scheme is advantageous for problems in which minimum mixing of the marks at each step is important.

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1. C. B. Tompkins, "Machine attacks on problems whose variables are permutations," *Proceedings of Symposia in Applied Mathematics*, v. VI, *Numerical Analysis*, McGraw-Hill, New York, 1956, pp. 195–211.
2. D. H. Lehmer, "Teaching combinatorial tricks to a computer," *Proceedings of Symposia in Applied Mathematics*, v. X, *Combinatorial Analysis*, American Mathematical Society, Providence, R. I., 1960, p. 179–193.

Chebyshev Approximations to the Gamma Function

By Helmut Werner and Robert Collinge

In this note several Chebyshev approximations are given for the function $y = \Gamma(x + 2)$ for x in the $0 \leqq x \leqq 1.0$ range. The approximations were obtained from a table of $\Gamma(x + 2)$, employing well-known methods as described in numerous papers; see for instance [1] and the literature quoted there. The table of $\Gamma(x + 2)$ was calculated from the asymptotic expansion of $\log \Gamma(z)$ as given in [2] to provide data accurate to at least 10^{-21}. Compare also [3].

The asymptotic expansion of $\ln \Gamma(z)$ is given by

$$\ln \Gamma(z) = (z - \tfrac{1}{2}) \ln z - z + \ln \sqrt{2\pi} + \Phi(z)$$

where

$$\Phi(z) = \sum_{r=1}^{n} \frac{(-1)^{r-1} B_r}{2r(2r - 1)} \frac{1}{z^{2r-1}} + R_n(z),$$

and B_r is the rth Bernoulli number.

It can be shown [2] that for $z > 0$ the value of $\Phi(z)$ always lies between the sum of n terms and the sum of $(n + 1)$ terms of the series, for all values of n. In terminating this series with the nth term the error $R_n(z)$ will be less than

$$\frac{B_{n+1}}{2(n + 1)(2n + 1)} \cdot \frac{1}{z^{2n+1}}.$$

By truncating $\Phi(z)$ at the 10th term it is easily shown that for values of $z \geqq 13$, the error in the expansion is less than 5.5×10^{-22}. We therefore replace $\Phi(z)$ by $\sum_{i=1}^{10} A_i/z^{2i-1}$ and calculate $\ln \Gamma(z)$ for values of z in the range $13 \leqq z \leqq 14$.

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TABLE 1

Table of Coefficients

n	7	8	10
$\epsilon_{max}^{(n)}$	0.25×10^{-7}	0.16×10^{-9}	0.74×10^{-11}
$\nu = 0$	0.99999 99758	0.99999 99998 452	0.99999 99999 9269
1	0.42278 74605	0.42278 43662 730	0.42278 43369 6202
2	0.41177 41955	0.41183 92935 920	0.41184 02517 9616
3	0.08211 17404	0.08159 03449 474	0.08157 82187 8492
4	0.07211 01567	0.07416 00915 535	0.07423 79076 0629
5	0.00445 11400	0.00007 55964 181	−0.00021 09074 6731
6	0.00515 89951	0.01033 20685 065	0.01097 36958 4174
7	0.00160 63118	−0.00157 80074 635	−0.00246 67479 8054
8		0.00079 62464 760	0.00153 97681 0472
9			−0.00034 42342 0456
10			0.00006 77105 7117

n	13	15
$\epsilon_{max}^{(n)}$	0.96×10^{-14}	0.97×10^{-16}
$\nu = 0$	0.99999 99999 99990 44	0.99999 99999 99999 9032
1	0.42278 43351 02334 79	0.42278 43350 98518 1178
2	0.41184 03301 66781 29	0.41184 03304 21981 4831
3	0.08157 69261 24155 46	0.08157 69194 01388 6786
4	0.07424 89154 19444 74	0.07424 90079 43401 2692
5	−0.00026 61865 94953 06	−0.00026 69510 28755 5266
6	0.01114 97143 35778 93	0.01115 38196 71906 6992
7	−0.00283 64625 20372 82	−0.00285 15012 43034 6494
8	0.00206 10918 50225 54	0.00209 97590 35077 0629
9	−0.00083 75646 85135 17	−0.00090 83465 57420 0521
10	0.00037 53650 52263 07	0.00046 77678 11496 4956
11	−0.00012 14173 48706 32	−0.00020 64476 31915 9326
12	0.00002 79832 88993 83	0.00008 15530 49806 6373
13	−0.00000 30301 90810 28	−0.00002 48410 05384 8712
14		0.00000 51063 59207 2582
15		−0.00000 05113 26272 6698

