

Evaluation of the Incomplete Gamma Function of Imaginary Argument by Chebyshev Polynomials

By Richard Barakat

During the course of some work on the diffraction theory of aberrations it was necessary to evaluate numerically the incomplete gamma function of imaginary argument $\gamma(\nu, ix)$ for certain values of the parameter ν . These integrals are special cases of the confluent hypergeometric function, and in the standard notation [1]

$$(1) \quad \begin{aligned} \gamma(\nu, ix) &= \int_0^{ix} e^{-t} t^{\nu-1} dt & \text{Re}(\nu) > 0 \\ &= (ix)^\nu \nu^{-1} {}_1F_1(\nu, 1 + \nu, -ix) \end{aligned}$$

where ${}_1F_1$ is the confluent hypergeometric function. Only the case of ν real is considered.

When ν is an integer, ${}_1F_1$ is simply a polynomial in x . It is also possible to evaluate (1) in terms of the Fresnel integrals when $(\nu - 1)$ is a half-integer. For $\nu = \frac{1}{2}$, (1) is proportional to the Fresnel integral. For other half-integer values, we can use the recurrence relation

$$(2) \quad \gamma(1 + \nu, ix) = \nu \gamma(\nu, ix) - (ix)^\nu e^{-ix}$$

to generate the necessary formulas. An alternate procedure utilizes the Lommel functions of two variables [2]; unfortunately, the Lommel functions have not been extensively tabulated.

The remaining values of ν can be treated by a Taylor series expansion for small values of x and by asymptotic developments for large values of x . The main difficulty is the intermediate range where x is neither large nor small. A possible method is to use Nielsen's [3] representation

$$(3) \quad \gamma(\nu, ix + iy) - \gamma(\nu, ix) = e^{-ix} (ix)^{\nu-1} \sum_{n=0}^{\infty} (1 - \nu)_n (-ix)^{-n} [1 - e^{-iy} e_n(iy)]$$

where $e_n(iy)$ is the truncated exponential series

$$(4) \quad e_n(iy) = \sum_{m=0}^n \frac{(iy)^m}{m!}.$$

This method seems unnecessarily complicated.

A more powerful method is to utilize the Chebyshev polynomials. Instead of expanding the exponential in (1) into a Taylor series, we expand into a series of Chebyshev polynomials. As Lanczos [4] has pointed out, "While the expansion into powers on the basis of the Taylor's series gives the slowest convergence, the expansion into the Chebyshev polynomials gives the fastest convergence."

It is convenient to transform the integral slightly by setting $t = ixq$ in (1)

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$$(5) \quad \gamma(\nu, ix) = (iq)^\nu \int_0^1 e^{-ixq} q^{\nu-1} dq.$$

Expanding the exponential into the series [5]

$$(6) \quad e^{-ixq} = \sum_{n=0}^{\infty} \epsilon_n (-i)^n J_n(x) T_n(q) \quad |q| \leq 1$$

where ϵ_n is Newmann's factor ($\epsilon_0 = 2$; $\epsilon_n = 1$ for $n \geq 1$) and $T_n(q)$ is the Chebyshev polynomial of the first kind

$$(7) \quad T_0(q) = 1$$

$$T_n(q) = \frac{n}{2} \sum_{s=0}^{\lfloor n/2 \rfloor} \frac{(-1)^s (n-s-1)!}{s!(n-2s)!} (2q)^{n-2s}.$$

Substituting (6) into (5) and interchanging sum and integral (this is permissible because of uniform convergence of the series) leads to the integrals

$$(8) \quad B_0(\nu) \equiv \frac{1}{\nu}$$

$$B_n(\nu) \equiv \int_0^1 T_n(q) q^{\nu-1} dq$$

$$= \frac{n}{2} \sum_{s=0}^{\lfloor n/2 \rfloor} \frac{(-1)^s (n-s-1)! 2^{n-2s}}{s!(n-2s)!(n-2s+\nu)}.$$

Finally (5) becomes

$$(9) \quad \gamma(\nu, ix) = (iq)^\nu \sum_{n=0}^{\infty} \epsilon_n (-i)^n J_n(x) B_n(\nu).$$

To obtain an error estimate on B_n for large n , we use the relation

$$(10) \quad T_n(q) = \cos n\theta, \quad q = \cos \theta$$

so that (8) becomes

$$(11) \quad B_n(\nu) = \int_0^{\pi/2} \cos n\theta (\cos \theta)^{\nu-1} \sin \theta d\theta$$

$$\sim 0(1).$$

The asymptotic expression for $J_n(x)$ when n is large and x is fixed is given by [6]

$$(12) \quad J_n(x) \sim \frac{(x/2)^n}{n!} \quad \left(\frac{x^2}{4} \ll n \right).$$

Hence the dominating factor is $J_n(x)$.

