

Expansion of the Confluent Hypergeometric Function in Series of Bessel Functions

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Abstract. An expansion of the confluent hypergeometric function $\Phi(a, c, z)$ in series of functions of the same kind has been given by Buchholz [1]. By specialization of some quantities, there is obtained an expansion in series of modified Bessel functions of the first kind, $I_\nu(z)$, where ν depends on the parameter a . Tricomi [2, 3] has developed two expansions of similar type where both the order and argument of the Bessel functions depend on the parameters a and c . In the present paper, we derive an expansion in series of Bessel functions of integral order whose argument is independent of a and c . Our expansion is advantageous for many purposes of computation since the parameters and variable of $\Phi(a, c, z)$ appear in separated form. Also, for desk calculation, extensive tables of $I_n(z)$ are available, while for automatic computation Bessel functions are easy to generate [4].

Special cases of the confluent function, such as the incomplete gamma function, are also studied. For the class of transcendents known as the error functions, including the Fresnel integrals, it is shown that our expansion coincides with that of Buchholz [1]. By specializing a parameter and passing to a limit, we derive expansions for the exponential integral and related functions. Other expansions for the error and exponential integrals are derived on altogether different bases. Finally, some numerical examples are presented to manifest the efficiency of our formulas.

1. Representation of the Confluent Hypergeometric Function in Series of Bessel Functions of Integral Order. The confluent hypergeometric function is defined by the series

$$(1.1) \quad \Phi(a, c, z) \equiv {}_1F_1(a; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k z^k}{(c)_k k!},$$

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)},$$

where $\Gamma(z)$ is the gamma function. We also have the integral representation

$$(1.2) \quad \Phi(a, c, z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 e^{zt} t^{a-1} (1-t)^{c-a-1} dt, \quad R(c) > R(a) > 0,$$

and Kummer's formula

$$(1.3) \quad \Phi(a, c, z) = e^z \Phi(c-a, c, -z).$$

For these and other properties of the confluent function, see [5].

Let $t = \sin^2 \theta$ in the integrand of (1.2). Using the Jacobi expansion [6]

$$(1.4) \quad e^{(z/2)\cos 2\theta} = \sum_{k=0}^{\infty} \epsilon_k \cos 2k\theta I_k(z/2), \quad \epsilon_0 = 1, \quad \epsilon_k = 2, \quad k > 0,$$

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(1.2) becomes

$$(1.5) \quad \Phi(a, c, z) = \frac{2\Gamma(c)}{\Gamma(a)\Gamma(c-a)} e^{z/2} \sum_{k=0}^{\infty} (-1)^k \epsilon_k I_k(z/2) \cdot \int_0^{\pi/2} \cos 2k\theta (\sin \theta)^{2a-1} (\cos \theta)^{2c-2a-1} d\theta .$$

Since

$$(1.6) \quad \cos 2k\theta = (-1)^k \sum_{m=0}^k \frac{(-1)^m \binom{k}{m} (\cos \theta)^{2m}}{(1/2)_m} ,$$

where $\binom{k}{m}$ is the binomial coefficient, and

$$(1.7) \quad \int_0^{\pi/2} (\sin \theta)^{2x-1} (\cos \theta)^{2y-1} d\theta = \frac{1}{2} \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} , \quad R(x) > 0, \quad R(y) > 0,$$

we find that

$$(1.8) \quad \Phi(a, c, z) = e^{z/2} \sum_{k=0}^{\infty} (-1)^k \epsilon_k R_k(a, c) I_k(z/2),$$

where $R_k(a, c)$ is conveniently expressed in hypergeometric form [7] as

$$(1.9) \quad R_k(a, c) = {}_3F_2(-k, k, a; c, \frac{1}{2}; 1).$$

An alternative expression for $R_k(a, c)$ is readily reduced from a result in [8]. We have

$$(1.10) \quad \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)} R_k(a, c) = G_k(a, c) + (-1)^k G_k(c-a, c),$$

$$G_k(a, c) = \frac{(-1)^k \cos(c-a) \pi \Gamma(k-c+1) \Gamma(2c-2a)}{2^{2c-2a-1} \Gamma(k+1+c-2a)} \cdot {}_2F_1\left(1-2a, 2c-2a; k+1+c-2a; \frac{1}{2}\right).$$

