New Tables of Howland's and Related Integrals

By C. W. Nelson

1. Introduction. In an earlier paper by C. B. Ling and the present author [1], values of the four integrals,

$$\text{(1)} \qquad \begin{matrix} I_k \\ I_k^* \end{matrix} = \frac{1}{2(k!)} \int_0^\infty \frac{w^k\, dw}{\sinh w \pm w} = \frac{2^k}{k!} \int_0^\infty \frac{x^k\, dx}{\sinh 2x \pm 2x}, \qquad \begin{matrix} (k \geqq 1) \\ (k \geqq 3) \end{matrix}$$

$$\text{(2)} \qquad \begin{matrix} II_k \\ II_k^* \end{matrix} = \frac{1}{2(k!)} \int_0^\infty \frac{w^k e^{-w}\, dw}{\sinh w \pm w} = \frac{2^k}{k!} \int_0^\infty \frac{x^k e^{-2x}\, dx}{\sinh 2x \pm 2x}, \qquad \begin{matrix} (k \geqq 1) \\ (k \geqq 3) \end{matrix}$$

were tabulated to 6D. These four integrals are called Howland's integrals because they first appeared in Howland's papers dealing with the stresses in a perforated strip [2, 3]. In another paper, C. B. Ling has reproduced the 6D tables of Howland's integrals from [1] and added tables of other integrals derived from them [4].

2. Values of Howland's Integrals to 9D. For most purposes, 6D tables of Howland's integrals are adequate. This is usually true in plane-stress problems of elasticity involving straight bars of rectangular cross-section, for example. However, the author has found that even in such problems the stresses are more easily evaluated in all regions of interest in the bar without resorting to contour integration if 9D rather than 6D tables of Howland's integrals are available. The 9D values of Howland's integrals given in Table 1 were computed by the author and have been checked by using the following checking formulas [1]:

$$\text{(3)} \qquad \begin{cases} \sum_{k=0}^{\infty} (1 - I_{2k+1}) = I_1 - \frac{1}{4}, & \sum_{k=1}^{\infty} 2k(1 - I_{2k}) = 2I_2 - \frac{1}{8}, \\ \sum_{k=1}^{\infty} (I_{2k+1}^* - 1) = \frac{5}{4}, & \sum_{k=2}^{\infty} 2k(I_{2k}^* - 1) = \frac{17}{8}, \\ \sum_{k=0}^{\infty} II_{2k+1} = \frac{1}{2} - II_1, & \sum_{k=1}^{\infty} 2kII_{2k} = \frac{1}{2} - 2II_2, \\ \sum_{k=1}^{\infty} II_{2k+1}^* = \frac{1}{2}, & \sum_{k=2}^{\infty} 2kII_{2k}^* = \frac{1}{2}. \end{cases}$$

The method of computing the integrals is explained in [1] and this explanation will not be repeated here.

3. Values of I_{2k}, I_{2k}^*, and $\begin{bmatrix} 2 \\ n \end{bmatrix}$ to 18D. Recently the author has encountered a need for still more precise values of some of the Howland's integrals. This occurs in the Hankel transform or Fourier-Bessel integral solutions [5, 6, 7] for problems of elasticity involving axially symmetric loading of a thick plate of infinite radius,

