Note on $\int_0^\infty e^{-x} J_0\left(\frac{\eta x}{\xi}\right) J_1\left(\frac{x}{\xi}\right) x^{-n}\, dx$

By Henry E. Fettis

In a recent paper this integral has been calculated numerically by Weeg [1] for the cases $n = 0, 1$. The following analytic expression for this integral when $n = 0$ in terms of elliptic integrals is given by Byrd and Friedman ([2], formula 563.01):

$$\begin{aligned} \tilde{I}(p, r, s) &= \int_0^\infty e^{-pt} J_0(rt)\, J_1(st)\, dt = \frac{1}{s}\,[1 - \Lambda_0(\beta, k)] \\ &= \frac{1}{s}\left[1 - \frac{2}{\pi}\,\{E(k)\, F(\beta, k') + K(k)\,[E(\beta, k') - F(\beta, k')]\}\right], \end{aligned} \tag{1}$$

with

$$k^2 = \frac{[N - p^2 - r^2 + s^2]\,[N - p^2 + r^2 - s^2]}{[N + p^2 + r^2 - s^2]\,[N + p^2 - r^2 + s^2]},$$

$$\sin^2 \beta = \frac{1}{2}\left[1 + \frac{p^2 + r^2 + s^2}{N}\right],$$

$$N = \sqrt{(p^2 + r^2 + s^2)^2 + 4p^2 s^2}.$$

$\Lambda_0(\beta, k)$ is known as Heuman's Lambda Function.

Calculations based on the above expressions do not agree with the ones made numerically in [1], and since the latter values have been verified by an independent relationship given in Weeg's paper,* the author was led to attempt to verify the above analytic expression independently. A closer examination of Eq. (1) leads one to question its validity by consideration of various special cases for which simpler expressions can be found. For example, it is known ([3], Chapter III, Art. 7) that

$$\tilde{I}(p, o, s) = \frac{\sqrt{p^2 + s^2} - p}{s\sqrt{p^2 + s^2}}, \tag{2}$$

whereas (1) fails to reduce to this simple form when $r = 0$. It is also possible to demonstrate directly the relations:

$$s\tilde{I}(p, r, s) + r\,\tilde{I}(p, s, r) = 1 - \frac{2pK(k)}{\pi\sqrt{p^2 + (r + s)^2}} = 1 - \frac{p}{\pi\sqrt{rs}}\, Q_{-\frac{1}{2}}(x) \tag{3}$$

where

$$k^2 = 4rs/[p^2 + (r + s)^2], \qquad x = (p^2 + r^2 + s^2)/2rs$$

and

$$\begin{aligned} \tilde{I}(0, r, s) &= 1/s, & r &< s, \\ &= 1/2s, & r &= s, \\ &= 0, & r &> s, \end{aligned} \tag{4}$$

Received December 8, 1959; revised April 28, 1960.

* Eq. 2 of [1] should read $I(\eta, \xi, 0) + I(\eta, \xi, 1) = \frac{1}{\pi\sqrt{\eta}}\,[Q_{-\frac{1}{2}}(x) - \eta Q_{\frac{1}{2}}(x)]$.

neither of which can be obtained as a special case of (1). In fact, for $p = 0$, the right side of (1) reduces identically to zero, irrespective of the relative magnitudes of r and s. A simpler expression for this integral has been derived by the present author in terms of complete elliptic integrals of the first and third kinds, by making use of the representation ([3], Chap. III, Art. 6):

$$\pi J_0(rt)J_1(st) = \int_0^\pi J_0[t\sqrt{r^2 + s^2 - 2rs\cos\phi}]\, d\phi. \tag{5}$$

Differentiation of (5) gives

$$\pi J_0(rt)J_1(st) = \int_0^\pi J_1[t\sqrt{r^2 + s^2 - 2rs\cos\phi}]\, \frac{(s - r\cos\phi)\, d\phi}{\sqrt{r^2 + s^2 - 2rs\cos\phi}}. \tag{6}$$

Thus

$$\begin{aligned} \pi \int_0^\infty e^{-pt}J_0(rt)J_1(st)\, dt &= \int_0^\infty e^{-pt}\, dt \int_0^\pi J_1[t\sqrt{r^2 + s^2 - 2rs\cos\phi}] \\ &\qquad \cdot \frac{(s - r\cos\phi)\, d\phi}{\sqrt{r^2 + s^2 - 2rs\cos\phi}} \\ &= \int_0^\pi \frac{(s - r\cos\phi)\, d\phi}{\sqrt{r^2 + s^2 - 2rs\cos\phi}} \int_0^\infty e^{-pt}J_1[t\sqrt{r^2 + s^2 - 2rs\cos\phi}]\, dt. \end{aligned} \tag{7}$$