As an example of the power of the method, we consider the case of the Fresnel integrals ($\nu = \frac{1}{2}$), using as a standard the recent table by Pearcey [7]. The first thirteen values of $B_n(.5)$ are:

$B_0 = 2.0$	$B_7 = -0.49696970$
$B_1 = 0.66666667$	$B_8 = 0.42976370$
$B_2 = -1.20$	$B_9 = 0.40710869$
$B_3 = -0.85714286$	$B_{10} = -0.40752711$
$B_4 = 0.57777778$	$B_{11} = -0.38709818$
$B_5 = 0.52813853$	$B_{12} = 0.35566473$
$B_6 = -0.54358974$	

Omitting the superfluous factor $(iq)^{-5}$ in (5) yields

$$\begin{aligned}
 \gamma_1(.5, ix) &= \gamma(.5, ix)(iq)^{-5} \\
 (13) \qquad &= \sum_{n=0}^{\infty} \epsilon_n (-i)^n J_n(x) B_n(.5) \\
 &= \left(\frac{2\pi}{x}\right)^{1/2} [C(x) - iS(x)].
 \end{aligned}$$

The results for various values of x are tabulated below. The numbers in parentheses indicate the number of terms used in the series expansion.

	$x = 1$	$x = 2$	$x = 3$	$x = 4$
Pearcey series	1.8090490 (4)	1.335193 (5)	0.811910 (6)	0.461462 (7)
	1.80902564 (3)	1.3351744 (4)	0.8118993 (5)	0.4614576 (6).

Similar results hold for the imaginary part of γ_1 . The small number of terms required for six-place accuracy is remarkable compared with the usual Taylor series expansion.

Using this method, integrals have been computed corresponding to $\nu = .2, .4, .6, .8$ in the interval $0(.1)5$. The integrals in question are

$$\begin{aligned}
 (14) \qquad \gamma_1(\nu, ix) &= \int_0^1 e^{-ixq} q^{\nu-1} dq \\
 &= (iq)^{-\nu} \gamma(\nu, ix)
 \end{aligned}$$

(see equation 5).

The first fifteen values of $B_n(\nu)$ are listed in Table 1, while in Table 2 we give the values of the real and imaginary values of $\gamma_1(\nu, ix)$ to six decimals.

TABLE 1

B_n/ν	.2	.4	.6	.8
B_0	5.0000000	2.5000000	1.6666667	1.2500000
B_1	.8333333	.7142857	.6250000	.5555556
B_2	-4.0909090	-1.6666667	-.8974359	-.53571429
B_3	-1.2500000	-.9663865	-.7638889	-.61403509
B_4	3.26839827	.9848484	.32887403	.05952381
B_5	.99358973	.65203859	.42658730	.27324056
B_6	-3.08546292	-.9090909	-.32988751	-.11554623
B_7	-.98290598	-.62150387	-.39839182	-.25726340
B_8	2.86921706	.76839826	.23742520	.05315763
B_9	.90301003	.53065850	.31171679	.18010639
B_{10}	-2.76714376	-.72594072	-.23077198	-.06744618
B_{11}	-.88701757	-.50877478	-.29504438	-.17180411
B_{12}	2.65095170	.65829331	.19072285	.04254734
B_{13}	.8439206	.4620702	.2532754	.1368987
B_{14}	-2.581791	-.630248	-.185156	-.048825
B_{15}	-.624846	-.335736	-.187154	-.108511

