

side, the yellow result light will flash on, indicating the truth value of  $KNpNq$  for  $p = \text{wrong}$  and  $q = \text{wrong}$  according to the left position of the switches for the variables  $p$  and  $q$  on the right side of the keyboard.

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## Evaluation at Half Periods of Weierstrass' Elliptic Function with Rectangular Primitive Period-Parallelogram

By Chih-Bing Ling

The purpose of this paper is to evaluate the following Weierstrass' elliptic function at half periods [1],

$$(1) \quad e_1 = \wp(\omega_1), \quad e_2 = \wp(\omega_2), \quad e_3 = \wp(\omega_3),$$

where  $2\omega_1$  and  $2\omega_2$  are double periods of the function and  $\omega_3$  is defined by

$$(2) \quad \omega_1 + \omega_2 + \omega_3 = 0.$$

This paper tabulates only the values of the function whose primitive period-parallelogram is a rectangle with  $2\omega_1 = 1$  and  $2\omega_2 = ai$ , where  $a \geq 1$ .

The three functions in (1) form a set of distinct roots of the cubic [1]

$$(3) \quad x^3 - px - q = 0,$$

where

$$(4) \quad p = 15\sigma_4, \quad q = 35\sigma_6,$$

and

$$(5) \quad \begin{aligned} \sigma_{2k} &= \sum'_{m, n=-\infty}^{\infty} \frac{1}{(2m\omega_1 + 2n\omega_2)^{2k}} \\ &= 2 \sum_{m=1}^{\infty} \frac{1}{m^{2k}} + 2 \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} \frac{1}{(m + nai)^{2k}}. \end{aligned}$$

The accent on the summation sign denotes the omission of simultaneous zero values of  $m$  and  $n$  from the double summation.

The cubic (3) indicates that

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$$(6) \quad e_1 + e_2 + e_3 = 0.$$

Also, since  $e_1$ ,  $e_2$  and  $e_3$  are distinct, the discriminant  $(4p^3 - 27q^2)$  of the cubic does not vanish. As will be seen later, in the present case both  $\sigma_4$  and  $\sigma_6$  are real and  $(4p^3 - 27q^2)$  is positive. This implies that all the roots of the cubic are real.

The evaluation of  $\sigma_4$  and  $\sigma_6$  is facilitated by using the known relation [2]

$$(7) \quad \cot x = \frac{1}{x} + \sum'_{m=-\infty}^{\infty} \left( \frac{1}{m\pi + x} - \frac{1}{m\pi} \right)$$

where the accent on the summation sign denotes the omission of the zero value of  $m$  from the summation. By repeated differentiation of Equation (7) and substitution of  $ix$  for  $x$ , it is found that

$$(8) \quad \sum_{m=-\infty}^{\infty} \frac{1}{(m\pi + ix)^4} = \frac{2}{3 \sinh^2 x} + \frac{1}{\sinh^4 x}$$

$$\sum_{m=-\infty}^{\infty} \frac{1}{(m\pi + ix)^6} = -\frac{2}{15 \sinh^2 x} - \frac{1}{\sinh^4 x} - \frac{1}{\sinh^6 x}.$$

Hence we have

$$(9) \quad \sigma_4 = \frac{\pi^4}{45} + \frac{4\pi^4 K_1}{3 \sinh^2 \pi a}$$

$$\sigma_6 = \frac{2\pi^6}{945} - \frac{4\pi^6 K_2}{15 \sinh^2 \pi a}$$

where

$$(10) \quad K_1 = \sinh^2 \pi a \sum_{n=1}^{\infty} \left( \frac{1}{\sinh^2 n\pi a} + \frac{3}{2 \sinh^4 n\pi a} \right)$$

$$K_2 = \sinh^2 \pi a \sum_{n=1}^{\infty} \left( \frac{1}{\sinh^2 n\pi a} + \frac{15}{2 \sinh^4 n\pi a} + \frac{15}{2 \sinh^6 n\pi a} \right).$$

Consequently, we find

$$(11) \quad \frac{4p^3 - 27q^2}{16\pi^{12}} = \frac{5K_1 + 7K_2}{3 \sinh^2 \pi a} + \frac{100K_1^2 - 147K_2^2}{\sinh^4 \pi a} + \frac{2000K_1^3}{\sinh^6 \pi a}.$$

With the aid of known tables [3, 4], values of  $K_1$ ,  $K_2$ , and then  $\sigma_4$ ,  $\sigma_6$  and  $(4p^3 - 27q^2)^{\frac{1}{3}}$  are computed to 16D for  $a = 1(0.25)2(1)6$  and  $\infty$  as shown in Table 1.

