Numerical Quadrature over a Rectangular Domain in Two or More Dimensions

Part 2. Quadrature in several dimensions, using special points

By J. C. P. Miller

1. Introduction. In Part 1 [1], several approximate formulas were developed for quadrature over a rectangular domain (reduced to a square by scale changes) in the form

$$I = \int_{-rh}^{rh} \int_{-rh}^{rh} f(x, y)\, dx\, dy \doteqdot \sum A_{s,t}\, f(sh, th) \tag{1.1}$$

using some or all of a lattice of 9 or 16 points equally spaced over the square.

In this note, the restriction to equal spacing is relaxed, to allow points to be chosen to give greater accuracy in the approximation to the integral. The restriction to two dimensions or variables is also removed in some cases.

It is still assumed that the integrand $f(x_1, x_2, \cdots, x_n)$ may be expanded in a power series as far as we please in all variables—in other words, that for the precision desired, the integrand may be replaced by a polynomial of suitable degree. Considerations of symmetry once again ensure that we need concern ourselves only with terms of even degree in each variable separately. By scale changes we also make the range of integration $-h$ to $+h$ in each variable.

2. Expansion in Taylor Series. We need then, to find values $x_{s,t}$, A_t such that

$$\begin{aligned} I &= \int_{-h}^{h} dx_1 \cdots \int_{-h}^{h} f(x_1, x_2, \cdots, x_n)\, dx_n \\ &\doteqdot (2h)^n \sum A_t f(x_{1,t}, x_{2,t}, \cdots, x_{n,t}) \end{aligned} \tag{2.1}$$

where the points $\{x_{s,t}\}$ are chosen in symmetrical groups which are such that $\sum f(x_{1,t}, x_{2,t}, \cdots, x_{n,t})$ is free of all odd powers, just as the integral I is free of such powers by symmetry.

We expand $f(x_1, x_2, \cdots, x_n)$ as a Taylor series and evaluate the integral. We use the notation of Bickley [2], somewhat extended, namely,

$$\begin{aligned} S_2 &= \sum_t \frac{\partial^2 \phi}{\partial x_t^2} \equiv \nabla^2 \phi \qquad S_{2,2} = \sum_{t,u}{}^{*} \frac{\partial^4 \phi}{\partial x_t^2 \partial x_u^2} \equiv \mathfrak{D}^4 \phi \\ S_{2,2,2} &= \sum_{t,u,v}{}^{*} \frac{\partial^6 \phi}{\partial x_t^2 \partial x_u^2 \partial x_v^2} \equiv \mathfrak{Z}^6 \phi \qquad S_{2,2,2,2} = \sum_{t,u,v,w}{}^{*} \frac{\partial^8 \phi}{\partial x_t^2 \partial x_u^2 \partial x_v^2 \partial x_w^2} \equiv \mathfrak{Q}^8 \phi \end{aligned} \tag{2.2}$$

where the asterisk indicates that the suffixes t, u, v, w are unequal in pairs throughout.

Received October 19, 1959.

This gives, with f_0 for $f(0, 0, 0, \cdots, 0)$

$$J \equiv I/(2h)^n = f_0 + \frac{h^2}{3!}\nabla^2 f_0 + \frac{h^4}{5!}\left(\nabla^4 + \frac{4}{3}\mathfrak{D}^4\right) f_0$$

$$(2.3) \qquad + \frac{h^6}{7!}\left(\nabla^6 + 4\Delta^2\mathfrak{D}^4 + \frac{16}{3}\mathfrak{J}^6\right) f_0$$

$$+ \frac{h^8}{9!}\left(\nabla^8 + 8\nabla^4\mathfrak{D}^4 + \frac{16}{5}\mathfrak{D}^8 + \frac{128}{5}\nabla^2\mathfrak{J}^6 + \frac{192}{5}\mathfrak{Q}^8\right) f_0 + \cdots$$