If a and c are fixed and k is sufficiently large, then the ${}_2F_1$'s in (1.10) are of order unity and

$$(1.11) \quad \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)} R_k(a, c) \approx \frac{(-1)^k \cos(c-a) \pi \Gamma(2c-2a) k^{2a-2c}}{2^{2c-2a-1}} + \frac{\cos a \pi \Gamma(2a) k^{-2a}}{2^{2a-1}} .$$

Here we have used the fact that for fixed α and β ,

$$(1.12) \quad \frac{\Gamma(k+\alpha)}{\Gamma(k+\beta)} \approx k^{\alpha-\beta}, \quad k \text{ large} .$$

Again, if z is fixed and k is large, then

$$(1.13) \quad I_k(z) \approx \frac{(z/2)^k}{k!}.$$

It follows that (1.8) converges like

$$(1.14) \quad \sum_{k=1}^{\infty} \frac{(Ak^{2a-2c} + Bk^{-2a})z^k}{2^{2k}k!},$$

A and B are constants. A similar argument shows that (1.1) converges like

$$(1.15) \quad \sum_{k=1}^{\infty} \frac{k^{a-c}z^k}{k!},$$

and so (1.8) converges more rapidly than (1.1).

Some general properties of $R_k(a, c)$ are next of interest. The combination of (1.3) and (1.8) shows that

$$(1.16) \quad R_k(a, c) = (-1)^k R_k(c - a, c); \quad R_k(a, a) = (-1)^k.$$

Further relations follow by applying known contiguous relations of $\Phi(a, c, z)$ to (1.8). For example, since

$$(1.17) \quad \frac{d}{dz} \Phi(a, c, z) = \frac{az}{c} \Phi(a + 1, c + 1, z),$$

it follows that

$$(1.18) \quad R_k(a + 1, c + 1) = -\frac{c}{4a} [R_{k+1}(a, c) - 2R_k(a, c) + R_{k-1}(a, c)],$$

and

$$(1.19) \quad (a - c + 1)R_k(a, c) - aR_k(a + 1, c) + (c - 1)R_k(a, c - 1) = 0,$$

results from a contiguous formula of the form (1.19) with $R_k(a, c)$ replaced by $\Phi(a, c, z)$. To obtain a pure recursion formula for $R_k(a, c)$, apply the differential equation satisfied by $\Phi(a, c, z)$ to (1.8). Employ the difference-differential properties of $I_n(z)$ and equate to zero the coefficient of $I_k(z)$. We get

$$(1.20) \quad (k + c)R_{k+1}(a, c) = 2(c - 2a)R_k(a, c) + (k - c)R_{k-1}(a, c).$$

For certain values of the parameters, the coefficients $R_k(a, c)$ can be expressed in simple form. Thus from (1.10),

$$(1.21) \quad R_k\left(\frac{1}{2}, c\right) = \frac{[\Gamma(c)]^2}{\Gamma(c + k)\Gamma(c - k)}, \quad c \neq 0, -1, -2, \dots, R(c) > 0.$$

This also follows from (1.9) and the known expression for ${}_2F_1$ with unit argument. If $c = 2a$, then from (1.9) and (1.20), $R_k(a, 2a)$ is zero if k is odd, and

$$(1.22) \quad R_{2k}(a, 2a) = \frac{(\frac{1}{2} - a)_k}{(\frac{1}{2} + a)_k}, \quad a \neq -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2} \dots$$

Since

$$(1.23) \quad \Phi\left(\nu + \frac{1}{2}, 2\nu + 1, -z\right) = \Gamma(\nu + 1)(z/4)^{-\nu} e^{-z/2} I_\nu(z/2),$$

it follows that

$$(1.24) \quad I_\nu(z) = (z/2)^\nu \Gamma(\nu + 1) \sum_{k=0}^{\infty} \frac{(-1)^k \epsilon_k I_{2k}(z)}{\Gamma(\nu - k + 1) \Gamma(\nu + k + 1)}$$

provided ν is not a negative integer. The latter is a special case of a known result [9].