TABLE 2

Re[$\gamma_1(\nu, iz)$]					Im[$\gamma_1(\nu, iz)$]				
x/ν	.2	.4	.6	.8	x/ν	.2	.4	.6	.8
0	5.000000	2.500000	1.666667	1.250000	0	0	0	0	0
.1	4.997728	2.497918	1.664744	1.248215	.1	-.083281	-.071380	-.062454	-.055512
.2	4.990925	2.491682	1.658989	1.242871	.2	-.166251	-.142465	-.124630	-.110761
.3	4.979626	2.481327	1.649432	1.233999	.3	-.248598	-.212966	-.186254	-.165486
.4	4.963889	2.466908	1.636128	1.221650	.4	-.330016	-.282593	-.247052	-.219430
.5	4.943798	2.448505	1.619153	1.205896	.5	-.410206	-.351063	-.306759	-.272340
.6	4.919457	2.426217	1.598600	1.186830	.6	-.488874	-.418102	-.365115	-.323971
.7	4.890992	2.400165	1.574586	1.164560	.7	-.565736	-.483443	-.421868	-.374084
.8	4.858551	2.370489	1.547245	1.139217	.8	-.640519	-.546831	-.476778	-.422454
.9	4.822300	2.337349	1.516730	1.110945	.9	-.712964	-.608020	-.529616	-.468863
1.0	4.782427	2.300922	1.483209	1.079908	1.0	-.782825	-.666783	-.580166	-.513108
1.1	4.739134	2.261403	1.446869	1.046282	1.1	-.849872	-.722903	-.628226	-.554999
1.2	4.692643	2.219001	1.407910	1.010259	1.2	-.913890	-.776182	-.673611	-.594363
1.3	4.643186	2.173939	1.366545	.972044	1.3	-.974687	-.826440	-.716151	-.631041
1.4	4.591014	2.126454	1.322998	.931850	1.4	-1.032085	-.873513	-.755697	-.664892
1.5	4.536384	2.076793	1.277506	.889904	1.5	-1.085929	-.917259	-.792115	-.695794
1.6	4.479566	2.025210	1.230312	.846438	1.6	-1.136085	-.957555	-.825293	-.723642
1.7	4.420838	1.971971	1.181667	.801691	1.7	-1.182439	-.994298	-.855137	-.748351
1.8	4.360484	1.917344	1.131828	.755908	1.8	-1.224902	-1.027407	-.881576	-.768857
1.9	4.298791	1.861603	1.081054	.709337	1.9	-1.263406	-1.056824	-.904557	-.788113
2.0	4.236050	1.805022	1.029608	.662227	2.0	-1.297904	-1.082511	-.924050	-.803095
2.1	4.172552	1.747879	.977750	.614827	2.1	-1.328374	-1.104452	-.940044	-.814797
2.2	4.108588	1.690447	.925742	.567385	2.2	-1.354815	-1.122653	-.952550	-.823233
2.3	4.044445	1.632998	.873840	.520143	2.3	-1.377251	-1.137140	-.961600	-.828439
2.4	3.980406	1.575798	.822206	.473342	2.4	-1.395725	-1.147963	-.967244	-.830468
2.5	3.916747	1.519107	.771356	.427214	2.5	-1.410303	-1.155190	-.969553	-.829390
2.6	3.853735	1.463177	.721257	.381983	2.6	-1.421070	-1.158907	-.968616	-.825297
2.7	3.791628	1.408251	.672227	.337863	2.7	-1.428133	-1.159223	-.964540	-.818295
2.8	3.730674	1.354558	.624483	.295059	2.8	-1.431615	-1.156262	-.957449	-.808507
2.9	3.671107	1.302318	.578228	.253763	2.9	-1.431659	-1.150165	-.947483	-.796071
3.0	3.613147	1.251735	.533655	.214153	3.0	-1.428423	-1.141088	-.934797	-.781139
3.1	3.556998	1.203000	.490939	.176394	3.1	-1.422082	-1.129203	-.919558	-.763875
3.2	3.502850	1.156285	.450241	.140635	3.2	-1.412821	-1.114693	-.901946	-.744456
3.3	3.450873	1.111749	.411707	.107010	3.3	-1.400841	-1.097754	-.882152	-.723067
3.4	3.401222	1.069530	.375462	.075636	3.4	-1.386353	-1.078591	-.860375	-.699904
3.5	3.354030	1.029750	.341618	.046611	3.5	-1.369574	-1.057416	-.836824	-.675167
3.6	3.309413	.992512	.310265	.020018	3.6	-1.350732	-1.034451	-.811711	-.649064
3.7	3.267467	.957899	.281477	-.004080	3.7	-1.330060	-1.009920	-.785256	-.621806
3.8	3.228268	.925976	.255308	-.025637	3.8	-1.307793	-.984051	-.757678	-.593607
3.9	3.191871	.896787	.231794	-.044625	3.9	-1.284170	-.957077	-.729200	-.564681
4.0	3.158312	.870358	.210951	-.061033	4.0	-1.259431	-.929226	-.700045	-.535242
4.1	3.127608	.846696	.192780	-.074871	4.1	-1.233815	-.900728	-.670431	-.505501
4.2	3.099754	.825790	.177259	-.086163	4.2	-1.207556	-.871809	-.640578	-.475667
4.3	3.074730	.807608	.164354	-.094951	4.3	-1.180887	-.842691	-.610695	-.445944
4.4	3.052494	.792104	.154010	-.101294	4.4	-1.154033	-.813590	-.580990	-.416526
4.5	3.032988	.779212	.146158	-.105264	4.5	-1.127213	-.784712	-.551660	-.387604
4.6	3.016138	.768853	.140713	-.106951	4.6	-1.100636	-.756258	-.522895	-.359357
4.7	3.001852	.760931	.137576	-.106457	4.7	-1.074503	-.728417	-.494873	-.331955
4.8	2.990025	.755335	.136634	-.103895	4.8	-1.049002	-.701367	-.467764	-.305558
4.9	2.980539	.751945	.137763	-.099392	4.9	-1.024311	-.675274	-.441722	-.280310
5.0	2.973261	.750625	.140826	-.093084	5.0	-1.000593	-.650291	-.416892	-.256347

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