where we have used such formulas as

$$S_4 = \sum_t \frac{\partial^4\phi}{\partial x_t^4} = (\nabla^4 - 2\mathfrak{D}^4)\phi \qquad S_6 = \sum_t \frac{\partial^6\phi}{\partial x_t^6} = (\nabla^6 - 3\nabla^2\mathfrak{D}^4 + 3\mathfrak{J}^6)\phi$$

$$(2.4) \qquad S_{4,2} = \sum_{\substack{t,u \\ t \neq u}} \frac{\partial^6\phi}{\partial x_t^4 \partial x_u^2} = (\nabla^2\mathfrak{D}^4 - 3\mathfrak{J}^6)\phi.$$

See David & Kendall [3] for extended tables of coefficients.

3. Summation over Sets of Points. We now choose appropriate sets of symmetrically placed points. We tabulate and label a few sets below.

(3.1)

Label	Coordinates	Number of Points
0	$(0, 0, \cdots, 0)$	1
$\alpha(a)$	$(\pm ah, 0, \cdots, 0)$ with all permutations	$2n$
$\beta(b)$	$(\pm bh, \pm bh, 0, \cdots, 0)$ with all permutations	$2n(n-1)$
$\gamma(c, d)$	$(\pm ch, \pm dh, 0, \cdots, 0)$ with all permutations	$4n(n-1)$
$\epsilon(e)$	$(\pm eh, \pm eh, \pm eh, 0, \cdots, 0)$ with all permutations	$4n(n-1)(n-2)/3$

In the formula (2.1) we have one arbitrary constant for the point 0, two constants for each set of type α, β, or ϵ, three constants for each set of type γ, and so on. We naturally wish to minimize the total number of points, but we note also that sets of type 0 and α alone are useless when dealing with terms involving $\mathfrak{D}^4$, $\mathfrak{J}^6$, $\mathfrak{Q}^8$, etc., while sets of 0, α, β, γ, only are useless for $\mathfrak{J}^6$ or $\mathfrak{Q}^8$, and so on.

If we expand $f(x_1, x_2, \cdots, x_n)$ at each point and sum over the set, we obtain the following expressions for the sums:

(3.21) 0 $\quad f_0$

$$(3.22) \quad \alpha(a) \qquad 2nf_0 + \frac{2h^2}{2!}a^2\nabla^2 f_0 + \frac{2h^4}{4!}a^4 S_4 + \frac{2h^6}{6!}a^6 S_6 + \frac{2h^8}{8!}a^8 S_8 + \cdots$$

$$= 2nf_0 + \frac{2h^2}{2!}a^2\nabla^2 f_0 + \frac{2h^4}{4!}a^4(\nabla^4 - 2\mathfrak{D}^4)f_0$$

$$+ \frac{2h^6}{6!}a^6(\nabla^6 - 3\nabla^2\mathfrak{D}^4 + 3\mathfrak{J}^6)f_0$$

$$+ \frac{2h^8}{8!} a^8 (\nabla^8 - 4\nabla^4 \mathfrak{D}^4 + 2\mathfrak{D}^8 + 4\nabla^2 \mathfrak{J}^6 - 4\mathfrak{Q}^8) f_0 + \cdots$$

(3.23) $\beta(b)$

$$2n(n-1)f_0 + \frac{4(n-1)h^2}{2!} b^2 \nabla^2 f_0$$

$$+ \frac{4h^4}{4!} b^4 \{(n-1)\nabla^4 - 2(n-4)\mathfrak{D}^4\} f_0$$

$$+ \frac{4h^6}{6!} b^6 \{(n-1)\nabla^6 - 3(n-6)\nabla^2 \mathfrak{D}^4 + 3(n-16)\mathfrak{J}^6\} f_0$$

$$+ \frac{4h^8}{8!} b^8 \{(n-1)\nabla^8 - 4(n-8)\nabla^4 \mathfrak{D}^4 + 2(n+6)\mathfrak{D}^8$$