2. The Incomplete Gamma Function. If $c = a + 1$, we write

$$(2.1) \quad \gamma(a, z) = a^{-1} z^a e^{-z} \Phi(1, 1 + a, z) = a^{-1} z^a \Phi(a, a + 1, -z),$$

$$(2.2) \quad \gamma(a, z) = \int_0^z e^{-t} t^{a-1} dt = \Gamma(a) - \int_z^{\infty} e^{-t} t^{a-1} dt, \quad R(a) > 0.$$

Thus

$$(2.3) \quad \gamma(a, z) = a^{-1} z^a e^{-z/2} \sum_{k=0}^{\infty} \epsilon_k R_k(a) I_k(z/2),$$

$$(2.4) \quad \begin{aligned} R_k(a) &\equiv R_k(a, a + 1) = {}_3F_2(-k, k, a; a + 1, \tfrac{1}{2}; 1) \\ &= (-1)^k {}_3F_2(-k, k, 1; a + 1, \tfrac{1}{2}; 1) \\ &= \frac{-(-1)^k a}{2(k-a)(k-a+1)} {}_2F_1(1-2a, 2; k-a+2; \tfrac{1}{2}) \\ &\quad + a \frac{\cos a\pi \Gamma(2a) k \Gamma(k-a)}{2^{2a-1} \Gamma(k+a+1)}, \end{aligned}$$

and

$$(2.5) \quad (k + a + 1) R_{k+1}(a) = 2(1 - a) R_k(a) + (k - a - 1) R_{k-1}(a),$$

$$(2.6) \quad R_k(a + 1) = -\frac{(a + 1)}{4a} [R_{k+1}(a) - 2R_k(a) + R_{k-1}(a)].$$

If $a = \frac{1}{2}$, we deduce the following expressions for the error function and related functions.

$$(2.7) \quad \operatorname{Erf} z = \int_0^z e^{-t^2} dt = -z \sum_{k=0}^{\infty} \frac{(-1)^k \epsilon_k e^{-z^2/2} I_k(z^2/2)}{4k^2 - 1}.$$

$$(2.8) \quad e^{-z^2} \operatorname{Erfi} z = e^{-z^2} \int_0^z e^{t^2} dt = -z \sum_{k=0}^{\infty} \frac{\epsilon_k e^{-z^2/2} I_k(z^2/2)}{4k^2 - 1}.$$

$$(2.9) \quad (2\pi)^{\frac{1}{2}} [C(z) + iS(z)] = \int_0^z t^{-\frac{1}{2}} e^{it} dt = -2z^{\frac{1}{2}} e^{iz/2} \sum_{k=0}^{\infty} \frac{\epsilon_k i^k J_k(z/2)}{4k^2 - 1}.$$

A more general expansion for the error function can be found as follows. Now

$$(2.10) \quad \int_0^{z \sin \theta} e^{-t^2} dt = z e^{-z^2/2} \int_0^{\theta} e^{(z^2/2) \cos 2t} \cos t dt,$$

and using (1.4), we obtain

$$(2.11) \quad \operatorname{Erf}(z \sin \theta) = \frac{z}{2} \sum_{k=0}^{\infty} \epsilon_k \left\{ \frac{\sin(2k+1)\theta}{2k+1} + \frac{\sin(2k-1)\theta}{2k-1} \right\} e^{-z^2/2} I_k(z^2/2),$$

and from the latter follows the interesting representation

$$(2.12) \int_0^{\frac{\pi}{2}} \operatorname{Erf}(z \sin \theta) \sin (2k + 1) \theta \, d\theta = \frac{\pi z}{4(2k + 1)} e^{-z^2/2} [I_k(z^2/2) + I_{k+1}(z^2/2)].$$

3. The Exponential Integral. This is a limiting form of the incomplete gamma function. From (2.2), we have

$$(3.1) \int_z^\infty t^{-1} e^{-t} \, dt = \lim_{a \rightarrow 0} \{ \Gamma(a) - \gamma(a, z) \}.$$

Another useful form is

$$(3.2) \int_0^z t^{-1} (1 - e^{-t}) \, dt = \lim_{a \rightarrow 0} \{ a^{-1} z^a - \gamma(a, z) \}.$$

Using (1.4) and (2.3), we write

$$(3.3) e^{z/2} z^{-a} \int_z^\infty e^{-u} u^{a-1} \, du = \left\{ \frac{\Gamma(a + 1) z^{-a} - 1}{a} \right\} e^{z/2} + 2 \sum_{k=1}^\infty \left\{ \frac{1 - R_k(a)}{a} \right\} I_k(z/2).$$

