, $k = 1, \cdots, n$, we see that the value of $D[f(x) - P_n^{(0)}(x)]$ at $x_k^{(1)} + \delta x_k^{(1)}$ must be the same as the value of $D[\delta P_n^0(x)]$ at $x_k^{(1)} + \delta x_k^{(1)}$, and this is the same, to the first order of infinitesimals, as the value of $D[\delta P_n^{(0)}(x)]$ at $x_k^{(1)}$. Thus, to the first order of infinitesimals, the value of $D[f(x) - P_n^{(0)}(x)]$ at $x_k^{(1)}$ plus the value of $D^2[f(x) - P_n^{(0)}(x)]$ at $x_k^{(1)}$ times $\delta x_k^{(1)}$ is equal to the value of $D[\delta P_n^{(0)}(x)]$ at $x_k^{(1)}$, so that $\delta x_k^{(1)}$ is the negative of the quotient of the value of $D[f(x) - P_n^{(1)}(x)]$ at $x_k^{(1)}$ by the value of $D^2[f(x) - P_n^{(0)}(x)]$ at $x_k^{(1)}$. We denote the value of $D[f(x) - P_n^{(1)}(x)]$ at $x_k^{(1)}$ by $\epsilon_k^{(2)}$, $k = 1, \cdots, n$, so that $\delta x_k^{(1)}$ is the negative of the quotient of $\epsilon_k^{(2)}$ by the value of $D^2[f(x) - P_n^{(0)}(x)]$ at $x_k^{(1)}$. If $\epsilon_k^{(2)}$ is zero, $x_k^{(2)} = x_k^{(1)}$ and $D[f(x) - P_n^{(1)}(x)]$ is zero at $x_k^{(2)} = x_k^{(1)}$. In order to complete the second cycle, we calculate the $n + 2$ numbers $\delta_0^{(2)} = f(-1) - P_n^{(1)}(-1)$, $\delta_k^{(2)} = f(x_k^{(2)}) - P_n^{(1)}(x_k^{(2)})$, $(k = 1, \cdots, n)$, $\delta_{n+1}^{(2)} = f(1) - P_n^{(1)}(1)$, and determine, as before, the coefficients of $\delta P_n^{(1)}(x)$. If, to the number of decimals we are using, the $n + 2$ numbers $\delta_0, \cdots, \delta_{n+1}$ are equal in absolute value and alternating in sign, the coefficients of $\delta P_n^{(1)}(x)$ are all zero and the approximating polynomial, $P_n^{(2)}(x)$, with which we end the second cycle is the same as the approximating polynomial, $P_n^{(1)}(x)$, with which we began it.

In a previous publication [2] we determined the numbers $x_1, \cdots, x_n$ in each cycle by solving the equation $D[f(x) - P_n(x)] = 0$, where $P_n(x)$ is the approximating polynomial, of degree $\leqq n$, with which we begin the cycle, but the less exacting method of the present paper is equally effective.

It remains only to describe the selection of our entering polynomial approximation, $P_n^{(0)}(x)$, of degree $\leqq n$, and the determination of the approximations $x_1^{(1)}, \cdots, x_n^{(1)}$ to the n points of the interval $-1 < x < 1$ at which $D[f(x) - P_n^{(0)}(x)]$ is zero. On setting $x = \cos\theta$, we see that $f(x)$ becomes a function, $F(\theta)$, of θ defined over $0 \leqq \theta \leqq \pi$, and we write the Fourier cosine series of $F(\theta)$ as $\frac{1}{2}a_0 + a_1 \cos\theta + a_2 \cos 2\theta + \cdots$. Then $\cos m\theta$ is a polynomial function, $T_m(x)$, of x of degree m, which is known as the mth Chebyshev polynomial, $m = 0, 1, 2, \cdots$, and $\frac{1}{2}a_0 + a_1T_1(x) + a_2T_2(x) + \cdots$ is known as the Chebyshev expansion of $f(x)$. The sum of the first $n + 1$ terms of this Chebyshev expansion of $f(x)$ is a polynomial function, of degree $\leqq n$, of x, and it is this polynomial function that we take as $P_n^{(0)}(x)$. We say that $P_n^{(0)}(x)$ is furnished by the truncated Chebyshev expansion (the truncation taking place at the term which involves $T_n(x)$). Now $f(x) - P_n^{(0)}(x) = a_{n+1}T_{n+1}(x) + \cdots$, and we take as our approximations to the n points of the interval $-1 < x < 1$ at which $D[f(x) - P_n^{(0)}(x)] = 0$ the n points of this interval at which $D[T_{n+1}(x)] = 0$, it being assumed that $a_{n+1} \neq 0$. (If $f(x)$ is odd its Chebyshev expansion is of the form $a_1T_1(x) + a_3T_3(x) + \cdots$ and $n = 2m$ is even; then we truncate this Chebyshev expansion at the term involving $T_{2m-1}(x)$, and we take as our approximations to the m points of the interval $0 < x < 1$ at which $D[f(x) - P_n^{(0)}(x)]$ is zero the m points of this interval at which $D[T_{2m+1}(x)]$ is zero. Similar remarks apply to the case where $f(x)$ is even and $n = 2m + 1$ is odd.) Since