$$+ 4(n-43)\nabla^2 \mathfrak{J}^6 - 4(n-64)\mathfrak{Q}^8\} f_0$$

$$+ \cdots$$

(3.24) $\gamma(c, d)$

$$4n(n-1)f_0 + \frac{4(n-1)h^2}{2!} (c^2 + d^2)\nabla^2 f_0$$

$$+ \frac{4h^4}{4!} [(n-1)(c^4 + d^4)\nabla^4 - 2\{(n-1)(c^4 + d^4) - 6c^2 d^2\}\mathfrak{D}^4] f_0$$

$$+ \frac{4h^6}{6!} [(n-1)(c^6 + d^6)\nabla^6$$

$$- 3\{(n-1)(c^6 + d^6) - 5c^2 d^2 (c^2 + d^2)\}\nabla^2 \mathfrak{D}^4$$

$$+ 3\{(n-1)(c^6 + d^6) - 15c^2 d^2 (c^2 + d^2)\}\mathfrak{J}^6] f_0$$

$$+ \frac{4h^8}{8!} [(n-1)(c^8 + d^8)\nabla^8$$

$$- 4\{(n-1)(c^8 + d^8) - 7c^2 d^2 (c^4 + d^4)\}\nabla^4 \mathfrak{D}^4$$

$$+ 2\{(n-1)(c^8 + d^8) - 28c^2 d^2 (c^4 + d^4) + 70c^4 d^4\}\mathfrak{D}^8$$

$$+ 4\{(n-1)(c^8 + d^8) - 7c^2 d^2 (c^4 + d^4) - 70c^4 d^4\}\nabla^2 \mathfrak{J}^6$$

$$- 4\{(n-1)(c^8 + d^8) - 28c^2 d^2 (c^4 + d^4) - 70c^4 d^4\}\mathfrak{Q}^8] f_0$$

$$+ \cdots$$

(3.25) $\epsilon(e)$

$$\frac{4}{3} n(n-1)(n-2)f_0 + \frac{4(n-1)(n-2)}{2!} h^2 e^2 \nabla^2 f_0$$

$$+ \frac{4(n-2)}{4!} h^4 e^4 \{(n-1)\nabla^4 - 2(n-7)\mathfrak{D}^4\} f_0$$

$$+ \frac{4h^6}{6!} e^6 \{(n-1)(n-2)\nabla^6 - 3(n-2)(n-11)\nabla^2 \mathfrak{D}^4$$

$$+ 3(n^2 - 33n + 122)\mathfrak{J}^6\} f_0$$

$$+ \frac{4h^8}{8!} e^8\{(n-1)(n-2)\nabla^8 - 4(n-2)(n-15)\nabla^4 \mathfrak{D}^4$$
$$+ 2(n-2)(n+13)\mathfrak{D}^8 + 4(n^2 - 87n + 380)\nabla^2 \mathfrak{Z}^6$$
$$- 4(n^2 - 129n + 1094)\mathfrak{Q}^8\}f_0 + \cdots$$

We use parts of (3.22), (3.23) and (3.25) in the present note; the rest is given for reference.

4. Formulas Accurate to Terms in h^4. We now seek a formula to give J accurately as far as terms in h^4, that is, including the first four distinct terms in (2.3). We need $\alpha(a)$ and $\beta(b)$ and shall retain the point 0; this gives five disposable constants, counting the three multipliers, which we shall now denote by A_0, A_α, A_β. This is apparently more than we need, but in fact, the fifth constant turns out to provide an essential extra degree of freedom.