Application of L'Hospital's theorem gives

$$(3.4) \lim_{a \rightarrow 0} \left\{ \frac{\Gamma(a + 1) z^{-a} - 1}{a} \right\} = -(\gamma + \ln z)$$

where γ is Euler's constant. Define

$$(3.5) f_k = \lim_{a \rightarrow 0} \frac{1 - R_k(a)}{a} = - \sum_{m=1}^k \frac{(-1)^m \binom{k}{m} (k)_m}{(\frac{1}{2})_m m}.$$

Then

$$(3.6) \begin{aligned} -Ei(-z) + (\gamma + \ln z) &= \int_z^\infty t^{-1} e^{-t} \, dt + (\gamma + \ln z) \\ &= \int_0^z t^{-1} (1 - e^{-t}) \, dt = 2 \sum_{k=1}^\infty f_k e^{-z/2} I_k(z/2). \end{aligned}$$

In view of (2.5) and (3.5), we have the recurrence formula

$$(3.7) (k + 1)f_{k+1} = 2f_k + (k - 1)f_{k-1} + 4,$$

and by induction or otherwise, we can prove that

$$(3.8) \begin{cases} f_{k+1} = f_k + \frac{1}{k} \{1 - (-1)^k\} + \frac{1}{k+1} \{1 + (-1)^k\}, & f_0 = 0, f_1 = 2, \\ f_{2k} = 4 \left(1 + \frac{1}{3} + \dots + \frac{1}{2k-1} \right), \\ f_{2k+1} = f_{2k} + \frac{2}{2k+1}, & f_{2k+2} = f_{2k+1} + \frac{2}{2k+1}. \end{cases}$$

Employing the theory of the $\psi(z)$ function, the logarithmic derivative of the

gamma function [10], we find that

$$(3.9) \quad \lim_{k \rightarrow \infty} (f_k - 2 \ln 2k) = 2\gamma,$$

and so (3.6) converges like

$$(3.10) \quad \sum_{k=1}^{\infty} u_k z^k, \quad u_k = (\ln k)/2^{2k}k!$$

An alternative form of (3.6) is

$$(3.11) \quad \begin{cases} \int_0^z t^{-1}(1 - e^{-t}) dt = 2[1 - e^{-z/2}I_0(z/2)] + 4 \sum_{k=0}^{\infty} g_{k+2} e^{-z/2} I_{k+2}(z/2), \\ g_k = \frac{1}{2}f_k - 1. \end{cases}$$

Now

$$(3.12) \quad Ei(z) = \int_{-\infty}^z t^{-1}e^t dt = Ei(ze^{\pm i\pi}) \mp i\pi,$$

and so

$$(3.13) \quad e^{-z} \int_{-\infty}^z t^{-1}e^t dt = e^{-z}(\gamma + \ln z) + 2[e^{-z/2}I_0(z/2) - e^{-z}] - 4 \sum_{k=0}^{\infty} (-1)^k g_{2k+2} e^{-z/2} I_{k+2}(z/2)$$

We also have

$$(3.14) \quad \begin{aligned} \int_0^z t^{-1}(1 - e^{-it}) dt &= (\gamma + \ln z) - Ci(z) + iSi(z) \\ &= 2e^{-iz/2} \sum_{k=1}^{\infty} f_k i^k J_k(z/2). \end{aligned}$$

4. A Second Expansion and Special Cases. Buchholz [1] has given the expansion

$$(4.1) \quad \Phi(a, c, z) = e^{z/2}(z/4)^{\frac{1}{2}-a}\Gamma(a - \frac{1}{2}) \cdot \sum_{k=0}^{\infty} \frac{(k + a - \frac{1}{2})(2a - 1)_k(2a - c)_k}{k!(c)_k} I_{k+a-\frac{1}{2}}(z/2), \quad R(a) > 0,$$

which coalesces with (1.8) when $a = \frac{1}{2}$. If a , c and z are fixed, then an analysis similar to (1.11)–(1.14) shows that (4.1) converges like

$$(4.2) \quad \sum_{k=1}^{\infty} \frac{k^{3a-2c-1/2} z^k}{2^{2k}k!},$$

and from this point of view, the convergence of (1.8) and (4.1) are nearly alike when $R(c - a) > 0$.

