On Numbers of the Form $n^4 + 1$

By Daniel Shanks

1. The Number of Primes. Let $Q_1(N)$ be the number of primes of the form $n^4 + 1$ for $1 \leq n \leq N$. By a double sieve argument similar to that used for primes of the form $n^4 + a$, [1], and for Gaussian twin primes, [2], one is led to the following conjecture:

(1) \[ Q_1(N) \sim \frac{1}{4} s_1 \int_1^N \frac{dn}{\log n} \]

where

(2) \[ s_1 = \prod_{p=3}^{\infty} \left[ 1 - \frac{(-1)^{p-1}}{p-1} \right] \]

the product being taken over all odd primes with $\left( \frac{a}{p} \right)$ the Legendre symbol. Now

(3) \[ \frac{s_1 L_1(1)L_2(1)L_3(1)}{\zeta^2(2)} = \prod_{p=3m+1} \left( 1 - \frac{4}{p} \right) \left( \frac{p+1}{p-1} \right)^2 \]

where this product is taken over all primes of the form $8m + 1$ and $L_a(s)$ and $\zeta_a(s)$ are as defined in [1, p. 323]. We may therefore write

(4) \[ s_1 = \frac{\pi^2}{4 \log (1 + \sqrt{2})} \prod_{p=3m+1} \left( 1 - \frac{4}{p} \right) \left( \frac{p+1}{p-1} \right)^2. \]

To evaluate this slowly convergent product we use the identity

(5) \[ 1 - 4x = \left( \frac{1 - x}{1 + x} \right)^2 \left( \frac{1 - x^2}{1 + x^2} \right)^4 \left( \frac{1 - x^4}{1 + x^4} \right)^8 \left( \frac{1 - x^8}{1 + x^8} \right)^{\cdots}, \]

which is valid for $x < \frac{1}{2}$, and the identity

(6) \[ \frac{\zeta^2(2s)}{\zeta(s)L_1(s)L_2(s)L_3(s)} = \prod_{p=3m+1} \left( \frac{p^s-1}{p^s+1} \right)^2, \]

which is valid for $s > 1$. From tables of $\zeta_a(s)$ and $L_a(s)$ we thus obtain

(7) \[ s_1 = 2.67896 \cdots \]

and therefore

(8) \[ Q_1(N) \sim \hat{Q}_1(N) = 0.66974 \int_2^N \frac{dn}{\log n}. \]

It is interesting to compare this formula with that for the conjectured number [1] of primes of the form $n^2 + 1$,

(9) \[ P_1(N) \sim \hat{P}_1(N) = 0.68641 \int_2^N \frac{dn}{\log n}. \]

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ON NUMBERS OF THE FORM $n^4 + 1$  

Table 1

<table>
<thead>
<tr>
<th>$N$</th>
<th>$Q_1(N)$</th>
<th>$\hat{Q}_1(N)$</th>
<th>$Q_1/\hat{Q}_1$</th>
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<td>100</td>
<td>18</td>
<td>19.5</td>
<td>0.924</td>
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<td>32.9</td>
<td>0.911</td>
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<td>52</td>
<td>56.5</td>
<td>0.920</td>
</tr>
<tr>
<td>500</td>
<td>63</td>
<td>67.5</td>
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</tr>
<tr>
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<td>75</td>
<td>78.1</td>
<td>0.960</td>
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<td>80</td>
<td>88.4</td>
<td>0.905</td>
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<tr>
<td>800</td>
<td>94</td>
<td>98.6</td>
<td>0.934</td>
</tr>
<tr>
<td>900</td>
<td>98</td>
<td>108.5</td>
<td>0.903</td>
</tr>
<tr>
<td>1000</td>
<td>109</td>
<td>118.3</td>
<td>0.922</td>
</tr>
</tbody>
</table>

The coefficients are nearly equal and have analogous formulae:

$$0.68641 = \frac{1}{2} \prod_{p \equiv 3 \mod 4} \left[ 1 - \left( \frac{-1}{p} \right) \right]$$

(10)

$$0.66974 = \frac{1}{4} \prod_{p \equiv 3 \mod 4} \left[ 1 - \left( \frac{-1}{p} \right) + \left( \frac{2}{p} \right) + \left( \frac{-2}{p} \right) \right] .$$

2. A Table. A comparison of $\hat{Q}_1(N)$ with the actual counts $Q_1(N)$ is handicapped by the very rapid increase in $n^4 + 1$. The 109th prime is already 984 095 744 257, nearly a trillion. A. Gloden [3] has completed the factorization of all $n^4 + 1$ up to $n = 1000$, following the work of Cunningham and others. He has kindly counted the primes for us, where $400 < n \leq 1000$, and using his results we present Table 1. The deviations of $Q_1/\hat{Q}_1$ from unity are not unduly large considering the relatively small upper limit for $N$. For $P_1(N)$ and for the ordinary prime count $\pi(N)$ we have similar deviations for $N = 1000$; $\pi(1000)/\zeta(1000) = 0.951$ and $P_1(1000)/\hat{P}_1(1000) = 0.924$.