We attempt, then, to satisfy as well as possible

$$J = A_0 f_0 + \sum A_\alpha f(x_\alpha) + \sum A_\beta f(x_\beta) \tag{4.1}$$

using x_α and x_β for typical *sets* of coordinates. If we now equate coefficients of f_0, $\nabla^2 f_0$, $\nabla^4 f_0$ and $\mathfrak{D}^4 f_0$, we obtain:

$$\begin{cases} A_0 + 2nA_\alpha + 2n(n-1)A_\beta = 1 \\ 2a^2 A_\alpha + 4(n-1)b^2 A_\beta = \frac{1}{3} \\ 2a^4 A_\alpha + 4(n-1)b^4 A_\beta = \frac{1}{5} \\ -4a^4 A_\alpha - 8(n-4)b^4 A_\beta = \frac{4}{15}. \end{cases} \tag{4.2}$$

We write $A_0 = \lambda$ and retain this, and solve for a, b, A_α, A_β. It may be noted that the second and third equations are no longer inconsistent, as they were for equally spaced points in Part 1 [1].

We use the last two equations to give A_α and A_β in terms of a^4 and b^4; then substitute in the first two equations and eliminate a^2 to give

$$\frac{1}{b^4}\frac{5n+4}{2} - \frac{30}{b^2} + \frac{18(5n-7) + 9(14-5n)\lambda}{n(n-1)} = 0 \tag{4.3}$$

whence

$$\frac{1}{b^2} = \frac{30}{5n+4} \pm \frac{6}{5n+4}\sqrt{\frac{(5n-14)\{(5n+4)\lambda - 4\}}{2n(n-1)}}. \tag{4.4}$$

The values of $1/b^2$ are real for $\lambda = 0$ only if $n \leqq 2.8$, that is, if $n = 1$ or 2. When $n \geqq 3$, we must have $\lambda > 4/(5n+4)$ for real values of b^2, indicating that the fifth disposable constant is essential.

We next consider several special cases; error terms will be considered individually.

5. Quadrature over a Square. Case $n = 2$. We take $\lambda = 0$ in (4.3); then

$$\frac{7}{b^4} - \frac{30}{b^2} + 27 = 0 \tag{5.1}$$

whence

$$\frac{1}{b^2} = 3 \qquad \frac{1}{a^2} = 0 \qquad A_\alpha = 0 \qquad A_\beta = \frac{1}{4} \tag{5.21}$$

or

$$\frac{1}{b^2} = \frac{9}{7} \qquad \frac{1}{a^2} = \frac{15}{7} \qquad A_\alpha = \frac{10}{49} \qquad A_\beta = \frac{9}{196}. \tag{5.22}$$

The first of these (5.21) is a degenerate case; it is in fact the product-Gauss two-point formula, with error term of order h^4 instead of the expected order h^6.

The second case (5.22) gives a useful approximate formula

$$\begin{aligned} J = I/(4h)^2 = \frac{10}{49}\Bigg\{ & f\left(\sqrt{\frac{7}{15}}\,h, 0\right) + f\left(-\sqrt{\frac{7}{15}}\,h, 0\right) \\ & + f\left(0, \sqrt{\frac{7}{15}}\,h\right) + f\left(0, -\sqrt{\frac{7}{15}}\,h\right)\Bigg\} \\ + \frac{9}{196}\Bigg\{ & f\left(\frac{\sqrt{7}}{3}h, \frac{\sqrt{7}}{3}h\right) + f\left(\frac{\sqrt{7}}{3}h, -\frac{\sqrt{7}}{3}h\right) \\ & + f\left(-\frac{\sqrt{7}}{3}h, \frac{\sqrt{7}}{3}h\right) + f\left(-\frac{\sqrt{7}}{3}h, -\frac{\sqrt{7}}{3}h\right)\Bigg\} \end{aligned} \tag{5.3}$$

with the main error term

$$\frac{h^6}{6!}\left(-\frac{212}{14175}\nabla^6 + \frac{1612}{4725}\nabla^2\mathfrak{D}^4\right) f_0. \tag{5.4}$$