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TABLE 1

Values of Howland's Integrals to 9D

k	I_k	I_k^*	II_k	II_k^*
1	0.768 574 538		0.220 119 581	
2	0.767 847 439		0.087 927 235	
3	0.827 710 296	2.038 710 667	0.043 347 862	0.460 713 719
4	0.883 506 807	1.353 294 115	0.022 583 004	0.099 315 532
5	0.925 475 998	1.156 864 366	0.011 923 473	0.032 412 690
6	0.954 191 562	1.076 729 764	0.006 287 972	0.012 616 908
7	0.972 698 993	1.039 251 312	0.003 295 013	0.005 391 117
8	0.984 124 180	1.020 537 600	0.001 713 298	0.002 433 000
9	0.990 949 179	1.010 870 147	0.000 884 147	0.001 135 997
10	0.994 922 440	1.005 784 842	0.000 453 215	0.000 542 202
11	0.997 188 575	1.003 084 774	0.000 231 010	0.000 262 590
12	0.998 459 958	1.001 644 976	0.000 117 209	0.000 128 431
13	0.999 163 823	1.000 876 180	0.000 059 254	0.000 063 239
14	0.999 549 306	1.000 465 841	0.000 029 870	0.000 031 283
15	0.999 758 559	1.000 247 139	0.000 015 025	0.000 015 525
16	0.999 871 321	1.000 130 809	0.000 007 546	0.000 007 722
17	0.999 931 719	1.000 069 073	0.000 003 785	0.000 003 847
18	0.999 963 903	1.000 036 390	0.000 001 897	0.000 001 919
19	0.999 980 979	1.000 019 128	0.000 000 950	0.000 000 958
20	0.999 990 006	1.000 010 034	0.000 000 476	0.000 000 478
21	0.999 994 762	1.000 005 252	0.000 000 238	0.000 000 239
22	0.999 997 261	1.000 002 744	0.000 000 119	0.000 000 119
23	0.999 998 570	1.000 001 431	0.000 000 060	0.000 000 060
24	0.999 999 255	1.000 000 745	0.000 000 030	0.000 000 030
25	0.999 999 613	1.000 000 388	0.000 000 015	0.000 000 015
26	0.999 999 799	1.000 000 201	0.000 000 007	0.000 000 007
27	0.999 999 896	1.000 000 104	0.000 000 004	0.000 000 004
28	0.999 999 946	1.000 000 054	0.000 000 002	0.000 000 002
29	0.999 999 972	1.000 000 028	0.000 000 001	0.000 000 001
30	0.999 999 986	1.000 000 014	0.000 000 000	0.000 000 000
31	0.999 999 993	1.000 000 007		
32	0.999 999 996	1.000 000 004		
33	0.999 999 998	1.000 000 002		
34	0.999 999 999	1.000 000 001		
35	0.999 999 999	1.000 000 001		
36	1.000 000 000	1.000 000 000		

where integrals such as

$$F(\rho, \alpha) = \int_0^\infty \frac{\left(\sinh \frac{x}{2} + \frac{x}{2} \cosh \frac{x}{2}\right) J_0(\rho x) J_1(\alpha x)\, dx}{\alpha(\sinh x \pm x)} \tag{4}$$

must be evaluated. In equation (4), $J_0(\rho x)$ and $J_1(\alpha x)$ are Bessel functions of order zero and unity, respectively. Sneddon [5, 6] describes an approximate method of evaluating such integrals. He and his associates appear to have treated only cases where the double sign in the denominator has the plus value, but, presumably, their method could be extended to cases where the double sign takes the minus value. Sneddon's approximate method does not involve the use of Howland's integrals.

For a limiting case of the integral in equation (4), namely $\alpha = 0$ and double sign taken as plus, Sneddon found that his approximate method gave errors of the order of 1% for $\rho = 0$ to $\frac{1}{2}$. However, the author found that Sneddon's approximate method applied to the integral in equation (4) gave an error of about 7% for $\rho = 0$, $\alpha = 1$, and double sign taken plus. Comparable errors in certain other integrals were obtained by the approximate method so that, in a certain thermal stress problem (radius of heated region equal to thickness of plate), the errors in normal stresses on the axis of the plate ranged up to 21.8% of the greatest normal stress occurring anywhere on the axis or 9.3% of the greatest normal stress occurring anywhere in the plate.

The author has felt that it is desirable to be able to obtain the values of integrals such as those considered in the foregoing two paragraphs with an accuracy better than that obtainable by Sneddon's approximate method. The more accurate values may at least be used to check a few values obtained by the approximate method in cases of doubt, and the availability of accurate values of the integrals occurring in thick-plate problems may even be found essential in the extension of the Hankel transform method to problems not yet considered. The author first attempted to evaluate the stresses in a thick-plate problem [7] with the aid of the 9D values of Howland's given in Table 1 and found that only a very limited range of values of the parameters ρ and α in integrals such as that of equation (4) could be successfully dealt with. For this reason, the 18D values of two of the four Howland's integrals, I_k and I_k^*, for even integral k, were computed. These are given in Table 2 and discussed further in the following. Green and Willmore [9] encountered integrals similar to those discussed in the present paper except that theirs contained only one Bessel function. They used Howland's integrals for the simpler integrals they had to evaluate and Sneddon's approximate method for the rest. Apparently the Fourier-Bessel integral method was first applied to thick plate problems by Lamb [10], who made no attempt to evaluate the integrals. Dougall [11] proposed that integrals such as the example in equation (4), but containing only one Bessel function, be evaluated by contour integration, but apparently no one has cared to perform the task involved which appears to be considerable even though the roots of $\sinh z \pm z = 0$ are known.