$$D\,[T_{n+1}(x)] = (n + 1)\,\frac{\sin\,(n + 1)\,\theta}{\sin\theta},$$

we have

$$x_k^{(1)} = \cos\left(\pi - \frac{k\pi}{n + 1}\right), \qquad k = 1, \cdots, n.$$

Example 1. $f(x) = \arctan x$, $n = 6$.

There are three positive interior critical points, which we denote by x_1^*, x_2^*, x_3^*, and

$$P_6^{(0)}(x) = 0.994949366x - 0.287060636x^3 + 0.078937176x^5,$$

since the Chebyshev expansion of arc tan x is

$$2\left[pT_1(x) - \frac{p^3}{3}T_3(x) + \frac{p^5}{5}T_5(x) - \cdots\right],$$

where $p = 2^{\frac{1}{2}} - 1 = 0.414213562$, to 9 decimals. The first-cycle approximations to x_1^*, x_2^*, x_3^* are

$$x_1^{(1)} = 0.222520934, \qquad x_2^{(1)} = 0.623489802, \qquad x_3^{(1)} = 0.900968868,$$

and the polynomial approximation with which we end the first cycle is

$$P_6^{(1)}(x) = 0.995383022x - 0.288700440x^3 + 0.079313307x^5,$$

the values of arc tan $x - P_n^{(0)}(x)$ at the points $x_1^{(1)}$, $x_2^{(1)}$, $x_3^{(1)}$, 1 being

$$\delta_1^{(1)} = 0.000676851, \qquad \delta_2^{(1)} = -0.000604555,$$
$$\delta_3^{(1)} = 0.000546760, \qquad \delta_4^{(1)} = -0.000527744,$$

respectively. The corresponding results for the second cycle are

$$x_1^{(2)} = 0.205422893, \qquad x_2^{(2)} = 0.593832571, \qquad x_3^{(2)} = 0.888813502,$$

$$\delta_1^{(2)} = 0.000603543, \qquad \delta_2^{(2)} = -0.000619441,$$
$$\delta_3^{(2)} = 0.000607728, \qquad \delta_4^{(2)} = -0.000597725,$$

and

$$P_6^{(2)}(x) = 0.995357994x - 0.288690417x^3 + 0.079339173x^5.$$

In the third cycle we find

$$x_1^{(3)} = 0.205218790, \qquad x_2^{(3)} = 0.593469973, \qquad x_3^{(3)} = 0.888196372,$$

$$\delta_1^{(3)} = 0.000608588, \qquad \delta_2^{(3)} = -0.000608590,$$
$$\delta_3^{(3)} = 0.000608612, \qquad \delta_4^{(3)} = -0.000608588,$$

and

$$P_6^{(3)}(x) = 0.995357955x - 0.288690238x^3 + 0.079339041x^5.$$

Finally, in the fourth cycle we obtain

$$x_1^{(4)} = 0.205219373, \qquad x_2^{(4)} = 0.593470162, \qquad x_3^{(4)} = 0.888196289,$$

$$\delta_1^{(4)} = 0.000608595, \qquad \delta_2^{(4)} = -0.000608595, \qquad \delta_3^{(4)} = 0.000608595,$$
$$\delta_4^{(4)} = -0.000608595, \qquad P_6^{(4)}(x) = P_6^{(3)}(x).$$