We may also seek a formula with $a = b$; this turns out to be the "product-Gauss" formula, see §6. Another idea that comes to mind is to choose λ so that a and b are both rational; this does not seem to be possible, although one or other of a, b may take any value we please. Thus

$$a = 1 \quad \text{gives} \quad b^2 = \frac{5}{11} \quad \text{with} \quad A_0 = \frac{64}{225} \qquad A_\alpha = \frac{2}{45} \qquad A_\beta = \frac{121}{900} \tag{5.5}$$

$$\text{and main error term} \quad -\frac{h^6}{6!}\left(\frac{4\nabla^6 + 268\nabla^2\mathfrak{D}^4}{1155}\right) f_0; \tag{5.6}$$

$$b = 1 \quad \text{gives} \quad a^2 = \frac{2}{5} \quad \text{with} \quad A_0 = -\frac{2}{9} \qquad A_\alpha = \frac{5}{18} \qquad A_\beta = \frac{1}{36} \tag{5.7}$$

$$\text{and main error term} \quad +\frac{h^6}{6!}\left(\frac{2\nabla^6 + 344\nabla^2\mathfrak{D}^4}{525}\right) f_0. \tag{5.8}$$

Note the possibly useful opposition of signs in the two error terms.

6. Quadrature over Cube and Hypercube. We now consider general n.

Equation (4.4) gives the following restrictions for $A_0 = \lambda$ if a, b are to be real.

	n	1	2	3	4	5
(6.1)	A_0	$\leqq \frac{4}{9}$	$\leqq \frac{2}{7}$	$\geqq \frac{4}{19}$	$\geqq \frac{1}{6}$	$\geqq \frac{4}{29}$.

Since we have one disposable constant, and $\lambda = 0$ is excluded for $n \geqq 3$, it is not as easy as in §5 to pick out an obviously convenient formula—there remains scope for investigation.

One choice is of interest. This is to put $a = b$. We then find

$$(6.2) \qquad a^2 = b^2 = \frac{3}{5} \qquad A_0 = \frac{25n^2 - 115n + 162}{162}$$
$$A_\alpha = \frac{5(14 - 5n)}{162} \qquad A_\beta = \frac{25}{324}.$$

The points given by $a = \pm\sqrt{3/5}$, $b = \pm\sqrt{3/5}$ are the "Product-Gauss" points with at most two non-zero coordinates. In detail we have

$$(6.31) \qquad n = 1 \qquad A_0 = \frac{4}{9} \qquad A_\alpha = \frac{5}{18} \qquad \left(A_\beta = \frac{25}{324}\right) \qquad \text{3 points}$$

$$(6.32) \qquad n = 2 \qquad A_0 = \frac{16}{81} \qquad A_\alpha = \frac{10}{81} \qquad A_\beta = \frac{25}{324} \qquad \text{9 points}$$

$$(6.33) \qquad n = 3 \qquad A_0 = \frac{7}{27} \qquad A_\alpha = -\frac{5}{162} \qquad A_\beta = \frac{25}{324} \qquad \text{19 points}$$

$$(6.34) \qquad n = 4 \qquad A_0 = \frac{17}{27} \qquad A_\alpha = -\frac{5}{27} \qquad A_\beta = \frac{25}{324} \qquad \text{33 points}$$

We observe that for $n = 1, 2$ these are respectively the Gauss and Product-Gauss formulas. For $n \geqq 3$, although the error term is still of *order* h^6 in $I/(2h)^n$, the number of points is fewer than 3^n, (it is, in fact, $2n^2 + 1$) and the Gauss coefficients are lost.

The leading error term in $I/(2h)^n$ is

$$(6.4) \qquad \frac{4h^6}{25.7!}\left(-\nabla^6 + 3\nabla^2\mathfrak{D}^4 - \frac{893}{6}\mathfrak{Z}^6\right) f_0$$

which does not depend on n. Of course, $\mathfrak{Z}^6 f_0 = 0$ if $n = 1$ or 2.