If the integral taken as an example in equation (4) is expressed as a series of Howland integrals by expanding the numerator of the integrand in powers of x, only even powers of x will occur in this expansion. Thus the integral can be evaluated if adequate values of I_k or I_k^* are available for even integral k, and if the series converges or can somehow be summed. From a consideration of various loading conditions of a thick plate, it appears to the author that as long as the axially-symmetric loading of the plate consists only of boundary loads on the plane faces (i.e. no body forces), all other integrals needed can also be evaluated from tables of I_k and I_k^* for even integral k. Accordingly, the values of I_k and I_k^* given to 18D in Table 2 are believed to be the only Howland integrals needed in a fairly broad class of thick-plate problems. Again the reader is referred to [1] for the general method of computing the integrals in Table 2. However, it should be stated here that the method depends on the basic equation

TABLE 2

Values of the Howland Integrals I_{2k} and I^*_{2k} to 18D

k	I_k	I_k^*
2	0.767 847 439 133 919 047	
4	0.883 506 806 508 692 590	1.353 294 115 170 484 009
6	0.954 191 561 826 139 064	1.076 729 763 674 217 127
8	0.984 124 180 148 424 614	1.020 537 600 064 959 340
10	0.994 922 439 853 445 374	1.005 784 842 224 523 505
12	0.998 459 957 947 832 383	1.001 644 976 253 520 390
14	0.999 549 305 562 626 177	1.000 465 841 012 174 418
16	0.999 871 321 426 371 778	1.000 130 808 809 455 335
18	0.999 963 903 069 165 323	1.000 036 389 774 380 024
20	0.999 990 005 851 207 354	1.000 010 033 628 030 387
22	0.999 997 260 778 826 414	1.000 002 744 455 935 026
24	0.999 999 255 282 148 058	1.000 000 745 402 213 094
26	0.999 999 798 878 365 743	1.000 000 201 210 025 403
28	0.999 999 945 988 927 609	1.000 000 054 022 369 865
30	0.999 999 985 565 214 614	1.000 000 014 436 216 216
32	0.999 999 996 158 384 196	1.000 000 003 841 795 571
34	0.999 999 998 981 377 142	1.000 000 001 018 645 284
36	0.999 999 999 730 790 958	1.000 000 000 269 211 822
38	0.999 999 999 929 059 585	1.000 000 000 070 940 758
40	0.999 999 999 981 355 380	1.000 000 000 018 644 662
42	0.999 999 999 995 111 469	1.000 000 000 004 888 537
44	0.999 999 999 998 721 023	1.000 000 000 001 278 977
46	0.999 999 999 999 666 045	1.000 000 000 000 333 955
48	0.999 999 999 999 912 959	1.000 000 000 000 087 041
50	0.999 999 999 999 977 351	1.000 000 000 000 022 649
52	0.999 999 999 999 994 116	1.000 000 000 000 005 884
54	0.999 999 999 999 998 473	1.000 000 000 000 001 527
56	0.999 999 999 999 999 604	1.000 000 000 000 000 396
58	0.999 999 999 999 999 898	1.000 000 000 000 000 102
60	0.999 999 999 999 999 974	1.000 000 000 000 000 026
62	0.999 999 999 999 999 993	1.000 000 000 000 000 007
64	0.999 999 999 999 999 998	1.000 000 000 000 000 002
66	1.000 000 000 000 000 000	1.000 000 000 000 000 000

(5)
$$\begin{matrix} I_k \\ I_k^* \end{matrix} = \sum_{n=1}^{\infty} (\mp 1)^{n+1} \begin{bmatrix} k \\ n \end{bmatrix}$$

where

(6)
$$\begin{bmatrix} k \\ n \end{bmatrix} = \frac{1}{2(k!)} \int_0^{\infty} \frac{x^{n+k-1}\, dx}{\sinh^n x}$$

and that the values in Table 2 have been checked by using the second and fourth of equations (3). When these checks were first applied, errors were indicated. To eliminate these errors, they had to be located and this was facilitated by applying another checking formula†