3. Four Classes of Numbers. When we consider that Euler determined $P_1(N)$ up to $N = 1500$ over two hundred years ago [4], the present table of $Q_1(N)$ up to $N = 1000$ seems rather meager. The much greater difficulty of factoring the $n^4 + 1$ numbers is fundamentally due to their much greater magnitude—but there are interesting technical differences also. The sieve method for $n^4 + 1$ used by Gloden, Cunningham, and others has three phases.

A. Compile a list of primes of the form $8m + 1$

B. For each such prime solve the congruence

$$\begin{cases} 
  x^4 \equiv -1 \pmod{p} \\
  x < p
\end{cases}$$

for its four roots. (Given one solution $x_1$, the remaining three are congruent to $-x_1$, $x_1^2$, and $-x_1^2$.)

C. With each $x$ and each $p$ divide out a factor of $p$ for each $n = x_1 + mp$. Similarly determine those $n^4 + 1$ divisible by $p^2$, $p^3$, etc.
Now unfortunately there is much waste computation here. For instance, the hundred \( n^4 + 1 \) for \( n \leq 100 \) have 122 different primes of the form \( 8m + 1 \) as factors.

Yet all 295 of the \( 8m + 1 \) primes \( < 100^2 \) must be examined in phases A and B, since \textit{a priori} any such prime may be a factor of the \( n^4 + 1 \). And clearly this waste increases rapidly with \( N \),—for \( N = 1000 \) we must examine all 19552 of the \( 8m + 1 \) primes \( < 1000^2 \) to factor out the (approximately) 1300 distinct actual prime factors.

On the contrary, in the author's sieve \cite{5} for \( n^2 + 1 \) there is no waste computation and no phases A and B, either. The primes arise automatically in the sieve itself, together with the corresponding solutions of the congruence, \( x^2 = -1 \ (\text{mod} \ p) \).

This significant difference comes about as follows. For every \( n \), \( n^4 + 1 \) either has no new prime factor (\( n \) is "reducible") or it has precisely one new prime factor—and that to the first power (\( n \) is "irreducible"). Therefore, if all prime factors corresponding to smaller values of \( n \) have already been sieved out, each new prime stands exposed at the smallest \( n \) which satisfies \( n^4 = -1 \ (\text{mod} \ p) \). But for \( n^4 + 1 \) we have not two but \textit{four} classes of \( n \); there are either 0, 1, 2, or 3 new prime factors in \( n^4 + 1 \). It is the occurrence of the "double" and "triple" irreducibles (i.e., 2 and 3 new primes) which prevents the use of the automatic, \( n^2 + 1 \) type sieve for \( n^4 + 1 \).

Already for \( n = 10 \) we have a double irreducible

\[
10^4 + 1 = 73 \cdot 137,
\]

with the two new primes 73 and 137.

Let \( R(N), I_1(N), I_2(N) \) and \( I_3(N) \) be the number of "reducibles" (no new prime) and single, double, and triple irreducibles respectively which are \( \leq N \). For example, \( I_1(120) = 92 \) and \( I_2(120) = 28 \). Further, \( R(120) = I_3(120) = 0 \), since neither reducibles nor triple irreducibles arise for \( n \leq 120 \). For larger \( n \) (from Gleden's tables) we find both reducibles

\[
29588^4 + 1 = 17^2 \cdot 41 \cdot 113 \cdot 1249 \cdot 16073 \cdot 28513
\]

and triple irreducibles

\[
23762^4 + 1 = 637489 \cdot 693569 \cdot 721057,
\]

but they are rare.

The mean number of new primes is

\[
\nu(N) = \frac{I_1(N) + 2I_2(N) + 3I_3(N)}{N},
\]

and in analogy with the situation for \( n^2 + 1 \) the question arises whether \( \nu(N) \) has a limit for \( N \to \infty \). For \( n^2 + 1 \), John Todd \cite[p. 83]{5} has conjectured \( \nu(N) \to \log 2 = 0.693 \). For \( n^4 + 1 \) and a modest \( N \) we have \( \nu(N) \approx 1.3 \). Analogy with Todd's results concerning \( n^2 + 1 \) and \( \log 2 \) would suggest a limit of \( \log 4 \) for \( n^4 + 1 \), but there is no serious evidence in favor of this.

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4. L. E. Dickson, History of the Theory of Numbers, Stechert, New York, 1934, v. 1, p. 381. According to Dickson, Euler (1752) gave $P_1(1500) = 161$, which is correct, and $Q_1(34) = 8$, which is incorrect—he omits the prime $28^4 + 1$.