For comparison we quote the 3-dimensional Product-Gauss formula, which uses the sets of points 0, $\alpha(\sqrt{3/5})$, $\beta(\sqrt{3/5})$ and $\epsilon(\sqrt{3/5})$.

$$(6.5) \quad n = 3 \qquad A_0 = \frac{64}{729} \qquad A_\alpha = \frac{40}{729} \qquad A_\beta = \frac{25}{729} \qquad A_\gamma = \frac{125}{5832} \quad \text{27 points}$$

with leading error term

$$(6.6) \qquad \frac{4h^6}{25.7!}\left(-\nabla^6 + 3\nabla^2\mathfrak{D}^4 - 3\mathfrak{Z}^6\right) f_0 \equiv -\frac{4h^6}{25.7!} S_6 .$$

The form of error term for (6.33) and (6.5) suggests further the complete elimination of $\mathfrak{Z}^6 f_0$ to yield

$$(6.7) \quad n = 3 \qquad A_0 = \frac{430}{5103} \qquad A_\alpha = \frac{289}{5103} \qquad A_\beta = \frac{341}{10206} \qquad A_\gamma = \frac{893}{40824}$$

27 points

with leading error term

$$(6.8) \qquad \frac{4h^6}{25.7!} (-\nabla^6 + 3\nabla^2\mathfrak{D}^4) f_0$$

which vanishes for a harmonic function, and, in fact, also for the example of the next section with $f(x, y, z) = \cos x \cos y \cos z$, since this is a solution of the partial differential equation

$$\nabla^4\phi - 3\mathfrak{D}^4\phi = 0.$$

In this 3-dimensional case, the further relevant error terms are

$$(6.9) \qquad \frac{4h^8}{1125.8!} (-11\nabla^8 - 16\nabla^4\mathfrak{D}^4 + 98\mathfrak{D}^8 + 70\nabla^2\mathfrak{Z}^6) f_0 .$$

7. Numerical Examples.

(i) As in Part 1 [1, §9] we use

$$J = \frac{1}{4} I = \frac{1}{4} \int_{-1}^{1} \int_{-1}^{1} \cos x \cos y \, dxdy \doteqdot 0.70807\ 342$$

for a non-harmonic illustrative example for two-dimensional formulas, and list the integral (J), its error (E), and the estimated correction (C), which is minus the error estimate.

Formula (5.3) with $h = 1$ gives

$$J \doteqdot 0.707362 \quad \text{with} \quad E \doteqdot -0.000711 \quad \text{and} \quad C \doteqdot +0.000782,$$

and with $h = \frac{1}{2}$

$$J \doteqdot 0.70806\ 42 \quad \text{with} \quad E \doteqdot -0.00000\ 92 \quad \text{and} \quad C \doteqdot +0.00000\ 94.$$

The Gauss-product formula (6.32) with $h = 1$ gives

$$J \doteqdot 0.708125 \quad \text{with} \quad E \doteqdot +0.000052 \quad \text{and} \quad C \doteqdot -0.000064,$$

and with $h = \frac{1}{2}$

$$J \doteqdot 0.70807\ 415 \quad \text{with} \quad E \doteqdot +0.00000\ 073 \quad \text{and} \quad C \doteqdot -0.00000\ 076.$$

(ii) For an example with a harmonic integrand, as in Part 1 [1], we use

$$J' = \frac{1}{4} I = \frac{1}{4} \int_{0}^{1.2} \int_{0}^{1.2} \sin x \sinh y \, dxdy \doteqdot 0.12922\ 70591.$$

Formula (5.3) with $h = 1.2$ gives

$$J' \doteqdot 0.12922\ 71000 \quad \text{with} \quad E \doteqdot +0.00000\ 00409 \quad \text{and} \quad C \doteqdot -0.00000\ 00409,$$

while formula (6.32), the Gauss-product formula, also with $h = 1.2$, gives

$$J' \doteqdot 0.12922\ 70778 \quad \text{with} \quad E \doteqdot +0.00000\ 00187 \quad \text{and} \quad C \doteqdot -0.00000\ 00188.$$

In each of these two cases our original error estimate vanishes since the integrand is harmonic, and the error is of order h^8; in fact, they are respectively about $\frac{69128}{91125} h^8\mathfrak{D}^8 f_0/8!$ and $\frac{392}{1145} h^8\mathfrak{D}^8 f_0/8!$.