† For $n = 1$, use

(7a)
$$\sum_{m=1}^{\infty} \left(\begin{bmatrix} 2m \\ 1 \end{bmatrix} - 1 \right) = \frac{3}{4} - \ln 2$$

$$\sum_{m=1}^{\infty} \frac{2m}{2m+1} \begin{bmatrix} 2m \\ n \end{bmatrix} = \frac{1}{2(n-1)}, \qquad n = 2, 3, 4, \cdots \tag{7}$$

Equation (7) is obtained by writing the series of integrals represented by the left member of the equation, interchanging the order of summation and integration, performing the summation which requires merely recognition of the Maclaurin expansion of $\cosh x - \dfrac{\sinh x}{x}$ and then performing the evaluation of the resulting elementary integral as follows:

$$\frac{1}{2} \int_0^{\infty} \frac{x^{n-2}(x \cosh x - \sinh x)\, dx}{\sinh^n x} = -\frac{x^{n-1}}{2(n-1)\sinh^{n-1} x}\Bigg|_0^{\infty} = \frac{1}{2(n-1)}. \tag{8}$$

TABLE 3

Values of $\begin{bmatrix} 2 \\ n \end{bmatrix}$ *to* 18D

n	$\begin{bmatrix} 2 \\ n \end{bmatrix}$	n	$\begin{bmatrix} 2 \\ n \end{bmatrix}$
1	1.051 799 790 264 645 000	21	0.036 396 239 976 639 766
2	0.450 771 338 684 847 857	22	0.034 712 213 240 700 944
3	0.283 656 164 817 032 302	23	0.033 177 094 523 417 174
4	0.206 411 435 020 280 449	24	0.031 771 973 852 978 737
5	0.162 091 502 769 567 468	25	0.030 481 012 454 106 163
6	0.133 389 997 047 394 398	26	0.029 290 843 691 027 776
7	0.113 302 756 992 243 931	27	0.028 190 108 942 707 765
8	0.098 463 515 074 533 508	28	0.027 169 094 240 899 480
9	0.087 055 830 293 630 464	29	0.026 219 443 043 962 235
10	0.078 014 012 373 164 903	30	0.025 333 927 165 280 539
11	0.070 671 880 431 666 952	31	0.024 506 262 569 448 302
12	0.064 591 720 775 032 384	32	0.023 730 960 108 696 094
13	0.059 474 138 718 298 180	33	0.023 003 203 705 225 661
14	0.055 107 451 486 079 519	34	0.022 318 750 267 406 497
15	0.051 337 774 971 088 869	35	0.021 673 846 947 047 716
16	0.048 050 565 748 631 057	36	0.021 065 162 331 082 628
17	0.045 158 809 284 589 149	37	0.020 489 728 904 903 199
18	0.042 595 221 342 283 185	38	0.019 944 894 690 604 315
19	0.040 306 957 521 379 715	39	0.019 428 282 397 575 541
20	0.038 251 937 970 666 201	40	0.018 937 754 758 470 797

TABLE 4

Values of $\int_0^{\infty} \dfrac{xJ_1(\rho x)\, dx}{\sinh x + x}$

ρ	$\int_0^{\infty} \frac{xJ_1(\rho x)\, dx}{\sinh x + x}$
0	0
1	0.471 282
2	0.250 403
3	0.166 664
4	0.125 000
5	0.100 000

The values of $\begin{bmatrix} 2 \\ n \end{bmatrix}$ are recorded in Table 3, since they are the starting point for obtaining I_k and I_k^*, and because they may be needed as in the example leading to Table 4 in the following. It was not felt necessary to include values of $\begin{bmatrix} m \\ n \end{bmatrix}$ for $m = 4, 6, 8, \cdots$ because, at worst, only a few such values might be needed by a reader, and these can be computed from Table 3 by using equation (16) given in the following.

4. Application of Howland's Integrals in Evaluating Related Integrals. Procedures for evaluating integrals such as those in equation (4) for moderate values of ρ and α, say for ρ and α each less than unity, have been amply discussed in connection with similar integrals treated in references [2], [3], [4] and [7]. If either ρ or α (or both) should be considerably greater than unity in integrals such as those of equation (4), some special study of the integral in question is usually required and an effort should be made to determine some simple expression which the value of the integral approaches asymptotically for large values of the parameter or parameters. Usually this can be done by considering the physical problem in which the integral arose and by examining the known approximate solution for a limiting case of the problem. For example, suppose the integral $\int_0^\infty \frac{xJ_1(\rho x)\,dx}{\sinh x + x}$ is to be evaluated for a series of values of ρ covering the range $0 \leqq \rho < \infty$. The integral arises in a three-dimensional elasticity problem involving axially symmetric loads on a thick plate. By considering a limiting case of the physical problem or by other methods, it can be shown that

$$\int_0^\infty \frac{xJ_1(\rho x)\,dx}{\sinh x + x} \approx \frac{1}{2\rho}, \qquad \text{for large } \rho. \tag{9}$$