Carrying out the integration with respect to "t" gives

$$\begin{aligned} \pi \int_0^\infty e^{-pt}J_0(rt)J_1(st)\, dt &= \int_0^\pi \frac{(s - r\cos\phi)}{r^2 + s^2 - 2rs\cos\phi} \\ &\qquad \left[1 - \frac{p}{\sqrt{p^2 + r^2 + s^2 - 2rs\cos\phi}}\right] d\phi. \end{aligned} \tag{8}$$

The first integral on the right side of (8) is known ([4], formula 860.2):

$$\begin{aligned} \int_0^\pi \frac{[s - r\cos\phi]\, d\phi}{r^2 + s^2 - 2rs\cos\phi} &= \pi/s, & r < s, \\ &= \pi/2s, & r = s, \\ &= 0, & r > s, \end{aligned} \tag{9}$$

and the second can be converted to elliptic integrals in standard form by substituting $\phi = \pi - 2\theta$ and by rearranging, thus,

$$\begin{aligned} \int_0^\pi \frac{(s - r\cos\phi)\, d\phi}{(r^2 + s^2 - 2rs\cos\phi)\sqrt{p^2 + r^2 + s^2 - 2rs\cos\phi}} &= \frac{1}{s\sqrt{p^2 + (r + s)^2}} \\ &\quad \cdot \left\{\frac{s - r}{s + r}\int_0^{\pi/2} \frac{d\theta}{(1 - \alpha^2\sin^2\theta)\sqrt{1 - k^2\sin^2\theta}} + \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2\sin^2\theta}}\right\} \\ &= \frac{1}{s\sqrt{p^2 + (r + s)^2}}\left\{\frac{s - r}{s + r}\Pi(\alpha, k) + K(k)\right\}, \end{aligned} \tag{10}$$

where

$$\alpha^2 = \frac{4rs}{(r+s)^2},$$

$$k^2 = \frac{4rs}{p^2 + (r+s)^2},$$

and $K(k)$ and $\prod(\alpha, k)$ are complete elliptic integrals of the first and third kinds, respectively. An alternative form which expresses the result in terms of Heuman's Lambda Function can be obtained from formula 413.01 of [2]:

$$\begin{aligned} p \int_0^\pi \frac{(s - r\cos\phi)\, d\phi}{(r^2 + s^2 - 2rs\cos\phi)\sqrt{p^2 + r^2 + s^2 - 2rs\cos\phi}} \\ = \frac{1}{s}\left\{\frac{\pi}{2}\frac{|s-r|}{(s-r)}\Lambda_0(\beta, k) + \frac{pK(k)}{\sqrt{p^2 + (r+s)^2}}\right\}, \end{aligned} \tag{11}$$

with

$$\sin\beta = \frac{p}{\sqrt{p^2 + (r-s)^2}}.$$

The final result is

$$\begin{aligned} \int_0^\infty e^{-pt} J_0(rt) J_1(st)\, dt &= \frac{1}{s}\left[1 - \frac{1}{2}\Lambda_0(\beta, k) - \frac{pK(k)}{\pi\sqrt{p^2 + (r+s)^2}}\right], && r < s, \\ &= \frac{1}{s}\left[-\frac{1}{2}\frac{pK(k)}{\pi\sqrt{p^2 + 4r^2}}\right] && r = s, \\ &= \frac{1}{s}\left[\frac{1}{2}\Lambda_0(\beta, k) - \frac{pK(k)}{\pi\sqrt{p^2 + (r+s)^2}}\right], && r > s, \end{aligned} \tag{12}$$

where

$$k^2 = \frac{4rs}{p^2 + (r+s)^2}, \qquad \sin\beta = \frac{p}{\sqrt{p + (r+s)^2}}.$$

Equation (12) when $r = s$, incidentally, gives a particularly simple expression for Weeg's integral in those cases where $\eta = 1$.

Applied Mathematics Research Laboratory
Wright-Patterson Air Force Base, Ohio

1. Gerald P. Weeg, "Numerical Integration of $\int_0^\infty e^{-x} J_0\left(\frac{\eta x}{\xi}\right) J_1\left(\frac{x}{\xi}\right) x^{-n}\, dx$," *MTAC*, v. 13, 1959, p. 312–313.
2. Paul F. Byrd & Morris D. Friedman, *Handbook of Elliptic Integrals for Engineers and Physicists*, Springer, Berlin, 1954.
3. W. Magnus & F. Oberhettinger, *Formeln und Satze fur die Speziellen Funktionen der Mathematischen Physik*, 2nd Edition, Springer, Berlin, 1948.
4. H. B. Dwight, *Tables of Integrals and Other Mathematical Data*, Third Edition, Macmillan, New York, 1957.