We now deduce from (4.1) series expansions for the exponential integral and related functions. The method of proof is akin to that in the previous section. If in (4.1), we put $a = c$, there follows a representation for $e^{z/2}$ since $\Phi(a, a, z) = e^z$. Use this and (4.1) with $c = a + 1$ and z replaced by $-z$. Then after the manner

of (3.3), we have

$$(4.3) \quad e^{z/2} \int_z^\infty t^{a-1} e^{-t} dt = (\pi z)^{1/2} \sum_{k=0}^\infty h_k(a) I_{k+a-1/2}(z/2),$$

and employing the duplication formula for gamma functions, the elements $h_k(a)$ take the form

$$(4.4) \quad h_k(a) = \frac{2^{k+2a-1} (k+a-1/2) \Gamma\left(\frac{k-1}{2} + a\right) \Gamma\left(\frac{k}{2} + a\right)}{\Gamma(\frac{1}{2})k!} \cdot \left\{ z^{-a} - \frac{(-1)^k}{\Gamma(a-1)(k+a)(k+a-1)} \right\}.$$

For $k = 0$ and 1 , application of L'Hospital's theorem gives

$$(4.5) \quad h_0(0) = -\frac{1}{2}(\gamma + \ln z), \quad h_1(0) = h_0(0) + 1.$$

Otherwise,

$$(4.6) \quad h_k(0) = \frac{2k-1}{k(k-1)}, \quad k > 1.$$

Thus

$$(4.7) \quad -Ei(-z) + (\gamma + \ln z) = \int_z^\infty t^{-1} e^{-t} dt + (\gamma + \ln z) = \int_0^z t^{-1} (1 - e^{-t}) dt \\ = (\pi z)^{1/2} e^{-z/2} I_{1/2}(z/2) + (\pi z)^{1/2} \sum_{k=0}^\infty \frac{(2k+3)}{(k+1)(k+2)} e^{-z/2} I_{k+3/2}(z/2),$$

and forms for the related functions follow readily enough after the manner of (3.12)–(3.14). By a familiar argument, (4.7) converges like

$$(4.8) \quad \sum_{k=1}^\infty v_k z^k, \quad v_k = [2^{2k} k \Gamma(k + \frac{1}{2})]^{-1},$$

and so (4.7) converges more rapidly than (3.6). However, the difference is not great since $u_k/v_k \sim k^3 \ln k$ (see (3.10)).

5. Further Expansions. The following representation is due to Tricomi [2, 3].

$$(5.1) \quad \Phi(a, c, z) = \Gamma(c) e^{hz} \sum_{k=0}^\infty A_k(a, c, h) z^k E_{c+k-1}(-az), \quad h \geq 0,$$

$$(5.2) \quad E_\nu(z) = z^{-\nu/2} J_\nu(2z^{1/2}), \quad E_\nu(-z) = z^{-\nu/2} I_\nu(2z^{1/2}),$$

where the coefficients $A_k(a, c, h)$ are defined by the generating function

$$(5.3) \quad \sum_{k=0}^\infty A_k(a, c, h) z^k = e^{-az} [1 + (h-1)z]^{-a} (1+hz)^{a-c}, \quad |z| < 1,$$

and satisfy the recurrence system

$$(5.4) \quad (k+1)A_{k+1} = [(1-2h)k - hc]A_k \\ + [a(1-2h) - h(h-1)(c+k-1)]A_{k-1} - h(h-1)A_{k-2}, \\ A_0 = 1, \quad A_1 = -hc, \quad A_2 = \frac{1}{2}h^2c(c+1) + a(\frac{1}{2} - h).$$

A second expansion given by Tricomi [2, 3] is

$$(5.5) \quad \Phi(a, c, z) = \Gamma(c)e^{z/2} \sum_{k=0}^{\infty} A_k^*(K, c/2)(z/2)^k E_{c+k-1}(Kz), \quad K = \frac{c}{2} - a,$$

where

$$(5.6) \quad \sum_{k=0}^{\infty} A_k^*(K, c/2)z^k = e^{2Kz}(1 - z)^{K-c/2}(1 + z)^{-K-c/2}, \quad |z| < 1,$$

$$(5.7) \quad (k + 1)A_{k+1}^* = (k + c - 1)A_{k-1}^* - 2KA_{k-2}^*,$$

$$A_0^* = 1, \quad A_1^* = 0, \quad A_2^* = \frac{c}{2}.$$

Both (5.1) and (5.5) are useful to study the behavior of Φ when the parameters are large [11], but are not too suitable for many purposes of computation, since the order and argument of the Bessel functions depend on the parameters a and c .