Thus, to seven decimals,

$$P_6^*(x) = 0.9953580x - 0.2886902x^3 + 0.0793390x^5,$$

and the maximum of $|\arctan x - P_6^*(x)|$ over $-1 \leqq x \leqq 1$ is 0.0006086. Hastings [3] has given $P_6^*(x)$ for arc tan x to six decimals as $0.995354x - 0.288679x^3 + 0.079331x^5$.

Example 2. $f(x) = \log \dfrac{a+x}{a-x}, \qquad a = \dfrac{10^{\frac{1}{2}}+1}{10^{\frac{1}{2}}-1}, \qquad n = 4.$

The number a must be greater than 1; we use the value indicated in order to check the work of Hastings. Setting $\xi = (a+x)/(a-x)$, the polynomial $P_4^*(x)$ which we determine will be an approximation to $\log \xi$ over the interval $10^{-\frac{1}{2}} \leqq \xi \leqq 10^{\frac{1}{2}}$.

The Chebyshev expansion of $\log (a+x)/(a-x)$ is

$$4M\left[pT_1(x) + \frac{p^3}{3}T_3(x) + \frac{p^5}{5}T_5(x) + \cdots\right],$$

where $M = \log e = 0.434294482$, to 9 decimals, and $p = a - (a^2-1)^{\frac{1}{2}} = 0.280130000$. The entering polynomial approximation is $P_4^{(0)}(x) = 0.448447982x + 0.050916894x^3$, and our first-cycle approximations to the two positive interior critical points are

$$x_1^{(1)} = 0.309016994; \qquad x_2^{(1)} = 0.809016994.$$

The values of $\log (a+x)/(a-x) - P_4^{(0)}(x)$ at the points $x_1^{(1)}$, $x_2^{(1)}$, 1 are

$$\delta_1^{(1)} = 0.000572828, \qquad \delta_2^{(1)} = -0.000607958, \qquad \delta_3^{(1)} = 0.000635124,$$

respectively, and $P_4^{(1)}(x) = 0.4483349355x + 0.051051305x^3$. In the second cycle

$$x_1^{(2)} = 0.321484228; \qquad x_2^{(2)} = 0.821954759$$

$$\delta_1^{(1)} = 0.000600487, \qquad \delta_2^{(2)} = -0.000602901, \qquad \delta_3^{(2)} = 0.000599339$$

$$P_4^{(2)}(x) = 0.448347007x + 0.051051766x^3,$$

and, in the third cycle,

$$x_1^{(3)} = 0.321320097; \qquad x_2^{(3)} = 0.821455202$$

$$\delta_1^{(3)} = 0.000601227; \qquad \delta_2^{(3)} = -0.000601233; \qquad \delta_3^{(3)} = 0.000601227$$

$$P_4^{(3)}(x) = 0.448346999x + 0.051051771x^3,$$

so that, to seven decimals, $P_4^{(3)}(x)$ coincides with $P_4^{(2)}(x)$, and $P_4^*(x) = 0.4483470x + 0.0510518x^3$, the maximum value of $|\log (a+x)/(a-x) - P_4^*(x)|$ over $-1 \leqq x \leqq 1$ being 0.0006012.

If x is replaced by $a(x-1)/(x+1)$, there results the approximation $0.8630458[(x-1)/(x+1)] + 0.3641410[(x-1)/(x+1)]^3$ to $\log x$ over the interval $10^{-\frac{1}{2}} \leqq x \leqq 10^{\frac{1}{2}}$. Hastings [3] gives as the coefficients in this approximation the numbers 0.86304 and 0.36415, respectively.

In a recent paper by Barth [4], $P_6^*(x)$ for $\ln (a+x)/(a-x)$, $a = (10^{\frac{1}{2}}+1)/(10^{\frac{1}{2}}-1)$, is given, to ten decimals, as

$$0.8690286986x + 0.2773833195x^3 + 0.2543282307x^5.$$

The correct formula, to seven decimals, for $P_6^*(x)$ is

$$0.8690285x + 0.2773864x^3 + 0.2543195x^5,$$

the maximum of $|\ln (a + x)/(a - x) - P_6^*(x)|$ over $-1 \leqq x \leqq 1$ being 0.0000337.