Thus the range of values of ρ which must be considered is reduced to the range from zero to the lowest value at which equation (9) gives the result with sufficient accuracy. The integral in question may be expressed as a series of integrals in the form

$$\int_0^\infty \frac{xJ_1(\rho x)\,dx}{\sinh x + x} = \sum_{n=1}^{\infty} (-1)^{n+1} \int_0^\infty \frac{x^n J_1(\rho x)\,dx}{\sinh^n x} \tag{10}$$

where the integrals in the right member may be evaluated by contour integration following the method described for a similar integral in [8]. For example, the first two integrals required are

$$\int_0^\infty \frac{xJ_1(\rho x)\,dx}{\sinh x} = \frac{1}{\rho} - 2\pi[\kappa_1(\pi\rho) - 2\kappa_1(2\pi\rho) + 3\kappa_1(3\pi\rho) - \cdots] \tag{11}$$

$$\int_0^\infty \frac{x^2 J_1(\rho x)\,dx}{\sinh^2 x} = \frac{1}{\rho} + 2\pi[\kappa_1(\pi\rho) + 2\kappa_1(2\pi\rho) + 3\kappa_1(3\pi\rho) + \cdots] - 2\pi^2\rho[\kappa_0(\pi\rho) + 4\kappa_0(2\pi\rho) + 9\kappa_0(3\pi\rho) + \cdots] \tag{12}$$

where $\kappa_0(x)$ and $\kappa_1(x)$ are modified Bessel functions of the second kind of order zero and one respectively.

By evaluating the first four integrals and applying an Euler transformation to the resulting alternating series in equation (10), the values of the left member of equation (10) for $\rho = 4$ and $\rho = 5$ were found to be as given in Table 4. The entries in Table 4 were completed as follows. For $\rho = 0$ and $\rho = 1$, the integral was evaluated from

$$\int_0^\infty \frac{xJ_1(\rho x)\,dx}{\sinh x + x} = 2\rho I_2 - 2\cdot\frac{3}{2}\rho^3 I_4 + 2\cdot\frac{3\cdot 5}{2\cdot 4}\rho^5 I_6 - \cdots \tag{13}$$

For $\rho = 2$ and $\rho = 3$, a Kummer transformation was applied to equation (13) so as to obtain

$$\int_0^\infty \frac{xJ_1(\rho x)\,dx}{\sinh x + x} = \sum_{n=1}^{4} (-1)^{n+1} \int_0^\infty \frac{x^n J_1(\rho x)\,dx}{\sinh^n x} + \text{Remainder Series} \tag{14}$$

where

$$\begin{aligned} \text{Remainder Series} &= 2\rho\left\{I_2 - \sum_{n=1}^{4} (-1)^{n+1}\begin{bmatrix}2\\ n\end{bmatrix}\right\} \\ &\quad - 2\cdot\tfrac{3}{2}\rho^3\left\{I_4 - \sum_{n=1}^{4} (-1)^{n+1}\begin{bmatrix}4\\ n\end{bmatrix}\right\} + \cdots \end{aligned} \tag{15}$$

The remainder series converges for $\rho < 5$ but the convergence is, of course, very slow as ρ approaches 5 and, even for $\rho = 2$ and $\rho = 3$, it is best to speed the convergence by applying an Euler transformation. To evaluate the series in equation (15), it is necessary to use Table 2, Table 3, and the equation

$$\begin{aligned} \begin{bmatrix}m\\ n\end{bmatrix} = \frac{1}{(m-1)m}\Bigg\{&\frac{(n+m-2)(n+m-1)}{n^2}\begin{bmatrix}m-2\\ n\end{bmatrix} \\ &- \frac{(n+1)}{n}\begin{bmatrix}m-2\\ n+2\end{bmatrix}\Bigg\} \end{aligned} \tag{16}$$

which is equivalent to the first of equations (4) in reference [1].

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Applied Physics Laboratory
GPD Development Laboratory
International Business Machines Corporation
Endicott, New York

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