In (5.1), put $a = 1$, replace c by $a + 1$ and set $h = 0$. Then

$$(5.8) \quad \gamma(a, z) = \Gamma(a)e^{-z}z^{a/2} \sum_{k=0}^{\infty} e_k(-1)z^{k/2}I_{k+a}(2z^{1/2}), \quad e_k(-1) = \sum_{m=0}^k \frac{(-1)^m}{m!}$$

follows from (2.1) and (5.3). The method of proof surrounding (1.11)–(1.14) shows that (5.8) converges like $\sum_{k=0}^{\infty} z^k/\Gamma(k + a + 1)$ which is the same as (1.1) for the above selection of parameters.

6. Further Expansions for the Error and Exponential Integrals. We start with the representation

$$(6.1) \quad \int_0^z t^\mu I_\nu(t) dt = \frac{z^\mu \Gamma\left(\frac{\nu + \mu + 1}{2}\right)}{\Gamma\left(\frac{\nu - \mu + 1}{2}\right)} \cdot \sum_{k=0}^{\infty} (-1)^k \frac{(\nu + 2k + 1)\Gamma\left(\frac{\nu - \mu + 1}{2} + k\right)}{\Gamma\left(\frac{\nu + \mu + 3}{2} + k\right)} I_{\nu+2k+1}(z), \quad R(\mu + \nu + 1) > 0$$

which is readily verified by differentiation. Put $\mu = 0$, and consider the formulas when $\nu = \frac{1}{2}$ and $\nu = -\frac{1}{2}$. Then by subtraction and addition, we get

$$(6.2) \quad \operatorname{Erf} z = (\pi/2)^{\frac{1}{2}} \sum_{k=1}^{\infty} (-1)^{[k/2]} I_{k-\frac{1}{2}}(z^2).$$

$$(6.3) \quad \operatorname{Erfi} z = (\pi/2)^{\frac{1}{2}} \sum_{k=0}^{\infty} (-1)^{[k/2]} I_{k+\frac{1}{2}}(z^2).$$

Here $[k/2]$ is the largest integer, including zero, contained in $k/2$. The latter two equations were also verified by Tricomi [3] by means of the Laplace transform*.

* There is a typographical error in [12] for (6.2). There the summation should start with $n = 1$, not $n = 0$.

Another representation which follows from (5.8) is

$$(6.4) \quad \text{Erf } z = \frac{1}{2}(\pi z)^{\frac{1}{2}} e^{-z^2} \sum_{k=0}^{\infty} e_k(-1) z^k I_{k+\frac{1}{2}}(2z).$$

By a familiar argument, the convergence of (2.7) is superior to that of (6.2)–(6.4).

To obtain another expansion for functions related to the exponential integral, it is convenient to again use (6.1) with z replaced by iz . The ensuing formula is of the same type as (6.1) with $I_\nu(z)$ replaced by $J_\nu(z)$ and the factor $(-1)^k$ omitted behind the summation sign. Now put $\mu = -\frac{1}{2}$ and $\nu = \frac{1}{2}$. Then for the sin integral, we have

$$(6.5) \quad \int_0^z \frac{\sin t}{t} dt = \left(\frac{\pi}{2z}\right)^{1/2} \sum_{k=0}^{\infty} \frac{(4k+3)k!}{\left(\frac{3}{2}\right)_k} J_{2k+3/2}(z)$$

Similarly, with $\nu = -\frac{1}{2}$, we can form an expression for

$$(6.6) \quad \int_0^z t^{\mu-1/2}(1-\cos t) dt. \quad \text{Let } \mu \rightarrow -\frac{1}{2}; \quad \text{then}$$

$$\int_0^z t^{-1}(1-\cos t) dt = \frac{1}{2} \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} \sum_{k=1}^{\infty} \frac{(4k+1)\left(\frac{1}{2}\right)_k}{k!} \cdot \{\psi(k+1) - \psi(1) + \psi(k+\frac{1}{2}) - \psi(\frac{1}{2})\} J_{2k+1}(z).$$

Our previous analysis shows that for z fixed and k large, the ratio of the k th term in the expansion (6.6) to the k th term of the Taylor series expansion for the cosine integral is proportional to $k(\ln k)/2^{2k}$. Similarly for (6.5), the ratio is proportional to $k/2^{2k}$. If (3.6) is compared with its corresponding Taylor series representation in like fashion, we again obtain the former ratio. Thus the proper combination of (6.5) and (6.6) gives about the same convergence as (3.6) and (4.7).