Example 3. $f(\xi) = \ln (1 + \xi), 0 \leqq \xi \leqq 1, n = 4$.

We denote in this example the independent variable by ξ, instead of x, since the interval, $0 \leqq \xi \leqq 1$, is not the standard interval $-1 \leqq x \leqq 1$. The linear transformation $x = 2\xi - 1$ transforms the interval $0 \leqq \xi \leqq 1$ into the interval $-1 \leqq x \leqq 1$, and the problem of determining $P_4^*(\xi)$ for $\ln (1 + \xi)$ over $0 \leqq \xi \leqq 1$ is the same as that of determining $P_4^*(x)$ for $\ln \frac{1}{2}(3 + x)$ over $-1 \leqq x \leqq 1$.

The Chebyshev expansion of $\ln \frac{1}{2}(3 + x)$ is

$$-2 \log_e 2p + 2\left[p^2T_1(x) - \frac{p^4}{2} T_2(x) + \frac{p^6}{3} T_3(x) \cdots\right],$$

where $p = 2^{\frac{1}{2}} - 1$, and our entering polynomial approximation, of degree 4, is $P_4^{(0)}(\xi) = 0.000069446 + 0.996261948\xi - 0.466442439\xi^2 + 0.218665484\xi^3 - 0.055459314\xi^4$. Our first-cycle approximations to the four interior critical points are

$$\xi_1^{(1)} = 0.095491503, \quad \xi_2^{(1)} = 0.345491503,$$
$$\xi_3^{(1)} = 0.654508497, \quad \xi_4^{(1)} = 0.904508497,$$

and the values of $\ln (1 + \xi) - P_4^{(0)}(\xi)$ at the points 0, $\xi_1^{(1)}$, $\xi_2^{(1)}$, $\xi_3^{(1)}$, $\xi_4^{(1)}$, 1 are

$$\delta_0^{(1)} = -0.000069446, \quad \delta_1^{(1)} = 0.000066650, \quad \delta_2^{(1)} = -0.000060948,$$
$$\delta_3^{(1)} = 0.000055988, \quad \delta_4^{(1)} = -0.000053017, \quad \delta_5^{(1)} = 0.000052055.$$

The polynomial approximation, of the fourth degree, with which we end the first cycle is, to eight decimals,

$$P_4^{(1)}(\xi) = 0.000059471 + 0.996558114\xi - 0.467864445\xi^2$$
$$+ 0.220882267\xi^3 - 0.056547698\xi^4.$$

In the second cycle we obtain

$$\xi_1^{(2)} = 0.084407707, \quad \xi_2^{(2)} = 0.318071278,$$
$$\xi_3^{(2)} = 0.629216597, \quad \xi_4^{(2)} = 0.895475308;$$
$$\delta_0^{(2)} = -0.000059471, \quad \delta_1^{(2)} = 0.000060703, \quad \delta_2^{(2)} = -0.000061938;$$
$$\delta_3^{(2)} = 0.000061315, \quad \delta_4^{(2)} = -0.000060043, \quad \delta_5^{(2)} = 0.000059471;$$
$$P_4^{(2)}(\xi) = 0.000060712 + 0.996540728\xi - 0.467834593\xi^2$$
$$+ 0.220891205\xi^3 - 0.056571583\xi^4;$$

and, in the third cycle,

$$\xi_1^{(3)} = 0.085058286, \quad \xi_2^{(3)} = 0.319106141, \quad \xi_3^{(3)} = 0.629174645,$$
$$\xi_4^{(3)} = 0.895122761;$$

$\delta_0^{(3)} = -0.000060712, \quad \delta_1^{(3)} = 0.000060716, \quad \delta_2^{(3)} = -0.000060716,$

