A Note on the Solution of Quartic Equations

By Herbert E. Salzer

For any quartic equation with real coefficients,

\[ X^4 + AX^3 + BX^2 + CX + D = 0, \]

the following condensation of the customary algebraic solution is recommended as quickest and easiest for the computer to follow (no mental effort required). It works in every exceptional case.

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Denote the four roots of (1), by \( X_1, X_2, X_3, \) and \( X_4 \). With the aid of (1), solve the "resolvent cubic equation" \( ax^3 + bx^2 + cx + d = 0 \) for the real root \( x_1 \) only, where

\[ (2) \quad a = 1, \quad b = -B, \quad c = AC - 4D, \quad \text{and} \quad d = D(4B - A^2) - C^2. \]

Find

\[ (3) \quad m = +\sqrt{\frac{1}{4}A^2 - B + x_1}, \quad n = \frac{Ax_1 - 2C}{4m}. \]

If \( m = 0 \), take \( n = \sqrt{\frac{1}{4}x_1^2 - D} \) and proceed according to the following Case I or Case II, depending upon whether \( m \) is real or imaginary.

**Case I:** If \( m \) is real, let \( \left(\frac{1}{4}A^2 - x_1 - B\right) = A, 4n - Am = \beta, \sqrt{\alpha + \beta} = \gamma, \sqrt{\alpha - \beta} = \delta \), and finally

\[ X_1 = \frac{-\frac{1}{2}A + m + \gamma}{2}, \quad X_2 = \frac{-\frac{1}{2}A - m + \delta}{2}, \]

\[ X_3 = \frac{-\frac{1}{2}A - m - \gamma}{2}, \quad \text{and} \quad X_4 = \frac{-\frac{1}{2}A - m - \delta}{2}. \]

**Case II:** If \( m \) is imaginary, say \( m = im' \), then \( n \) is also imaginary, say \( n = in' \). Let

\[ \left(\frac{1}{4}A^2 - x_1 - B\right) = A, \quad 4n' - Am' = \beta, \quad +\sqrt{\alpha^2 + \beta^2} = \rho, \]

\[ \sqrt{\frac{\alpha + \rho}{2}} = \gamma, \quad \frac{\beta}{2\gamma} = \delta, \]

and finally

\[ (4I) \quad \begin{cases} X_1 = \frac{-\frac{1}{2}A + \gamma + i(m' + \delta)}{2}, \\ X_3 = \frac{-\frac{1}{2}A - \gamma + i(m' - \delta)}{2} \end{cases}, \quad X_2 = \bar{X}_1, \quad \text{the complex conjugate of} \ X_1, \]

\[ X_4 = \bar{X}_3, \quad \text{the complex conjugate of} \ X_3. \]

If \( \gamma = 0 \), we must have \( \alpha = -\alpha', \alpha' \geq 0 \), and formula (4II) still holds provided that in it we replace \( \delta \) by \( +\sqrt{\alpha'} \).

As an example consider the quartic equation \( X^4 + X^3 + X^2 + X + 1 = 0 \), where \( A = B = C = D = 1 \), so that from (2) the resolvent cubic equation is \( x^3 - x^2 - 3x + 2 = 0 \). From (1) we find \( x_1 = 0.61803 \ 400 \). From (3), \( m = +\sqrt{-0.13196 \ 600} = +0.36327 \ 125i \), so that \( m' = +0.36327 \ 125 \). Then \( n = \frac{-1.38196 \ 600}{1.45308 \ 500i} = +0.95105 \ 655i \), so that \( n' = +0.95105 \ 655 \). Proceeding according to Case II, \( \alpha = -1.11803 \ 400, \beta = 3.44095 \ 495, \rho = 3.61803 \ 41, \gamma = 1.11803 \ 40 \) and \( \delta = 1.53884 \ 18 \). Then from (4II) we obtain \( X_1 = 0.30901 \ 70 + 0.95105 \ 65i, \ X_2 = \bar{X}_1 = 0.30901 \ 70 - 0.95105 \ 65i, \ X_3 = -0.80901 \ 70 - 0.58778 \ 53i \) and
A Conjugate Factor Method for the Solution of a Cubic

By D. A. Maguía

1. Introduction. This paper gives a simple method for computing the real roots of the reduced cubic equation with real coefficients,

\[ x^3 + Ax + B = 0, \]

having roots a, b, c. We assume a to be real, since every cubic equation has at least one real root.

The method consists in factoring B, and setting one factor equal to ±\( \sqrt{m} \), the other n. For all pairs m, n such that \( m + n = -A \), ±\( \sqrt{m} \) is a root. If no such pair exists, a method of interpolation is shown.

2. Proof of Method. The reduced cubic equation (1) can be transformed, by using the relations between the roots and coefficients, into a complete cubic,

\[ p^3 + 6Ap^2 + 9A^2p + 4A^3 + 27B^2 = 0, \]

where

\[ p = (-3a^2 - 4A). \]

Equation (2) can be written in the form:

\[ (p + A)^2(-p - 4A) = 27B^2 \]

or

\[ \frac{(p + A)}{3} \sqrt{\frac{(-p - 4A)}{3}} = ±B. \]

Let

\[ m = \frac{-p - 4A}{3} \quad \text{and} \quad n = \frac{p + A}{3} \]

and

\[ m + n = -A. \]

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