As a remark aside, using (6.1), we can produce

$$(6.7) \quad \lim_{\mu \rightarrow -1} \int_0^z t^\mu [1 - J_0(t)] dt$$

$$= \int_0^z t^{-1} [1 - J_0(t)] dt = (\gamma + \ln z/2) + \int_0^\infty t^{-1} J_0(t) dt$$

$$= 2z^{-1} \sum_{k=0}^{\infty} (2k+3)[\psi(k+2) - \psi(1)] J_{2k+3}(z)$$

$$= 1 - 2z^{-1} J_1(z) + 2z^{-1} \sum_{k=0}^{\infty} (2k+5)[\psi(k+3) - \psi(1) - 1] J_{2k+5}(z).$$

For tables of this integral when z is real, see [13].

We now derive yet another expression for the exponential integral. It is known [14, 15, 16]* that

* We take this occasion to point out some typographical errors in [15] and [16]. The formula $(\partial J_\nu(z)/\partial \nu)_{\nu=k}$ should read

$$\left. \frac{\partial J_\nu(z)}{\partial \nu} \right|_{\nu=k} = -G_k(z) + \frac{1}{2}k! \sum_{n=0}^{k-1} \frac{(\frac{1}{2}z)^{n-k} J_n(z)}{n!(k-n)}, \quad G_k(z) = -\frac{\pi}{2} Y_k(z).$$

$$(6.8) \quad Ei(-2z) = (\pi z/2)^{1/2} e^{-z} \left[\frac{\partial I_\nu(z)}{\partial \nu} \Big|_{\nu=1/2} + \frac{\partial I_\nu(z)}{\partial \nu} \Big|_{\nu=-1/2} \right],$$

where $Ei(-z)$ is given by (3.6). To evaluate the partial derivatives in (6.8), we use the expansion [17]

$$(6.9) \quad I_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu)} \sum_{k=0}^\infty \frac{(-1)^k z^k I_k(z)}{(\nu + k)k!},$$

and find that

$$(6.10) \quad \int_0^z t^{-1}(1 - e^{-t}) dt = 2e^{-z/2} \cosh z/2 + zg_1(z) - 2g_2(z),$$

$$(6.11) \quad e^{-z} \int_0^z t^{-1}(e^t - 1) dt = -2e^{-z/2} \cosh z/2 + zg_1(z) + 2g_2(z),$$

where

$$(6.12) \quad g_m(z) = \sum_{k=0}^\infty \frac{(-1)^k (z/4)^k e^{-z/2} I_k(z/2)}{k!(2k + 3 - 2m)^2}, \quad m = 1, 2$$

Of all the exponential integral expressions given, this converges the best.

7. Numerical Examples.*

EXAMPLE 1. Using (1.8) and (1.9), we illustrate the computation of

$$(7.1) \quad e^{-ix} \Phi(m + 1 - ia, 2m + 2, 2ix) = \sum_{k=0}^\infty \epsilon_k R_k i^k J_k(x),$$

$$(7.2) \quad R_k = {}_3F_2(-k, k, m + 1 + ia; 2m + 2, \frac{1}{2}; 1).$$

Suppose $a = 1, m = 2$. Then values of $i^k R_k$ are easily generated using (7.2) and (1.20). We have $R_0 = 1, iR_1 = \frac{1}{3}, i^2 R_2 = \frac{1}{2} \frac{9}{1}$, etc. Using standard tables of Bessel functions, if $x = 1$, seven terms of (7.1) give the six-decimal value 1.451140. If $x = 2$, nine terms give 1.293748. For tables of (7.1), see [18]. If $x = 2$, about 15 terms of the Taylor series are required to achieve six-decimal accuracy.

EXAMPLE 2. We employ (2.7) and (2.8) to compute the error integrals for $z = 2$. Using nine terms, we find $\text{Erf } 2 = 0.88208 \ 14$ and $e^{-4} \text{Erfi } 2 = 0.30134 \ 04$. About 19 terms of the Taylor series expansion are needed for the same accuracy.

EXAMPLE 3. To illustrate computation of the exponential integral, it is sufficient to consider (6.10). If $z = 4$ and we use six terms of each series in (6.12), then the value of the integral (6.10) is 1.96728 94. Similarly, if $z = 8$, only eight terms of each series in (6.12) are needed to yield the value 2.65669 49. For $z = 8$, about 32 terms of the Taylor series expansion would be required to achieve the same accuracy.

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