$\delta_3^{(3)} = 0.000060712, \quad \delta_4^{(3)} = -0.000060713, \quad \delta_5^{(3)} = 0.000060712;$

$$P_5^{(3)}(\xi) = 0.000060714 + 0.996540741\xi - 0.467834762\xi^2 + 0.220891541\xi^3 - 0.056571768\xi^4.$$

Beginning the fourth cycle, we compute

$\xi_1^{(4)} = 0.085060350, \quad \xi_2^{(4)} = 0.319112305,$

$\xi_3^{(4)} = 0.629171981, \quad \xi_4^{(4)} = 0.895123131,$

and we find that the corresponding values $\delta_i^{(4)}$ for $i = 0, 1, 2, 3$, and 4 are all numerically equal to 0.0000607141, to within a unit in the tenth decimal place.

Thus, to seven-decimal accuracy in the coefficients, we have

$$P_4^*(\xi) = 0.0000607 + 0.9965407\xi - 0.4678348\xi^2 + 0.2208915\xi^3 - 0.0565718\xi^4.$$

The discrepancy between this approximation and the similar one presented in our earlier paper [2] is attributable to the premature termination of the iterative procedure in that reference, which stemmed from the erroneous belief that the precision of the coefficients of the approximating polynomial was comparable to that of the maximum difference between that polynomial and the given function. In this example the quantities $\delta_i^{(4)}$ have all become stabilized to 10 decimal places, whereas the coefficients of the corresponding approximating polynomial are subject to errors of approximately a unit in the eighth decimal place. This behavior of the coefficients is due to the relatively small value of the determinant of the system of equations used for their evaluation. Calculation of such coefficients to ten-place accuracy generally will require double-precision operations.

Hastings [3] gives as an approximating polynomial of degree 4, whose graph is arbitrarily required to pass through the origin, the following

$$0.9974442x - 0.4712839x^2 + 0.2256685x^3 - 0.0587527x^4,$$

for which the maximum departure from $\ln(1 + x)$ over the interval $0 \leqq x \leqq 1$ is 0.0000710, in contrast to the value 0.0000607, attained by the Chebyshev approximating polynomial of the same degree.

Example 4. $f(x) = \cos(\pi/4)x$, $n = 3$.

The Chebyshev expansion of $\cos(\pi/4)x$ is $J_0(\pi/4) - 2J_2(\pi/4)T_2(x) + 2J_4(\pi/4)T_4(x) - \cdots$, where $J_{2k}(\pi/4)$, for $k = 0, 1, \cdots$, is the value at $\pi/4$ of the Bessel function of the first kind, of order $2k$. There is only one positive interior critical point x^*, and our first-cycle approximation to this is $x^{(1)} = \cos\pi/4 = 0.707106781$. Our entering polynomial approximation is $P_3^{(0)}(x) = 0.998068558 - 0.292873289x^2$. The values of $\cos(\pi/4)x - P_3^{(0)}(x)$ at the points 0, $x^{(1)}$, 1 are

$$\delta_0^{(1)} = 0.001931442, \quad \delta_1^{(1)} = -0.001921422, \quad \delta_2^{(1)} = 0.001911512,$$

and $P_3^{(1)}(x) = 0.998078551 - 0.292893219x^2$. The coefficient of x^2 remains the same in all the succeeding cycles, so that, in this example, we obtain the coefficient of x^2 in $P_3^*(x)$ before we begin the second cycle; this simplification is due to the fact that both the points 0 and 1 are critical points, and this implies that the coefficient of x^2 in $P_3^*(x)$ is $\cos \pi/4 - 1$. In the second cycle we obtain

$$x^{(2)} = 0.705276652, \quad \delta_0^{(2)} = 0.001921449, \quad \delta_1^{(2)} = -0.001921553,$$
$$\delta_2^{(2)} = \delta_0^{(2)}, \quad P_3^{(2)}(x) = 0.998078499 - 0.292893219x^2;$$

and, in the third cycle,

$$x^{(3)} = 0.705270859, \quad \delta_0^{(3)} = 0.001921501, \quad \delta_1^{(3)} = -0.001921500,$$
$$\delta_2^{(3)} = \delta_0^{(3)}, \quad P_3^{(3)}(x) = 0.998078499 - 0.292893219x^2.$$

Thus, to seven decimals, $P_3^*(x) = 0.9980785 - 0.2928932x^2$, the maximum of $| \cos (\pi/4)x - P_3^*(x) |$ over $-1 \leqq x \leqq 1$ being 0.0019215.