(iii) For a three-dimensional example, we consider

$$J = \frac{1}{8} I = \frac{1}{8} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \cos x \cos y \cos z \, dxdydz = \sin^3 1 \doteqdot 0.59582\,3237.$$

With $h = 1$, formula (6.33) gives

$$J \doteqdot 0.59987 \quad \text{with} \quad E \doteqdot +0.00405 \quad \text{and} \quad C \doteqdot -0.00473$$

while formula (6.5), the Gauss-product formula, gives

$$J \doteqdot 0.595889 \quad \text{with} \quad E \doteqdot +0.000066 \quad \text{and} \quad C \doteqdot -0.000095$$

and the special formula (6.7) gives

$$J \doteqdot 0.595806 \quad \text{with} \quad E \doteqdot -0.000017 \quad \text{and} \quad C \doteqdot +0.000020$$

The last is exceptional in having an error of order h^8, since, as remarked in §6, $\sin x \sin y \sin z$ satisfies $\nabla^4\phi - 3\mathfrak{D}^4\phi = 0$. It exhibits accidentally what may be expected with a harmonic integrand.

With $h = \frac{1}{2}$ these three formulas give

(6.33) $J \doteqdot 0.595871$ with $E \doteqdot +0.000048$ and $C \doteqdot -0.000050$

(6.5) $J \doteqdot 0.59582\,415$ with $E \doteqdot +0.00000\,091$ and $C \doteqdot -0.00000\,100$

(6.7) $J \doteqdot 0.59582\,319$ with $E \doteqdot -0.00000\,004$ and $C \doteqdot +0.00000\,0054$

(iv) For $n = 4$, we consider

$$J = \frac{1}{16} I = \frac{1}{16} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \cos x \cos y \cos z \cos w \, dxdydzdw$$

$$= \sin^4 1 \doteqdot 0.50136\,80.$$

With $h = 1$, we have

(6.34) $J \doteqdot 0.514$ with $E \doteqdot +0.013$ and $C \doteqdot -0.019$ 33 points

$(\text{Gauss})^4$ $J \doteqdot 0.501441$ with $E \doteqdot +0.000073$ 81 points

and with $h = \frac{1}{2}$, we have

(6.34) $J \doteqdot 0.50153$ with $E \doteqdot +0.00016$ and $C \doteqdot -0.00017$

$(\text{Gauss})^4$ $J \doteqdot 0.50136\,90$ with $E \doteqdot +0.00000\,10$.

These numerical examples illustrate that the extra points used in the Gauss-product formulas may, sometimes at any rate, have a useful effect in reducing error, even though the order of the error, as represented by the power of h in its leading term, remains the same. Comparison of (6.4) and (6.6) shows a much reduced coefficient of $\mathfrak{Z}^6 f_0$ in the latter, and (6.7) is a definite improvement on the Gauss-product in the present case. More investigation is clearly needed of other formulas and of other integrands, particularly those which do not so easily separate into a product of integrals.

Once again I am pleased to acknowledge help and advice received from Dr. J. W. Wrench, Jr. with the numerical examples and also to express my appreciation of

facilities placed at my disposal at the Digital Computer Laboratory in the University of Illinois, at Urbana, Illinois.

The University Mathematical Laboratory
Cambridge, England; and
The Digital Computer Laboratory
University of Illinois, Urbana, Illinois

1. J. C. P. MILLER, "Numerical quadrature over a rectangular domain in two or more dimensions, Part 1.," *Math. Comp.* (*MTAC*), v. 14, 1960, p. 13–20.
2. W. G. BICKLEY, "Finite difference formulae for the square lattice," *Quart. Jn. Mech. and Appl. Math.*, v. 1, 1948, p. 35–42.
3. F. N. DAVID & M. G. KENDALL, "Tables of symmetric functions, Parts II and III," *Biometrika*, v. 38, 1951, p. 435–462. See page 439.