n	17	18
$\epsilon_{max}^{(n)}$	0.10×10^{-17}	0.10×10^{-18}
$\nu = 0$	0.99999 99999 99999 99901 2	0.99999 99999 99999 99990 02
1	0.42278 43350 98467 79580 6	0.42278 43350 98467 21319 64
2	0.41184 03304 26367 20638 1	0.41184 03304 26430 62304 23
3	0.08157 69192 50260 90508 9	0.08157 69192 47528 84581 87
4	0.07424 90106 80090 41696 9	0.07424 90107 42094 91715 38
5	−0.00026 69810 33348 38176 8	−0.00026 69818 88740 38315 07
6	0.01115 40360 24034 39169 2	0.01115 40438 29069 91793 28
7	−0.00285 25821 44619 65607 6	−0.00285 26318 64702 11862 89
8	0.00210 36287 02459 83329 2	0.00210 38579 20672 20524 09
9	−0.00091 84843 69099 08014 2	−0.00091 92675 95039 95026 11
10	0.00048 74227 94476 75810 4	0.00048 94361 06998 14458 34
11	−0.00023 47204 01891 94985 9	−0.00023 86428 33752 63647 10
12	0.00011 15339 51966 59947 0	0.00011 73283 10224 09396 51
13	−0.00004 78747 98383 44672 4	−0.00005 43183 86280 13508 99
14	0.00001 75102 72717 90508 0	0.00002 28140 41153 66022 75
15	−0.00000 49203 75090 42313 2	−0.00000 80523 43363 48309 46
16	0.00000 09199 15640 71621 4	0.00000 21741 77495 45532 64
17	−0.00000 00839 94049 59039 7	−0.00000 03889 70057 38769 55
18		0.00000 00339 81801 01810 43

For the convenience of the reader the A_i coefficients are quoted below, to 25 significant figures.

$$
\begin{aligned}
A_1 &= 0.08333\ 33333\ 33333\ 33333\ 33333\ 3\\
A_2 &= -0.00277\ 77777\ 77777\ 77777\ 77777\ 78\\
A_3 &= 0.00079\ 36507\ 93650\ 79365\ 07936\ 508\\
A_4 &= -0.00059\ 52380\ 95238\ 09523\ 80952\ 381\\
A_5 &= 0.00084\ 17508\ 41750\ 84175\ 08417\ 508\\
A_6 &= -0.00191\ 75269\ 17526\ 91752\ 69175\ 27\\
A_7 &= 0.00641\ 02564\ 10256\ 41025\ 64102\ 56\\
A_8 &= -0.02955\ 06535\ 94771\ 24183\ 00653\ 6\\
A_9 &= 0.17964\ 43723\ 68830\ 57316\ 49385\\
A_{10} &= -1.39243\ 22169\ 05901\ 11642\ 7432\\
\ln \sqrt{2\pi} &= 0.91893\ 85332\ 04672\ 74178\ 03297
\end{aligned}
$$

A triple precision logarithm routine was used to evaluate $\ln z$, and then an exponential routine to calculate $\Gamma(z) = e^{\ln \Gamma(z)}$. Each of these routines produces results accurate to at least 24 significant digits.

After obtaining a table of $\Gamma(z)$ for z in the range $13 \leqq z \leqq 14$, we made use of the recursion formula $\Gamma(z + 1) = z\Gamma(z)$ in order to obtain a table of $\Gamma(x + 2)$ for x in the range $0 \leqq x \leqq 1.0$.

From the tests made on the results obtained, the values of $\Gamma(x + 2)$ were shown to be accurate to at least 21 significant figures.

Several Chebyshev approximations have been calculated to provide varying degrees of accuracy. Let

$$\Gamma(2 + x) = \sum_{\nu=0}^{n} a_\nu^{(n)} x^\nu + \epsilon_n(x)$$

and

$$\epsilon_{\max}^{(n)} = \max_{0 \leqq x \leqq 1} | \epsilon_n(x) |.$$

Table 1 gives the coefficients $a_\nu^{(n)}$ for $n = 7, 8, 10, 13, 15, 17, 18$ together with the corresponding $\epsilon_{\max}^{(n)}$.

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1. E. L. Stiefel, "Numerical methods of Tchebycheff approximation," *On Numerical Approximation* by R. E. Langer, The University of Wisconsin Press, Madison, 1959, p. 217–232.

2. E. T. Whittaker & G. N. Watson, *A Course of Modern Analysis*, Cambridge, 1958, p. 251–253.

3. M. E. Sherry & S. Fulda, "Calculation of gamma functions to high accuracy," *MTAC*, v. 13, 1959, p. 314–315.

4. H. S. Uhler, "Log π and other basic constants," *Proc. Nat. Acad. Sci. U. S. A.*, v. 24, 1938, p. 23–30.

5. H. S. Uhler, "The coefficients of Stirling's series for log $\Gamma(x)$," *Proc. Nat. Acad. Sci. U. S. A.*, v. 28, 1942, p. 59–62.