We observe in this example that the entering polynomial approximation, $P_3^{(0)}(x)$, is so good that the maximum of $| \cos (\pi/4)x - P_3^{(0)}(x) |$, over $-1 \leqq x \leqq 1$, is 0.0019314, which exceeds the corresponding maximum of $| \cos (\pi/4)x - P_3^*(x) |$ by less than 0.52 per cent.

Example 5. $f(x) = \cos (\pi/2)x$, $n = 5$.

There are two positive interior critical points, x_1^* and x_2^*, and our first-cycle approximations are $x_1^{(1)} = \cos \pi/3 = \frac{1}{2}$ and $x_2^{(1)} = \cos \pi/6 = 3^{\frac{1}{2}}/2$. The Chebyshev expansion of $\cos (\pi/2)x$ is $J_0(\pi/2) - 2J_2(\pi/2)T_2(x) + \cdots$, and our entering polynomial approximation, of degree four, is $P_5^{(0)}(x) = J_0(\pi/2) - 2J_2(\pi/2)T_2(x) + 2J_2(\pi/4)T_4(x) = 0.999396554 - 1.222743153x^2 + 0.223936637x^4$. The values of $\cos (\pi/2)x - P_5^{(0)}(x)$ at the points 0, $x_1^{(1)}$, $x_2^{(1)}$, 1 are

$$\delta_0^{(1)} = 0.000603446, \quad \delta_1^{(1)} = -0.000600024,$$
$$\delta_2^{(1)} = 0.000593320, \quad \delta_3^{(1)} = -0.000590037.$$

Since 0 and 1 are critical points, the coefficients of the approximating polynomial $P_5^{(k)}(x)$, with which we end the kth cycle ($k = 1, 2, \cdots$) satisfy the relation $2\alpha^{(k)} + \beta^{(k)} + \gamma^{(k)} = 1$, and this implies that the coefficients of $P_5^*(x)$ satisfy the relation $2\alpha^* + \beta^* + \gamma^* = 1$. We find that

$$P_5^{(1)}(x) = 0.999403304 - 1.22796880x^2 + 0.223990272x^4.$$

In the second cycle we obtain

$$x_1^{(2)} = 0.497202761, \quad x_2^{(2)} = 0.864404535;$$
$$\delta_0^{(2)} = 0.000596695, \quad \delta_1^{(2)} = -0.000596808,$$
$$\delta_2^{(2)} = 0.000596805, \quad \delta_3^{(2)} = -0.000596697;$$
$$P_5^{(2)}(x) = 0.999403231 - 1.222796733x^2 + 0.223990272x^4.$$

In the third cycle we obtain

$$x_1^{(3)} = 0.497195260, \quad x_2^{(3)} = 0.864395279;$$
$$\delta_0^{(3)} = 0.000596769, \quad \delta_1^{(3)} = -0.000596772,$$
$$\delta_2^{(3)} = 0.000596770, \quad \delta_3^{(3)} = -0.000596770;$$
$$P_5^{(3)}(x) = P_5^{(2)}(x) \text{ to 9 decimals.}$$

Hence, we conclude that

$$P_5^*(x) = 0.9994032 - 1.2227967x^2 + 0.2239903x^4,$$

the maximum of $|\cos(\pi/2)x - P_5^*(x)|$ over $-1 \leqq x \leqq 1$ being 0.0005968.

The iterative procedure described herein for the determination of the Chebyshev approximating polynomial $P_n^*(x)$, of degree not exceeding n, over the interval $-1 \leqq x \leqq 1$, to a given differentiable function $f(x)$ will converge if the difference between $f(x)$ and the initial approximating polynomial $P_n^{(0)}(x)$ assumes extreme values at $n + 2$ points of the interval $-1 \leqq x \leqq 1$ and if, furthermore, these extreme values alternate in sign. A proof of this based on the argument of Novodvorskii and Pinsker [5] has been given, and illustrated by a numerical example, in our previous publication [2] on this subject.

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