The subsequent evaluation of  $e_1$ ,  $e_2$ , and  $e_3$  requires the solution of the cubic (3). It appears that one of the roots,  $e_1$ , can be easily evaluated to 16D as shown in Table 2 by using Newton's method or otherwise, but difficulty arises in evaluating the other two roots for in most cases they are almost equal. However, they can be separated by forming a new cubic

$$(12) \quad x^3 + p'x - q' = 0$$

whose roots are the differences of the roots of the cubic (3). Let  $(e_1 - e_2)$ ,  $(e_2 - e_3)$  and  $(e_3 - e_1)$  be the roots of the new cubic. We have

$$(13) \quad p' = (e_1 - e_2)(e_2 - e_3) + (e_2 - e_3)(e_3 - e_1) + (e_3 - e_1)(e_1 - e_2) = -3p,$$

$$q'^2 = (e_1 - e_2)^2(e_2 - e_3)^2(e_3 - e_1)^2 = 4p^3 - 27q^2.$$

TABLE 1

$a$	$K_1$	$K_2$	$e_1$	$e_2$	$(4e^2 - 27e^3)^{1/2}$
1	1.01311, 06293, 09539	1.05851, 88947, 76779	3.15121, 20021, 53898	0	6.49955, 04200, 35218 $\times 10^2$
1.25	1.00271, 90816, 67748	1.01206, 13101, 78251	2.36702, 93923, 35617	1.63147, 62559, 94511	3.01664, 67968, 60964 $\times 10^2$
1.5	1.00056, 49682, 73190	1.00250, 28510, 82885	2.20660, 15468, 91272	1.95170, 97194, 76020	1.38049, 25551, 66708 $\times 10^2$
1.75	1.00011, 74335, 66488	1.00052, 00996, 05499	2.17336, 30560, 32070	2.01747, 33403, 07279	6.29902, 26479, 45028 $\times 10$
2	1.00002, 44115, 30272	1.00010, 81097, 89987	2.16645, 82514, 80805	2.03110, 95062, 61006	2.87242, 26104, 19235 $\times 10$
3	1.00000, 00455, 86885	1.00000, 02018, 84784	2.16464, 98507, 19257	2.03467, 94456, 07301	1.24133, 82088, 92023 $\times 10^{-3}$
4	1.00000, 00000, 85131	1.00000, 00003, 77008	2.16464, 64737, 40389	2.03468, 61114, 97443	5.36430, 92081, 07968 $\times 10^{-3}$
5	1.00000, 00000, 00159	1.00000, 00000, 00704	2.16464, 64674, 34075	2.03468, 61239, 45609	2.31812, 81969, 45459 $\times 10^{-3}$
6	1.00000, 00000, 00000	1.00000, 00000, 00001	2.16464, 64674, 22298	2.03468, 61239, 68855	1.00175, 40242, 77741 $\times 10^{-4}$
$\infty$	1.00000, 00000, 00000	1.00000, 00000, 00000	2.16464, 64674, 22276	2.03468, 61239, 68898	0

TABLE 2

$a$	$e_1$	$-(e_1 - e_2)$	$-e_2$	$-e_3$
1	6.87518, 58180, 20373	6.87518, 58180, 20373	6.87518, 58180, 20373	0
1.25	6.64106, 26950, 12943	3.11618, 71546, 58608	4.87862, 49248, 35775	1.76243, 77701, 77167
1.5	6.59248, 08531, 44224	1.41904, 24293, 36329	4.00576, 16412, 40277	2.58671, 92119, 03947
1.75	6.58238, 54370, 71645	6.46830, 14416, 89006 $\times 10^{-1}$	3.61460, 77906, 20273	2.96777, 76464, 51372
2	6.58028, 69683, 44880	2.94898, 84967, 54931 $\times 10^{-1}$	3.43759, 29090, 10186	3.14269, 40593, 34693
3	6.57973, 72957, 91816	1.27435, 57352, 40785 $\times 10^{-2}$	3.29624, 04265, 72112	3.28349, 68692, 19704
4	6.57973, 62693, 13382	5.50699, 03149, 79195 $\times 10^{-4}$	3.29014, 34841, 72440	3.28959, 27851, 40942
5	6.57973, 62673, 96492	2.37978, 62933, 93412 $\times 10^{-6}$	3.28988, 00326, 29713	3.28985, 62347, 66779
6	6.57973, 62673, 92912	1.02839, 89036, 79391 $\times 10^{-6}$	3.28986, 86478, 95908	3.28986, 76194, 97004
$\infty$	6.57973, 62673, 92906	0	3.28986, 81336, 96453	3.28986, 81336, 96453

Consequently, by taking a positive sign for  $q'$ , the new cubic is in the form

$$(14) \quad x^3 - 3px - (4p^3 - 27q^2)^{\frac{1}{2}} = 0.$$

From this cubic, values of  $(e_2 - e_3)$  and then  $e_2$  and  $e_3$  are computed to  $16D$  as shown in Table 2.

It is mentioned that the values of the function, for  $0 \leq a < 1$  or in general for  $\omega_2/\omega_1$  purely imaginary, can be computed from the tabulated values with the aid of the following relation [1]

$$(15) \quad \wp(\lambda z \mid \lambda\omega_1, \lambda\omega_2) = \lambda^{-2}\wp(z \mid \omega_1, \omega_2)$$

where  $\lambda$  is a constant, real or complex.

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## A Note on the Nonexistence of Certain Projective Planes of Order Nine

By Raymond B. Killgrove

**1. Introduction.** Every finite projective plane may be coordinatized in at least one way [1]. In this process some line is chosen to be the line at infinity, and the points not on this line are represented by an ordered pair of elements. The elements  $x$  and  $y$  for any point  $(x, y)$  on a given line of the plane satisfy the equation  $y = x \cdot m \cdot b$ , where  $m$  and  $b$  are specific elements for the given line. This ternary operation on  $x$ ,  $m$ , and  $b$  includes an additive loop in a special case.

A sequence of SWAC computer routines has been written to search for all planes having a specific additive loop in an appropriate ternary ring. Using these routines, a complete search had been made previously using the elementary Abelian group for the additive loop [2]. Now a complete search has been made using the

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